EME 303 - FLUID MECHANICS
SUMMARIZED EQUATIONS AND CONCEPTS

Some Useful Conversion Factors

<table>
<thead>
<tr>
<th></th>
<th>SI</th>
<th>BGS</th>
<th>EE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature:</td>
<td>( K =^\circ C + 273.15 )</td>
<td>( ^\circ R =^\circ F + 459.67 )</td>
<td>( ^\circ R =^\circ F + 459.67 )</td>
</tr>
<tr>
<td>Force:</td>
<td>( 1N = (1kg)(1m / s^2) )</td>
<td>( 1lb = (1slug)(1ft / s^2) )</td>
<td>( 1lb = (1lbm)(32.2 ft / s^2) )</td>
</tr>
<tr>
<td>Mass:</td>
<td>( kg )</td>
<td>( slug )</td>
<td>( lbm )</td>
</tr>
<tr>
<td>Density:</td>
<td>( 1 kg / m^3 )</td>
<td>( 0.00194 slug / ft^3 )</td>
<td>( 0.06243 lbm / ft^3 )</td>
</tr>
<tr>
<td>( \rho_{\text{water}} )</td>
<td>( 1000kg / m^3 )</td>
<td>( 1.94slugs / ft^3 )</td>
<td>( 62.4lbm / ft^3 )</td>
</tr>
<tr>
<td>Pressure:</td>
<td>( 1Pa = 1N / m^2 )</td>
<td>( 0.0209 lb / ft^2 )</td>
<td>( 0.0209 lbf / ft^2 )</td>
</tr>
<tr>
<td>Work, energy:</td>
<td>( 1J = 1N.m )</td>
<td>( 1 ft.lb = 778.2Btu )</td>
<td>( — )</td>
</tr>
<tr>
<td>Power:</td>
<td>( 1W = 1N.m / s )</td>
<td>( 1hp = 550 ft.lb / s )</td>
<td>( — )</td>
</tr>
</tbody>
</table>

General [Topic:1]

\[ \mathbf{F} = \mathbf{ma} \text{ or } \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

\[ p = \rho RT \]

\[ \begin{aligned} 
\text{Isothermal: } & \frac{p}{\rho} = \text{const.} \\
\text{Isentropic: } & \frac{p}{\rho^k} = \text{const.} 
\end{aligned} \]

\[ \tau = \mu \frac{du}{dy}; \quad \nu = \frac{\mu}{\rho} \]

\[ E_v = -\frac{dp}{d\varphi / \varphi} = \frac{dp}{d\rho / \rho} \]

\[ h = \frac{2\sigma \cos \theta}{\gamma R} \]
Recap
Definitions
Dimensional homogeneity
Fluid properties

Mass and Weight
\[ M = \rho V; \ W = Mg \]

Equations of State
\[ p = \rho RT \]

Compressibility
\[ E_v = -\frac{dp}{dV/V_0} = +\frac{dp}{d\rho/\rho_0} \]

Outline

Wave Speeds
\[ c = \sqrt{g\gamma} \text{ or } \sqrt{E_v/\rho} \]

Viscosity
\[ \tau = \mu \frac{\partial v_x}{\partial y} \]

Vapor pressure (airfoil)
Surface tension (vinometer)

Topic 2 Pressure

Fluid pressure at a point
Incompressible (water)
Compressible (atmosphere)

http://en.wikipedia.org/wiki/Supercritical_fluid
1.7.2 Compression/Expansion of Gases

Ideal gas law \[ p = \rho RT \]

Compression at constant temperature (Isothermal) \((dT=0)\)
\[ \frac{p}{\rho} = \text{constant} \quad (1.14) \]

Compression - Frictionless (Isentropic)

Adiabatic - no heat exchange to surroundings - no heat loss
\[ \frac{p}{\rho^k} = \text{constant} \quad (1.15) \]

\[ k = \frac{\text{Specific heat @ const. pressure}}{\text{Specific heat @ const volume}} \]

\(k\) = Specific heat @ const. pressure

Table 1.7 & 1.8.

Moduli of gases are given as, \(E_e\)
\[ E_e = - \frac{dp}{dA/V_0} = + \frac{dp}{dp/\rho_0} = \frac{dp}{dp} \cdot \rho_0 \]

Isothermal: \(p = \rho \cdot \text{const.}\) \(\Rightarrow \frac{dp}{dp} = \text{const.}\) \(\Rightarrow E_e = \text{const.} \cdot \rho \)

Also \(\rho = \frac{p}{\text{const}}\)

Isentropic: Similar procedure \(E_e = kp\)

Modulus, \(E_e\), is directly prop. to pressure \(\therefore\) compressible!!
The action of a fluid may be described as a series of very thin sheets each of which slip relative to the next.

Through experimentation it has been shown that the velocity gradient \((\frac{du}{dy})\) times the viscosity \((\mu)\) is equal to the shearing stress \((\tau)\) between the thin sheets.

\[
\tau = \mu \frac{dv}{dy}
\]

Experimentally determined

Note: \(dv\) may be represented by \(dv\)

Velocity may be represented as \(U\) or \(V\).

Shear stress, \(\tau = \frac{F}{A}\)

Pull at const. velocity

Units:

\[
\mu = \frac{\tau}{\frac{dv}{dy}} = \frac{ML^{-1}T^{-2}}{LT^{-1}/L} \Rightarrow \mu = ML^{-1}T^{-1}
\]
The capillary rise (or depression) as shown in the figure below is expressed as:

\[ h = \frac{2\sigma \cos \theta}{\gamma r} \]

\(\sigma\) = surface tension in units of force per unit length
\(\theta\) = wetting angle
\(\gamma\) = specific weight of the liquid
\(r\) = radius of the tube
\(h\) = capillary rise (from inflection point of meniscus)

For a clean tube, \(\theta = 0^\circ\) \(H_2O\), \(140^\circ\) \(Hg\)

For tube diameters > \(\frac{1}{2}\)" (12mm) capillary effects are negligible.
Fluid Statics [2,3]

\[ p_x = p_y = p_z = p \]

\[-\nabla p - \gamma \hat{k} = \rho a \quad \text{or} \quad - \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} p - \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \rho \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

Simplifies when: \( a_x = a_y = a_z = 0 \) to \( \frac{dp}{dx} = \frac{dp}{dy} = 0 \) and \( \frac{dp}{dz} = -\gamma \), \( \frac{dz}{dx} = -\frac{a_z}{g + a_z} \)

Rigid body rotation: \( \frac{dp}{dr} = \rho r\omega^2 \); \( \frac{dp}{\partial \theta} = 0 \); \( \frac{dp}{dz} = -\gamma \); \( \begin{cases} z = \frac{a_r}{2g} + c \\ p = \frac{\rho \omega^2 z^2}{2} - \gamma z + c \end{cases} \)

Incompressible fluid: \( p = \gamma h + p_0 \)

Compressible fluid: \( \frac{dp}{dz} = -\frac{g \rho}{RT} \) and integrate w.r.t \((p, z)\).

Manometer rules: \((\uparrow -ve) (\downarrow +ve); \frac{dp}{dx} = \frac{dp}{dy} = 0; p_v \) if evacuated; \( \gamma_{gas} \to 0 \).

\[ F_R = \gamma h A; \quad F_R = \sqrt{F_v^2 + F_v^2} \]

\[ F_R \text{ acts through center of pressure } \begin{cases} y_R = \frac{I_{xc}}{y_c A} + y_c \\ x_R = \frac{I_{xc}}{y_c A} + x_c \end{cases} \]

\[ F_B = \gamma V \]
PRESSURE AT A POINT

INCOMPRESSIBLE FLUID

\[ \frac{dp}{dz} = -y \]

Integrate as:
\[ \int_{p_1}^{p_2} dp = -\int_{z_1}^{z_2} y \, dz \]

Insert limits as:
\[ p_2 - p_1 = -y(z_2 - z_1) \]

or:
\[ p_1 = y(z_2 - z_1) + p_2 \]

Static head or head difference, \( h \).

Pressure, \( p_0 = 0 \)

Set \( p_2 = \xi_2 \) on surface \( p_2^0 \)

then
\[ p_1 = yh + p_2^0 \]

Note change in \( p \) is linear with depth.

Pressure head:
\[ h = \frac{p}{y} \]
Recap

Fluid pressure at a point (static) \( p_x = p_y = p_z = p \)

\[-\nabla p - \gamma \hat{k} = \rho \text{a} \text{ or } - \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} p - \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \rho \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

Simplifies when: \( a_x = a_y = a_z = 0 \) to \( \frac{dp}{dx} = \frac{dp}{dy} = 0 \) and \( \frac{dp}{dz} = -\gamma \)

Incompressible (water)

Compressible (atmosphere)

Outline

Pressure measurement (manometry)

Manometer rules: \((\uparrow -ve)(\downarrow +ve)\); \( \frac{dp}{dx} = \frac{dp}{dy} = 0; p_v \) if evacuated; \( \gamma_{\text{gas}} \rightarrow 0. \)
Pressure at a Point (Cont'd)

Compressible Fluid - Gases

\[ \rho \approx \gamma = f(p,T) \]

Important in column.

General equation:

\[ \frac{dp}{dz} = -\gamma \]

\[ \Delta p \text{ is small, even for large } dz \text{ since } \gamma \text{ is small.} \]

\[ \gamma_{\text{water}} = 62.4 \text{ lb/ft}^3 \]
\[ \gamma_{\text{air}} = 0.0763 \text{ lb/ft}^3 \]

For large height variations (\(dz \) large)

\[ p = \rho RT \]

\[ \frac{dp}{dz} = -\gamma = \gamma g \]

\[ -\gamma g = \frac{dp}{RT} \]

Separate variables:

\[ \int_{z_1}^{z_2} \frac{-g}{RT} \, dz = \int_{P_1}^{P_2} \frac{dp}{P} \]

\[ -g \int_{z_1}^{z_2} \frac{dz}{T} = (ln p_2 - ln p_1) = ln \left( \frac{P_2}{P_1} \right) \]

\[ \text{How does temperature vary, } T. \]
If $T$ is constant in the range $z_1 \rightarrow z_2$ (isothermal)

$$T = T_0$$

Then:

$$-\frac{g}{R T_0} \int_{z_1}^{z_2} dz = \ln \left( \frac{P_2}{P_1} \right)$$

$$\exp \left[ -\frac{g (z_2 - z_1)}{R T_0} \right] = \exp \left[ \ln \left( \frac{P_2}{P_1} \right) \right]$$

$$P_2 = P_1 \exp \left[ -\frac{g (z_2 - z_1)}{R T_0} \right]$$

Q.E.D.
WHAT DO WE KNOW?

\[ P_x = P_y = P_0 \quad \text{at a point} \]

\[ \frac{dp}{dx} = \frac{dp}{dy} = 0 \quad \frac{dp}{dz} = - \gamma \]

\[ p_i = \gamma h + P_0 \]

\[ p_A = p_B !! \]

\[ p_A = p_B ? \]
From Table 1.6

\[ \gamma_w = 9.8 \text{ kN/m}^3 \]
\[ \gamma_{Hg} = 133 \text{ kN/m}^3 \]

Rules: (-ve \uparrow) \quad (+ve \downarrow)

\[ P_A - 10 \gamma_w + 5 \gamma_w + 5 \gamma_w + 2 \gamma_w - 7 \gamma_{Hg} = P_{atm} \]

\[ P_A + 2 \gamma_w - 7 \gamma_{Hg} = 0 \]

\[ P_A = -2 \gamma_w + 7 \gamma_{Hg} = -2(9.8) + 7(133) \text{ kPa} \]

\[ P_A = 911.4 \text{ kPa} = \text{KN/m}^2 \]
Hydrostatic Forces Acting on Surfaces.

Why?

![Diagram of forces acting on a submerged object](image)

Questions:
1. What is the magnitude of force $F_H$?

$$S = N \tan \phi$$

Resolve vertically: $N = W - F_v$

Strength of base $S = (W - F_v) \tan \phi$

if $F_H > S$ Translational failure.

if $F_H \leq S$ Stable equilibrium.

... Need to know force magnitudes!!
2. Where does the force act?

\[ \Sigma H_0 = 0 \]

\[ F_H l_2 - (W - F_v) l_1 = 0 \]

\[ F_H = \frac{(W - F_v) l_1}{l_2} \]

If \( F_H \) is larger than this magnitude limit, then the dam “fails” by overturning.

"Need to know where the forces act!!"
CENTER OF PRESSURE

Centroid is the "balance" point of the plate!

Center of pressure is the "balance" point for the pressure distribution.

Determine C of Pressure by summing moments around 'x' axis.
As before: \[ F_R x_R = \int_A y \sin \theta \, xy \, dA \]

Pressure varies from surface \((y)\), but moment taken about \(\frac{x}{y}\):

\[ x_R = \frac{\int_A xy \, dA}{y_c A} = \frac{I_{xy}}{y_c A} \]

Parallel axes theorem:

\[ x_R = \frac{I_{xy_c}}{y_c A} + x_c \]

\[ y_R = \frac{I_{xc}}{y_c A} + y_c \]

\[ F_R = \gamma h_c A \]

\[ h_c = y_c \sin \theta \]
\textbf{FIGURE 2.18} Geometric properties of some common shapes.

For any shape symmetric w.r.t. $x=0$ then $I_{xyc}=0$.
The 6-ft-diameter drainage conduit of Fig. E2.9a is half full of water at rest. Determine the magnitude and line of action of the resultant force that the water exerts on a 1-ft length of the curved section BC of the conduit wall.

From Geometry

\[ F_1 = F_H \]
\[ F_V = W \]

\[ F_1 = \gamma h c A \Rightarrow F_1 = \frac{(62.4) \text{ lb}}{\text{ft}^3} \left( \frac{\pi}{2} \text{ ft}^2 \right)(3) \text{ ft}^2 \]

Unit length along pipe

\[ F_1 = 281 \text{ lb} = F_H \]

\[ W = \frac{\gamma \pi r^2}{4} = \frac{(62.4) \text{ lb}}{\text{ft}^3} \left( \frac{\pi}{2} \text{ ft}^2 \right)(1) \text{ ft} = 441 \text{ lb} = F_V \]

Resultant
\[ F_R = \sqrt{F_H^2 + F_V^2} = 523 \text{ lb} \]

Direction of resultant: Pressure \( F_H \) to conduit wall... all pressure "vectors" pass through 'O'!

Consequently Resultant passes through 'O'.

\[ 523 \cos \theta = 441 \]
\[ \cos \theta = \frac{441}{523} \]
\[ \theta = 32.5^\circ \]
HYDROSTATIC FORCE ON CURVED SURFACE

Isolate free body

\[ \Sigma F_H = 0 \]

\[ F_H = F_2 \]

\[ F_V = F_1 + W \]

\[ F_R = \sqrt{F_H^2 + F_V^2} \]

(Point of action, \( O \))

Determined by summing \( F_H, F_V \) and \( F_R \) (all known) about an appropriate axis.
2.84 The 9-ft-long cylinder of Fig. P2.84 floats in oil and rests against a wall. Determine the horizontal force the cylinder exerts on the wall at the point of contact, \( A \).

![Figure P2.84](image)

The horizontal forces acting on the free-body-diagram are shown on the figure. For equilibrium,

\[
F_A = F_1 - F_2
\]

where \( F_A \) is the horizontal force the wall exerts on the cylinder.

Since,

\[
F_1 = \gamma h c_1 \ A_1
\]

\[
= \left( 57.0 \ \frac{lb}{ft^3} \right) \left( \frac{6 ft}{2} \right) (6 ft \times 9 ft)
\]

\[
= 9230 \ lb
\]

and

\[
F_2 = \gamma h c_2 \ A_2
\]

\[
= \left( 57.0 \ \frac{lb}{ft^3} \right) \left( 3 ft + \frac{3}{2} ft \right) (3 ft \times 9 ft)
\]

\[
= 6930 \ lb
\]

then

\[
F_A = 9230 \ lb - 6930 \ lb = 2300 \ lb \rightarrow \text{on the wall}
\]
Determine the horizontal and vertical forces acting on this gate.

\[ Y = 10 \text{ kN/m}^3 \]
\[ W = 1 \text{ m} \text{- in. page} \]

These approaches to solve the same problem—they are equivalent.

**Centroid and Centre of Pressure Approach**

\[ F_R = YA h_c = 10 \text{ kN/m}^3 (1 \times \sqrt{2}) m^2 \times \frac{1}{2} m = 5 \sqrt{2} \text{ kN} \]
\[ Y_R = y_c + \frac{I_{zc}}{Y_c A} = \frac{1}{2} \sqrt{2} m + \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \sqrt{2} m} = \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{2} \]

**Pressure Prism Approach**

\[ F_R = \frac{1}{2} P \times A = \frac{10 \text{ kPa} \times \sqrt{2} \times 1}{A} = 5 \sqrt{2} \text{ kN} \]
\[ Y_R = \frac{2}{3} \sqrt{2} m = (\frac{1}{2} + \frac{1}{2}) \sqrt{2} \text{ m} \]

**Free Body Diagram Approach**

Resolving horizontally:
\[ F_{R_H} + F_h = F_R \]
\[ F_{R_H} = F_R = YAH_c \]
\[ F_{R_H} = F_R = 10 \text{ kN/m}^3 (1 \times 1) m^2 \times \frac{1}{2} m = 5 \text{ kN} \]

Resolving vertically:
\[ F_v = F_v + W \text{ or } F_v = F_R - W \]
\[ F_v \]
\[ F_v = F_v - W = YAH_c - YV \]
\[ = 10 \text{ kN/m}^3 (1 \times 1 \times 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) = 5 \text{ kN} \]

Resultant:
\[ F_R = \sqrt{F_{R_H}^2 + F_v^2} = \sqrt{5^2 + 5^2} = \sqrt{2 	imes 25} = 5 \sqrt{2} \text{ kN} \]

Assume \( F_{R_H} \) and \( F_v \) act at \( 2/3 \) depth and check moments:
\[ F_{R_H} \]
\[ \frac{1}{3} W \]
\[ 5 \frac{1}{3} + 5 \frac{2}{3} = 5 \frac{1}{2} \text{ QED.} \]
**Buoyancy, Flotation, Stability**

\[ \Delta F_B = (p_1 - p_2) \Delta A = \gamma h \Delta A = \gamma \int dA \]

Integrating over prism:

\[ F_B = \int \Delta F_B = \gamma \int dA = \gamma A \frac{A}{A} \]

- No lateral forces (all cancel)
- Buoyant force acts through centroid of displaced volume. Center of buoyancy
A spherical buoy has a diameter of 1.5 m, weighs 8.50 kN, and is anchored to the sea floor with a cable as is shown in Fig. E2.10a. Although the buoy normally floats on the surface, at certain times the water depth increases so that the buoy is completely immersed as illustrated. For this condition what is the tension of the cable?

\[ T + W = F_B \]

\[ F_B = \gamma A \]

\[ T = F_B - W = \gamma A - W \]

\[ = 10.1 \text{ kN/m}^3 \times \frac{4 \pi (0.75)^3}{3} m^3 - 8.5 \text{ kN} \]

\[ T = 17.85 \text{ kN} - 8.5 \text{ kN} \]

\[ T = 9.35 \text{ kN}. \]
LINEAR MOTION

Accelerate:
\[ a_y \neq 0; \quad a_z \neq 0 \]
\[ a_x = 0 \]
\[ \therefore \frac{dz}{dx} \text{ surface} = 0 \]

From basic relations:
\[ \frac{\partial p}{\partial x} = -\rho a_x = 0 \]
\[ \frac{\partial p}{\partial y} = -\rho a_y \]
\[ \frac{\partial p}{\partial z} = -\rho (g + a_z) \]
\[ \{1\} \]

Evaluate change in pressure, \( dp \), in \( x, y \) \& \( z \) directions

\[ dp = \frac{dp}{dx} dx + \frac{dp}{dy} dy + \frac{dp}{dz} dz \]
\[ \{2\} \]

Substitute \(1\) into \(2\):

\[ dp = -\rho a_y dy - \rho (g + a_z) dz \]
\[ \{3\} \]

Along a line of constant pressure, \( dp = 0 \). Setting \( dp = 0 \) in \(3\) gives slope of line of constant pressure \( \frac{dz}{dy} \), including free surface.

\[ \rho a_y dy = -\rho (g + a_z) dz \]
\[ \frac{dz}{dy} = -\frac{a_y}{(g + a_z)} \]
\[ \{2.28\} \]
Equation (2.28): \( \Delta \frac{dz}{dy} \) is constant for constant accelerations \( a_y \) and \( a_z \).

- Surfaces of constant pressure of inclination \( \frac{dz}{dy} = \text{const} \).

Free surface \( \frac{dz}{dy} \)

Pressure gradient down from the free surface is given by (1) as: \( \frac{dp}{dz} = -\rho (g + a_z) \)

- If \( a_z = 0 \) then \( \frac{dp}{dz} = -\rho g \) \( \Rightarrow \) "hydrostatic".

- If \( a_z \neq 0 \) then extra component.

\[ p = \rho gh \]
Rigid Body Rotation

Basic relations:
\[ \begin{align*}
\frac{\partial p}{\partial r} &= \rho r \omega^2 \\
\frac{\partial p}{\partial \theta} &= 0 \\
\frac{\partial p}{\partial z} &= -\gamma
\end{align*} \] (2.30)

Evaluate change in pressure, \( dp \).

\[ dp = \frac{dp}{dr} dr + \frac{dp}{d\theta} d\theta + \frac{dp}{dz} dz \] (1)

Setting \( dp = 0 \) for equi-pressures, isobars then

\[ dp = \rho r \omega^2 dr - \gamma dz \] (2)

Rearrange for

\[ \frac{dz}{dr} = \frac{\rho r \omega^2}{\gamma} = \frac{r \omega^2}{g} \] (3)

Integrating (3).

\[ \int dz = \frac{\rho \omega^2}{g} \int r dr \]

\[ z = \frac{\rho \omega^2}{g} \frac{1}{2} r^2 + \text{constant} \]

gives surface of equal pressure (parabolic)
With lines of constant pressure defined by
\[ t = \frac{w^2r^2 + c}{2g} \]
(parabolic).

Integrate equation (2):
\[ dp = prw^2 dr - r dz \]
\[ \int dp = pw^2 \int r dr - \int r dz \]
\[ p = \frac{pw^2r^2}{2} - r^2 + \text{Const} \]

To solve, define \( p \) at specified \( r_0, z_0 \) and define \( \text{Const} \).

Then resubstitute to solve \( p \) at any \( r \) and \( z \).

- For \( r = \text{constant} \), \( p \) varies linearly, since
  \[ p = 0 \text{ at surface and sets const. magnitude}. \]