

# Mathematical Analysis of Size-Frequency Distributions of Particles in the Subsieve Range

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## SYNOPSIS

A method is proposed for the calculation of surface area and weight distributions from microscopic size-frequency measurements wherein the necessity of assuming a mean diameter to represent the size interval has been obviated. This method permits the microscopic classification of particles into broader intervals than currently recommended, thereby reducing the time consumed in counting particles, especially when the sample contains a wide range of sizes.

In the method proposed, the size-frequency data are plotted in terms of the cumulative number of particles greater than a size  $x$ ,  $N$ , versus  $x$ , from which intermediate values of  $N$  can be interpolated for any value of  $x$ . In the interval from  $x$  to  $x + dx$ , the number of particles may be represented by  $-dN$ . On this basis, equations can be set up which may be integrated between any interval limits to determine the surface area or weight of that

interval. These equations have the form of  $A_{12} = \int_{N_1}^{N_2} \beta x^2 dN$  and  $W_{12} = \int_{N_1}^{N_2} \alpha x^3 dN$  for the surface area and weight, respectively, of the interval.

A statistical analysis of count data for coal samples with sizes ranging from 1 to 100  $\mu$  indicates that the reproducibility is sufficient to warrant calculations on the basis of the above method.

**T**ECHNIQUES have been proposed by various investigators where-  
 y properties such as size-frequency  
 distribution, surface area distribution,  
 weight distribution, and projected and  
 total surface areas can be directly  
 measured for subsieve size particles.  
 Most of these techniques have been  
 summarized in review papers (1, 2, 3,  
 4, 5).<sup>3</sup> In general, however, these  
 techniques require calibration against  
 some standard such as the direct classi-  
 fication of particles into size intervals  
 under the microscope. While the size-  
 frequency data are very convenient  
 for calculation purposes, certain as-  
 sumptions should be considered which  
 have been required regarding the density  
 of the particles, the shape factors, and  
 the mean diameter (6) representing  
 the size of the particles in each interval.

The density of the particles in each  
 size fraction of the sample is required  
 in order to determine the weight dis-  
 tribution. If the sample is homogene-  
 ous, no problem arises as the density is  
 either known or can be determined. In  
 the case of heterogeneous materials a  
 useful expedient is to assume the con-  
 stancy of density distribution through-

out all size ranges; however, the results  
 obtained are no better than the validity  
 of this assumption.

The volumetric shape factor is de-  
 fined as the ratio of the volume of a  
 particle to the cube of any linear dimen-  
 sion of that particle. Examples of the  
 linear dimensions commonly chosen  
 are length, width, thickness, and equiva-  
 lent Stokes diameter, but for each  
 dimension, there is a corresponding shape  
 factor defined by this relationship. As  
 herein used, the volumetric shape fac-  
 tor  $\alpha$  is defined as the ratio of the par-  
 ticle volume to the particle length  
 cubed. In addition there are surface  
 area shape factors which are defined  
 analogously as the ratio of surface area  
 to the square of the linear dimension.  
 As herein used, the surface area shape  
 factor  $\beta$  is related to the square of the  
 length of a particle. A common expedi-  
 ent used in the absence of specific  
 data is to assume the constancy of a  
 particular shape factor throughout the  
 range of particle sizes investigated.  
 The validity of this assumption de-  
 pends upon the nature of the material  
 being analyzed.

The mean diameter is usually chosen  
 as the arithmetic mean of the extremes  
 of the size interval into which the par-  
 ticles are classified. As the size inter-  
 vals are made smaller, this arithmetic  
 mean diameter approaches more nearly  
 the true mean diameter of each size  
 interval. However, the smaller the

size interval, the greater the time re-  
 quired to classify the particles under the  
 microscope. Hence, the microscopic  
 technique has lost favor with many  
 workers because of the long task of get-  
 ting the size-frequency distribution in  
 intervals small enough to get reason-  
 ably accurate estimates of the mean  
 diameter.

It is the purpose of this paper to pre-  
 sent a new method of interpretation  
 and calculation of size-frequency meas-  
 urements from the microscope, offering  
 two distinct advantages over methods  
 currently in use, namely, (a) the  
 problem of selecting a mean diameter  
 to represent the interval counted has  
 been obviated, and (b) the size inter-  
 vals into which the particles are classi-  
 fied under the microscope have been  
 greatly broadened, thus reducing the  
 time consumed at the microscope.

As a basis for the method proposed,  
 the microscopic counts are arranged in  
 terms of the number of particles greater  
 than a size  $x$ ,  $N(x)$  (or simply  $N$  since  
 it is clear that  $N$  is a function of  $x$ ),  
 versus the size  $x$ . Intermediate values  
 of  $N$  can be interpolated from a plot  
 of  $N$  versus  $x$ , and the number of par-  
 ticles in any size interval can be approxi-  
 mately predicted. The number of  
 particles in an interval from  $x$  to  $x + dx$   
 may be represented by the value of  
 $-dN$ . Equations can then be set up  
 which, when integrated between the  
 limits of  $x_1$  and  $x_2$ , will exactly represent  
 the surface area and weight distribu-  
 tions.

## DEVELOPMENT OF WEIGHT DISTRIBUTION EQUATIONS

Consider now the conversion of  
 size-frequency data into weight dis-  
 tribution. The weight of  $M$  particles  
 of a given size  $x$  is always given by the  
 equation:

$$W = \alpha \rho x^3 M \dots \dots \dots (1)$$

where:

- $W$  = weight of particles of size  $x$ ,
- $\alpha$  = shape factor,
- $\rho$  = density,
- $x$  = any characteristic dimension of a  
 particle, and
- $M$  = number of particles of size  $x$ .

Equation 1 is always true. However,  
 count data are usually obtained by  
 classifying the particles into size in-  
 tervals (that is, say 5 to 10 $\mu$ , 10 to 20 $\mu$ ,

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<sup>3</sup> The boldface numbers in parentheses refer to the list of references appended to this paper.

etc.). From such data an approximation of the weight of particles in a given size interval may be obtained using the equation:

$$W' = \alpha\rho(x_{\text{mean}})^2 M \dots \dots (2)$$

where:

- $W'$  = approximate weight of particles in the interval,
- $x_{\text{mean}}$  = arithmetic mean of extremities of the interval, and
- $M$  = number of particles in the size interval.

It is often desirable to express the results of a size distribution analysis as the weight fraction less than size  $x$  versus the size  $x$ . This follows readily by evaluating the weight of each fraction and expressing the sum of the fractions up to a size  $x$  and the weight of all the fractions as a ratio.

Equation 2 holds approximately if the  $\alpha\rho$  product remains constant and if the size interval is small. It is clear that if the true mean diameter were known and substituted for  $x_{\text{mean}}$ , Eq 2 would hold exactly for broad intervals. However, there is no simple method for determining the value of the true mean diameter because it is a function of  $x^3 dN$ . The nature of this function will be shown later in Eq 6.

It would be extremely desirable to calculate the weight distribution without using a mean diameter. Such a method will now be presented.  $N(x)$ , or simply  $N$  since its functional dependence on  $x$  is understood, is defined as the number of particles greater than a size  $x$ . Then  $N_1 - N_2$  is equal to the number of particles in the size range from  $x_1$  to  $x_2$  and  $-dN$  represents the number of particles having sizes between  $x$  and  $x + dx$ . Therefore, the following equation can be written:

$$dW = -\alpha\rho x^3 dN \dots \dots (3)$$

where  $dW$  is the weight of particles having sizes between  $x$  and  $x + dx$ .

Integrating:

$$W_{12} = - \int_{N_1}^{N_2} \alpha\rho x^3 dN \dots \dots (4)$$

where  $W_{12}$  = weight of particles in the interval from  $x_1$  to  $x_2$  corresponding to  $N_1$  and  $N_2$ .

Equation 4 is the most general form of the equation for the weight of particles between  $x_1$  and  $x_2$ . A modification of this general equation can be made which has proved for many size distributions to allow more accurate evaluation because of the nature of the graphical integration. When Eq 3 is integrated by parts, Eq 5 results:

$$W_{12} = - \int_{N_1}^{N_2} \alpha\rho x^3 dN =$$

$$-\alpha\rho N x^3 \Big|_{N_1, x_1}^{N_2, x_2} + 3\alpha\rho \int_{x_1}^{x_2} N x^2 dx \dots (5)$$

Equation 5 is a general form of the weight distribution equation involving only the assumption of constant  $\alpha\rho$  product.

In theory, Eq 4 can be solved by plotting  $\alpha\rho x^3$  versus  $N$  and evaluating the area under the curve from  $N_1$  to  $N_2$ . This will represent the weight of particles from the corresponding values of  $x_1$  to  $x_2$ . Unfortunately, in most cases  $N$  varies over such a wide range of values that a large error may be introduced for small values of  $N$  when using this method.

Equation 5 can be solved by plotting  $\alpha\rho N x^2$  versus  $x$ . The area under the curve from  $x_1$  to  $x_2$  represents the value of

$$\alpha\rho \int_{x_1}^{x_2} N x^2 dx$$

The interesting characteristic of the  $\alpha\rho N x^2$  versus  $x$  curve is that for the distributions with which the authors have worked, the magnitude of the  $\alpha\rho N x^2$  term did not vary over threefold. Because of the relatively small variation in  $\alpha\rho N x^2$ , the accuracy of the weight calculations for all intervals remains nearly constant.

The value of the true mean diameter can now be determined from Eqs 2 and 4:

$$W_{12} = \alpha\rho [\text{true mean diameter}]^2 M = -\alpha\rho \int_{N_1}^{N_2} x^3 dN$$

True mean diameter =

$$\left[ -\frac{1}{M} \int_{N_1}^{N_2} x^3 dN \right]^{1/2} \dots (6)$$

However, the determination of true mean diameter requires that the weight of the fraction be first calculated, and since this is the object of the calculation, it is more convenient to omit the true mean diameter concept.

*N Versus x Curve:*

No mention has been made in the preceding discussion of how  $N$  might vary with  $x$ . When the nature of the function is such that an equation cannot be conveniently determined, a graphical integration is perhaps the best solution. However, if an equation of  $N$  as a function of  $x$  can be easily obtained, an analytical solution may be the shortest solution. An attempt was made to assume a weight distribution which many materials have been observed to follow and calculate the size-frequency distribution. With one minor exception, these attempts lead to expressions for which generalized solutions are not

available. For example, assume a weight distribution to follow the generalized Rosin-Rammler equation:

$$r = e^{-(x/\bar{x})^n} \dots \dots (7)$$

where  $r$  = weight fraction greater than a size  $x$ , and  $\bar{x}$  and  $n$  are constants the distribution.

If one attempts to evaluate  $N$  as a function of  $x$  beginning with the generalized Rosin-Rammler relationship it results in an integral which has its solution in terms of gamma functions.<sup>4</sup> However, in many cases the distribution of the smallest material in the complement for repeated mild fracture may be expressed by the modified form of the Rosin-Rammler equation (5) as follows:

$$1 - r = kx^n \dots \dots (8)$$

where  $k$  and  $n$  are constants of the distribution.

Differentiating Eq 8:

$$dr = -nkx^{n-1} dx \dots \dots (9)$$

where  $dr$  is the differential weight fraction of particles in the size interval from  $x$  to  $x + dx$ . If  $K$  is defined as the total weight of particles then  $Kdr$  is the differential weight of particles of size  $x$  which we have previously defined in Eq 3 as  $dW$ . Therefore:

$$\begin{aligned} dW &= Kdr \\ \alpha\rho x^3 dN &= Knkx^{n-1} dx \\ dN &= \left( \frac{Knk}{\alpha\rho} \right) x^{n-4} dx \\ N &= \frac{Knk}{\alpha\rho(n-3)} x^{n-3} + C \end{aligned}$$

which can be written in the form:

$$N - C = Bx^n \dots \dots (10)$$

where:

$$\begin{aligned} B &= \frac{Knk}{\alpha\rho(n-3)} \\ p &= n - 3 \end{aligned}$$

Therefore, if from the count data,  $\log(N - C)$  is a linear function of  $\log x$ , the weight distribution will follow the modified form of the Rosin-Rammler equation. This section has dealt very briefly with an attempt to determine the size-frequency distribution if weight distribution was assumed. The authors believe that more investigation on this particular phase of the work may be of great value. It should be noted, however, that the failure to derive a completely general theoretical size distribution does not in any way affect the use of the proposed method, because the graphical integration can be performed without a knowledge of the equation of  $N$  as an analytic function of  $x$ .

<sup>4</sup>  $N = -\frac{Kn}{\alpha\rho} \int x^{n-4} e^{-(x/\bar{x})^n} dx$

The graphical solution is applicable in the general case when the distribution equation is unknown. Using the count data, a graphical solution of the last term of Eq 5 can be accomplished as follows: plot  $\alpha\rho Nx^2$  versus  $x$  as shown

Fig. 1. The area of the small strip equals  $\alpha\rho Nx^2 dx$ , and the sum of these small areas from  $x_1$  to  $x_2$  equals the total area under the curve. Therefore, in the proper units, this area can be used in Eq 5 to calculate the weight of particles in the interval from  $x_1$  to  $x_2$ .

#### DEVELOPMENT OF AREA DISTRIBUTION EQUATIONS

By analogy to the weight distribution equations, surface area and projected area distribution equations also can be developed.

The surface area of  $M$  particles of size  $x$  can be determined from the following equation:

$$A = \beta x^2 M \dots \dots \dots (11)$$

where  $\beta$  = area shape factor.<sup>5</sup>

From similar reasoning to the development of Eq 3:

$$dA = -\beta x^2 dN$$

$$A = - \int_{N_1}^{N_2} \beta x^2 dN \dots \dots (12)$$

A convenient form of this equation can be developed by a slight modification:

$$A = - \int_{N_1}^{N_2} \frac{\beta N x^2 dN}{N}$$

$$A = - \int_{N_1}^{N_2} \beta N x^2 d \ln N \dots \dots (13)$$

Utilizing a plot of  $\beta N x^2$  versus  $\ln N$  from  $N_1$  to  $N_2$  the surface area of the particles from  $x_1$  to  $x_2$  can be calculated.

The above equations apply to the projected area of the particles if the shape factor,  $\beta$ , is changed to the shape factor,  $\gamma$ .<sup>6</sup>

<sup>5</sup> By definition:  $\beta = A'/x^2$ , where  $A'$  is the surface area per particle of size  $x$ .  
<sup>6</sup> By definition:  $\gamma = A''/x^2$ , where  $A''$  is the projected area per particle of size  $x$ .

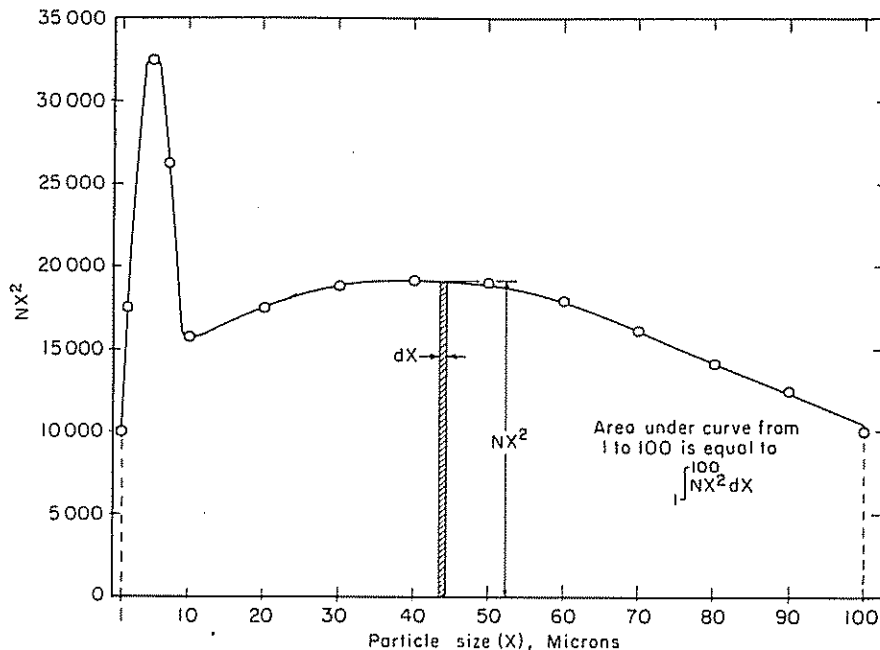


Fig. 1.—Auxiliary Plot Used in the Graphical Integration of Weight Distribution Equation for Run No. 1.

#### Applications of Equations to Theoretical Distribution:

The following example illustrates the use of the proposed method of calculation and compares the results with those calculated assuming an arithmetic mean diameter. The assumed distribution approximates that actually found for a sample of crushed bituminous coal. The distribution for the example is given in Table I and corresponds to a distribution found directly from microscope counts. The nature of the distribution was assumed so that the solution by the proposed method could be readily solved analytically, thus eliminating any possible source of error introduced by a graphical technique. For the sake of the example, assume the particles are spherical and have a specific gravity equal to one. The particular distribution chosen in this example conforms to Eq 10 previously derived. The value

of  $p$  is  $-2$  and represents the slope of the curve on a plot of  $\log N$  versus  $\log x$ . The value of  $B$  is  $10,000$  and can be interpreted as the number of particles counted that are greater than  $1 \mu$ .  $C$  for this case is equal to zero. Therefore:

$$N = Bx^p$$

$$N = 10,000x^{-2}$$

$$dN = -\frac{20,000}{x^3} dx \dots \dots (14)$$

Substituting into Eq 4 for  $dN$

$$W_{12} = \frac{20,000\pi}{(6)(10^{12})} \int_{x_1}^{x_2} dx \dots \dots (15)$$

$$= (1.046)(10^{-4})(x) \Big|_{x_1}^{x_2} \dots \dots (16)$$

where  $\alpha = \pi/6$  for spherical particles, and  $d = 10^{-12} g/(\mu)^3$ .

The area distribution can also be calculated for the size distribution

TABLE I.—CALCULATION OF WEIGHT AND SURFACE AREA DISTRIBUTIONS FOR ASSUMED DISTRIBUTION.

A Size Interval, $\mu$	B Number of Particles in Size Interval, $M$	C ( $x_{mean}$ )	D ( $x_{mean}$ ) <sup>2</sup>	E ( $x_{mean}$ ) <sup>3</sup>	F $\log(x_2/x_1)$	G Weight of Particles in Interval, g		H Area of Particles in Interval, sq. cm.	
						$x_{mean}$ Method	New Method	$x_{mean}$ Method	New Method
1 to 2	7500	1.5	2.25	3.37	0.301	$1.32 \times 10^{-6}$	$1.06 \times 10^{-6}$	$5.30 \times 10^{-4}$	$4.36 \times 10^{-4}$
2 to 5	2100	3.5	12.25	42.9	0.308	4.72	3.14	8.09	5.77
5 to 10	300	7.5	56.3	422	0.301	6.63	5.23	5.30	4.36
10 to 15	55.5	12.5	156.3	1953	0.176	5.68	5.23	2.73	2.55
15 to 20	19.5	17.5	306.3	5359	0.124	5.48	5.23	1.88	1.80
20 to 25	9.0	22.5	506.3	11300	0.0909	5.36	5.23	1.43	1.41
25 to 30	4.9	27.5	756.3	20797	0.0792	5.34	5.23	1.17	1.15
30 to 35	2.9	32.5	1056	34328	0.0671	5.21	5.23	0.96	0.97
35 to 40	1.9	37.5	1406	52734	0.0580	5.26	5.23	0.84	0.84
40 to 45	1.4	42.5	1806	76766	0.0492	5.63	5.23	0.80	0.72
45 to 50	0.9	47.5	2256	107172	0.0414	5.05	5.23	0.64	0.60
						55.68	52.30	29.14	24.53

$G = (0.523 \times 10^{-12}) (BE)$ .  $H = (1.046 \times 10^{-6}) (-A)$ .  $I = (3.14 \times 10^{-8}) (BD)$ .  $J = (1.45 \times 10^{-3}) (F)$ .

given in Eq 14. From Eq 12 the area of a fraction is equal to

$$A = - \int_{N_1}^{N_2} \beta x^2 dN$$

where  $\beta$  equals  $\pi$  for spherical particles.

Substituting Eq 14 into Eq 12:

$$A_{12} = \frac{20,000\pi}{10^6} \int_{x_1}^{x_2} \frac{dx}{x} \\ = 1.45 \times 10^{-3} \log \frac{x_2}{x_1} \dots (17)$$

The calculations for weight and area distributions by the method of arithmetic mean diameters are based upon Eqs 2 and 10, respectively. In the latter equations, the arithmetic mean diameter of the interval is substituted for  $x$ . Since the values calculated by the new method are exact, the deviation of any value calculated by the arithmetic mean diameters from the exact value is a measure of the error introduced by the calculation method. Inspection of Table I shows deviations as high as 50 per cent for the weight of intervals and 40 per cent for the area of intervals. Moreover, the total weight calculated by the arithmetic mean diameter method is  $6\frac{1}{2}$  per cent high and the total surface area is 19 per cent high.

The surface area calculations are of especial interest because many important properties of dusts are related to surface area. The largest contributions to the surface area are from the small size particles, as shown in Table I. Also, the largest calculation errors are introduced in the small sizes, when using the method of mean diameters. Therefore it is here that the new method is of greatest utility, because in obviating the selection of a mean diameter, it affords a means of calculating accurately the area of particles where the major part of the total surface area is concentrated, that is, in the small size particles.

#### Application of Method to Microscopic Count Data:

The reliability of the calculations based upon the new method depends upon the experimental reproducibility of the count data. Therefore, an investigation was conducted using coal samples to determine the magnitude of variations characteristic of the microscopic sizing.

Count data as taken directly from the microscope consist of a series of counts which represent the number of particles classified in arbitrarily selected size intervals. In the subsequent discussion, these data will be interpreted in an attempt to evaluate how reliably such data can be considered representative of the actual distribution in the original sample. Because the actual

distribution in the sample cannot be determined, the approach is of necessity based upon statistics.

There are many ways to compare count data statistically, but it seems sufficient here to treat the data in two ways. First, it is of value to compare the total number of particles counted in one field of the microscope with other fields of the same cell. A further comparison of the counts should then be made between fields of different cells. If the dispersion of particles within the cell is uniform and if the dispersion from which the cells were made up was uniform, the counts in each field should be identical. These counts will be referred to hereafter as the absolute count, and the number of particles counted in a given size interval will be likewise referred to as the absolute count of the size interval. A second comparison can be made between runs of the percentage of the particles counted greater than a size  $x$ . This is essentially comparing relative counts and will subsequently be referred to as such. Judgment as to the reproducibility of count data then will be based upon a statistical comparison of both the absolute and relative counts.

**Preparation and Counting of the Cells.**—Minus 200 mesh fractions from two similarly treated bituminous coal samples were each divided into three samples according to the ASTM Method of Sampling and Fineness Test of Powdered Coal (D 197-30).<sup>7</sup> These six 0.25-g composite samples were each subsequently analyzed microscopically.

The cells used for the microscopic analysis were each prepared by dispersing the 0.25-g sample in the quantity of n-propyl alcohol required to obtain a particle density suitable for microscopic sizing. One milliliter of this dispersion was then pipetted into each of five individual Sedwick-Rafter cells, allowed to settle overnight, and counts read the following day.

The particles in five fields of each cell were classified into intervals of 1 to 2, 2 to 5, 5 to 10, 10 to 20, . . . . , 90 to 100  $\mu$  where the particle dimension measured was the length. In addition, 50 fields per cell were counted classifying only those particles greater than 20 $\mu$  because the number of particles in this range was small so that additional counts were necessary to increase the accuracy. By scanning the field of the microscope, the observer can count the particles lying within a given size interval with the aid of a small hand counter.

All counts were made using a mine safety appliance "Dust-View" microprojector equipped with a Spencer

microscope. The lens system used was a 10 $\times$  apochromatic objective and 10 $\times$  and 30 $\times$  compensating oculars. With the higher magnification it is possible to resolve particles smaller than 1  $\mu$ ; however, no attempt was made to count particles smaller than 1  $\mu$ .

**Presentation of Results.**—Data presented in this section are the results of six microscopic analyses—three from each of the original two samples. Since these analyses are independent, a comparison of the magnitudes of the absolute counts is a measure of the experimental reproducibility of the analytical technique. However, the counts for a given size interval vary from field to field so that some statistical average values should be compared. Such a comparison was made wherein the arithmetic mean count was used and the standard deviation was utilized to determine the confidence limits of the count data (8).

The results obtained by applying this treatment to microscopic counts are presented in Table II. Analyses 1a, b, c and 2a, b, c are from the original samples 1 and 2, respectively. It should be noted that the data considered here are those less than 20  $\mu$ . There are two reasons for this: first, the counts fall off rapidly beyond 20  $\mu$  sizes; and, second, most of the surface area of the samples is contributed by particles less than 20  $\mu$  as was shown in Table I. The significance of the results presented is that  $\bar{M}$ , the arithmetic mean count of the interval, should represent the actual count of the interval to within  $\pm z\sigma$ , where  $z\sigma$  is the confidence limit of the data. In-

TABLE II.—MEAN COUNTS, STANDARD DEVIATIONS, AND CONFIDENCE LIMITS FOR COAL SAMPLES ANALYZED BY MICROSCOPE

Interval	1 to 2 $\mu$	2 to 5 $\mu$	5 to 10 $\mu$	10 to 20 $\mu$
1a				
$\bar{M}$ (1) . . .	34.5	18.5	4.0	0.6
$\sigma$ (2) . . .	7.2	4.1	2.0	0.8
$z\sigma$ (3) . . .	2.5	1.4	0.7	0.3
1b				
$\bar{M}$ . . . . .	24.4	16.5	5.4	0.4
$\sigma$ . . . . .	3.7	3.8	2.3	0.7
$z\sigma$ . . . . .	1.3	1.3	0.8	0.3
1c				
$\bar{M}$ . . . . .	24.3	16.8	6.3	1.0
$\sigma$ . . . . .	4.3	3.6	2.5	1.0
$z\sigma$ . . . . .	1.5	1.3	0.9	0.3
2a				
$\bar{M}$ . . . . .	24.6	19.1	6.1	0.7
$\sigma$ . . . . .	3.6	5.6	2.4	0.7
$z\sigma$ . . . . .	1.3	2.0	0.8	0.3
2b				
$\bar{M}$ . . . . .	23.1	15.3	5.2	0.7
$\sigma$ . . . . .	3.0	4.2	2.1	0.9
$z\sigma$ . . . . .	1.0	1.5	0.7	0.3
2c				
$\bar{M}$ . . . . .	22.8	15.5	6.3	0.6
$\sigma$ . . . . .	2.1	3.8	2.5	1.1
$z\sigma$ . . . . .	0.7	1.3	0.9	0.4
Mean $\bar{M}$				
avg. . . . .	23.8	16.6	5.9	0.7

<sup>a</sup> Omitting (a).

(1) Mean count. (2) Standard deviation. (3) Confidence limit (9 times out of 10).

<sup>7</sup> 1940 Book of ASTM Standards, Part 5, p. 610.

spection of Table II will show that with few exceptions the  $\bar{M}$ 's for the same interval compare within the confidence limit  $\pm z\sigma$ . The same data show that the ratio of  $z\sigma/\bar{M}$  increases with particle size. This illustrates that a representative count is more easily obtained when the number of particles in the field is large.

The complete sample analyses for the size runs are presented in Table III and are plotted in Fig 2. These data represent the relative counts based on 10,000 particles counted greater than 1  $\mu$  (or 100 times the percentage of particles greater than size  $x$ ). To increase the accuracy of the counts for the larger sizes, 250 additional fields were read in which all particles greater than 20  $\mu$  were classified. The deviations in the larger sizes of the samples are more pronounced, again due to the difficulty in dealing with a small number of particles per field. However, larger errors can be tolerated here because the surface area contributions to the total are small for particles in this range.

The ultimate purpose of the counts was to calculate the surface area distribution of the particles and to determine to what extent the uncertainties in the microscopic counts would be present in the calculated total surface area. An estimation can be made using the values of the surface contributions of each interval to the total surface area found in the example illustrated in Table I. If the confidence limit of each interval is multiplied by the interval surface area and summed up, the total error can be calculated. On this basis the maximum probable error would be less than 12 per cent if the errors are additive. However, the deviations in Table II are random; therefore, the actual error would most likely be less than 12 per cent.

#### SUMMARY AND CONCLUSIONS

The magnitude of errors involved in the method of mean diameters was shown by means of a hypothetical

example. The weight and area distribution calculations for the particular example chosen are known to be exact when the new method is used. Therefore, deviations from the exact values served as a measure of the errors possible by the method of mean diameters.

The weight and surface area distributions can be calculated from the distribution equations presented when a few values of  $N$  are accurately known as a function of  $x$ . Therefore, in using the proposed method the microscopic counts can be classified into broad intervals. Because the particles may be sized in broader intervals, the time required in microscopic sizing is greatly reduced. The calculations using the new method usually require more time than with the method of mean diameters, but this is more than offset by the reduced counting time.

It has been shown that it is desirable to utilize the new method of calculation

when the surface area of the sample is important, because the largest contributions to the total surface area are from the small size particles which cannot be accurately estimated by the method of mean diameters.

A statistical analysis has been made of count data for six analyses, the confidence limits established for intervals from 1 to 20  $\mu$ , and the standard deviations evaluated for particles from 1 to 100  $\mu$ .

On the basis of the statistical analysis, it is concluded that the maximum probable error in the calculation of total surface area would be less than 12 per cent. Therefore, the count data are reproducible to a degree of accuracy sufficient to warrant the use of the new method of calculation.

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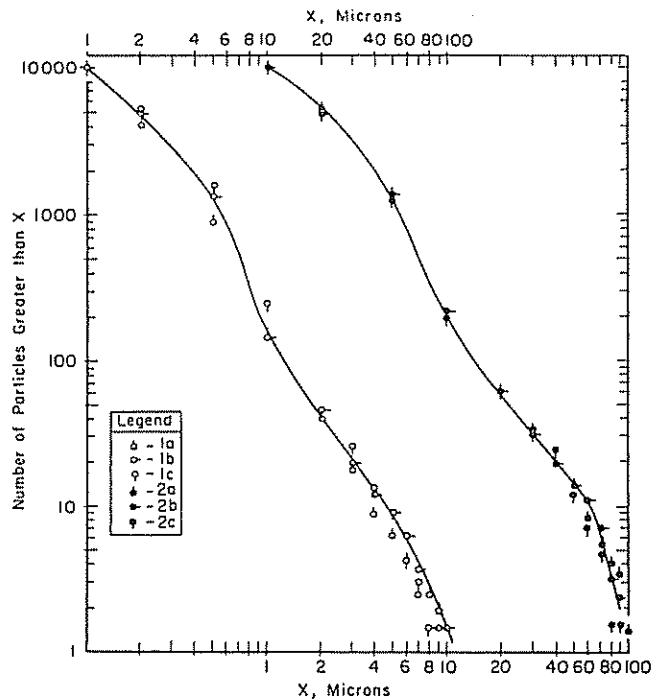


Fig. 2.—Plot of Number of Particles Greater Than Size  $x$  Versus Size  $x$ .

TABLE III.—RELATIVE MICROSCOPIC COUNTS.  
Number of Particles Counted Greater Than  $x \mu$  (Based on Total of 10,000 Particles).  
Particle Diameter,  $x$ .

Sample.....	1	2	5	10	20	30	40	50	60	70	80	90	100	>100
1a N <sup>a</sup> .....	10 000	3982	841.6	144.5	39.8	17.6	8.8	6.3	4.4	2.5	2.5	1.0	0	0
1b N.....	10 000	4846	1325	145.7	45.7	19.3	12.3	9.2	6.14	3.8	1.5	1.5	1.5	0
1c N.....	10 000	5015	1565	244.8	47.8	26.2	13.5	8.9	4.4	3.0	1.5	0.7	0.7	0
1 avg N <sup>b</sup> .....	10 000	4610	1250	179	44.4	21	11.5	7.9	5	3.1	1.8	1.2	0.7	0
σ <sup>c</sup> .....	0	474	305	55	3.4	3.7	1.6	1.2	0.81	0.54	0.46	0.48	0.61	0
2a N.....	10 000	5056	1337	192.5	60.2	32.6	19.8	14.1	8.40	5.6	4.2	3.5	1.4	0
2b N.....	10 000	4866	1376	211.2	60.2	31.2	19.8	14.2	11.2	7.2	3.2	2.4	0.8	0
2c N.....	10 000	4986	1526	196	62	34.0	24.3	12.1	7.2	4.8	1.6	1.0	0.8	0
2 avg N.....	10 000	4970	1414	200	61	32.6	21.3	13.5	8.9	5.9	3	2.5	0.1	0
σ.....	0	107	80	8.1	1.8	1.21	4.4	0.97	1.67	1.02	1.07	0.78	0.28	0

<sup>a</sup> N is the number of particles greater than a size  $x$ .

<sup>b</sup>  $\bar{N}$  is the arithmetic average of analyses N's from a, b, and c.

<sup>c</sup> σ is the standard deviation of N's from  $\bar{N}$ .

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