A Theoretical Calculation of Residual Stresses in Carbon Fibers

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Abstract. Theoretical equations for prediction of the residual stresses in radial, hoop and longitudinal directions were derived from linear elasticity for fibers which have a perfect onion skin or core/sheath structure. Core/sheath model calculations predicted the magnitude of the longitudinal stress observed in PAN base carbon fibers experimentally. No theoretical solution of residual stress was found for a spoke symmetry carbon fiber.

The theoretical equations for residual stresses in carbon fibers which have: 1) an onion-skin or 2) a skin-core microstructure are derived. The radial-spoke microstructure, often observed in pitch precursor fibers, could not be modeled because of a singularity at the center of the fiber.

The equations for a stress at any point in a fiber due to anisotropic thermal contraction upon cooling down from processing temperature can be developed based on the following assumptions: 1) the material is continuous and obeys Hooke's Law, 2) The Young's moduli and Poission's ratios are independent of temperature, and 3) the problem can be treated as a general plane strain state. The product of the coefficients of expansion and the temperature change is denoted by the strains ε_i^T for the hoop ($i=\theta$), radial ($i=r$) and axial ($i=z$) directions. Consider an element of the cylinder with the stresses acting on its principal planes due to anisotropic thermal contraction as shown in Fig. 1. A general equation for this case is derived as follows:

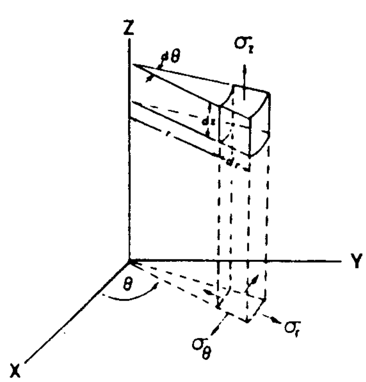

Fig. 1. Element of the cylinder.

A) Equations of equilibrium for cylindrical coordinates with principal stresses[1]:

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \quad (1) \qquad \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} = 0 \quad (2) \qquad \frac{\partial\sigma_z}{\partial z} = 0 \quad (3)$$

B) Equations of compatibility for linear strains [6,7] The net radial strain at any point in the material is related to the hoop strain:

$$2\pi\left(r + \int_0^r \varepsilon_r dr\right) = 2\pi r + \int_0^{2\pi} \varepsilon_\theta r d\theta \qquad (4)$$

or

$$\varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta) \qquad (4a)$$

C) Stress-strain relationships

$$\varepsilon_r = \varepsilon_r^T + \frac{\sigma_r}{E_r} - \nu_{r\theta}\frac{\sigma_\theta}{E_\theta} - \nu_{rz}\frac{\sigma_z}{E_z} \qquad (5)$$

$$\varepsilon_\theta = \varepsilon_\theta^T - \nu_{\theta r}\frac{\sigma_r}{E_r} + \frac{\sigma_\theta}{E_\theta} - \nu_{\theta z}\frac{\sigma_z}{E_z} \qquad (6)$$

$$\varepsilon_z = \varepsilon_z^T - \nu_{zr}\frac{\sigma_r}{E_r} - \nu_{z\theta}\frac{\sigma_\theta}{E_\theta} + \frac{\sigma_z}{E_z} \qquad (7)$$

The axial stress (σ_z) and hoop stress (σ_θ) can be expressed in terms of radial stress (σ_r) and its derivatives and the material's properties. Substituting (5) and (6) into eq (4a), a second order differential equation is obtained as follows:

$$r^2\frac{d^2\sigma_r}{dr^2} + A\,r\frac{d\sigma_r}{dr} + B\,\sigma_r = C \qquad (8)$$

Where

$$A = 3 + \frac{1}{1-\nu_{\theta z}\nu_{z\theta}}(\nu_{r\theta} + \nu_{rz}\nu_{z\theta} - \frac{E_\theta}{E_r}\nu_{\theta r} - \frac{E_\theta}{E_r}\nu_{\theta z}\nu_{zr}) + \frac{rE_\theta}{1-\nu_{\theta z}\nu_{z\theta}}[\frac{d}{dr}(\frac{1-\nu_{\theta z}\nu_{z\theta}}{E\theta})]$$

$$B = 1 + \frac{\nu_{r\theta}+\nu_{rz}\nu_{z\theta}}{1-\nu_{\theta z}\nu_{z\theta}} - \frac{E}{E_r}\,\frac{\nu_{\theta r}+\nu_{\theta z}\nu_{zr}+1-\nu_{rz}\nu_{zr}}{1-\nu_{\theta z}\nu_{z\theta}} - \frac{rE_\theta}{1-\nu_{\theta z}\nu_{z\theta}}\frac{d}{dr}[(\frac{\nu_{\theta r}+\nu_{\theta z}\nu_{zr}}{E_r}) - (\frac{1-\nu_{\theta z}\nu_{z\theta}}{E_\theta})]$$

$$C = \frac{E_\theta}{1-\nu_{\theta z}\nu_{z\theta}}[(\varepsilon_r^T - \varepsilon_\theta^T) - (\nu_{rz} - \nu_{\theta z})(\varepsilon_z^o - \varepsilon_z^T) - r\frac{d\varepsilon_\theta}{dr} + r\frac{d}{dr}[\nu_{\theta Z}(\varepsilon_Z^O - \varepsilon_z^T)]]$$

and the boundary condition for eq. (8) are as follows:

$$\sigma_r(r_o) = 0 \qquad (9)$$

$$\sigma_r(o) = \text{finite value} \qquad (10)$$

where r_o is the outside radius.

Solving Eq (8) requires knowledge of the variation of properties across the diameter of the fiber. This reduces to a managable level of complexity by using a simplified model: viz., a perfect onion skin model. It is assumed that the material is homogeneous; and properties are all constants (Fig. 2). Properties are assumed to be those of pyrolytic graphite such that:

$E_\theta = E_z = E_a = 31\text{GPa} = 4.5\text{x}10^6\text{psi}$

$E_r = E_c = 10.3\text{GPa} = 1.5\text{x}10^6$ psi

$\alpha_z = \alpha_\theta = \alpha_a = 1.5\text{x}10^{-6}\ {}^\circ C^{-1}$

$\alpha_r = \alpha_c = 20\text{x}10^{-6}\ {}^\circ C^{-1}$

$\nu_{zr} = \nu_{\theta r} = \nu_{ac} = .3$

$\nu_z = \nu_{\theta z} = \nu_{aa} = -.19$

$\nu_{rz} = \nu_{r\theta} = \nu_{ca} = .9$

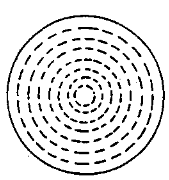

Fig. 2. Onion layers model.

Properties closer to those that might be observed in fibers could be assumed, but the signs of the stresses would not change. The stresses in radial, hoop and axial directions can be solved using eq (8) yielding the following relations:

$$\sigma_r = 1.26[1-\sqrt{\frac{r}{r_o}}] \text{ GPa} \qquad (11)$$

$$\sigma_\theta = 1.26[1-1.5\sqrt{\frac{r}{ro}}] \text{ GPa} \qquad (12)$$

and

$$\sigma_z = [.62-.77\sqrt{\frac{r}{ro}}] \text{ GPa} \qquad (13)$$

As shown in Fig. 3, the high compressive hoop stress on the surface will cause a crenulated surface to form on the fiber. Both hoop and radial stresses contribute to the development of axial stress in the fiber through Poisson's ratio effects, and so they will also be compressive on the surface and tensile inside of the fiber. The radial stress within the fiber will induce microcracks within it, and the stress relief due to cracking reduces hoop, radial and longitudinal stresses.

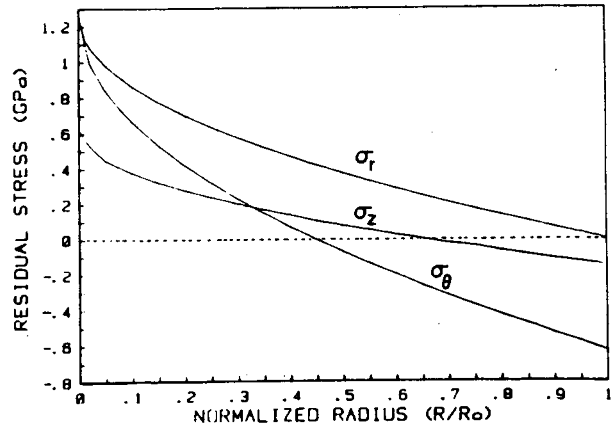

Fig. 3. Residual stresses profiles for perfect onion skin model.

PAN precursor carbon fibers are not homogeneous and a contribution from inhomogeneity must be added to any homogeneous orientation stress. The structure of the fiber is assumed to be skin and core. Each of these constituents is treated as having different size and properties (Fig. 4).

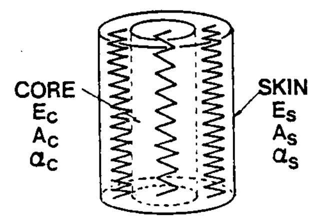

Fig. 4. Parallel spring model.

The residual stresses on the core and surface of the fiber, estimated from the parallel springs model based on purely physical reasoning are as follows[4]:

$\sigma_s = -.5\text{GPa}(-73.4\text{ksi})$ and $\sigma_c = .027\text{GPa}(3.81\text{ksi})$

By adding the contribution from homogeneous orientation, the stress in the surface is in between .5 to .65GPa (compressive). Since the stress relieves due to microcrack formation, the residual stress should be near the lower limit, which is very close to experimental values[5] (.35∿.5 GPa).

References

1. Saada, A.S., "Elasticity: Theory and Applications"

Pergamon Press Inc., p. 179, 1974.

2. Clark, T.J. and Larsen, R.J., Pl-WS-0232, General Electric Co., 1961.

3. LeMaistre, C.W., Ph.D. Thesis, Rensselaer Polytechnic Institute, 1971.

4. Chen, K-J., Ph.D. Thesis, Rensselaer Polytechnic Institute, 1984.

5. Chen, K-J. and Direfendorf, R.J., Progress in Science & Engineering of Composites, ICCM-IV, Tokyo, 1982, 97-105.