5.6 ANALYTICAL SOLUTIONS

Why? - Determine off-site migration
- Approximate geometries

What parameters are important?

1. Diffusion
2. Mechanical dispersion } hydrodynamic dispersion $D_L = \alpha_L v^a + D^*$
3. Groundwater velocity field, v^a

Solutions:

Fetter section 2.8 pp 56-64.

Analytical Solutions

1. Simplify geometry 1-D etc
2. Simplify heterogeneity (assume an equivalent magnitude of dispersion).
3. Simple solutions (but accurate).

Numerical Solutions

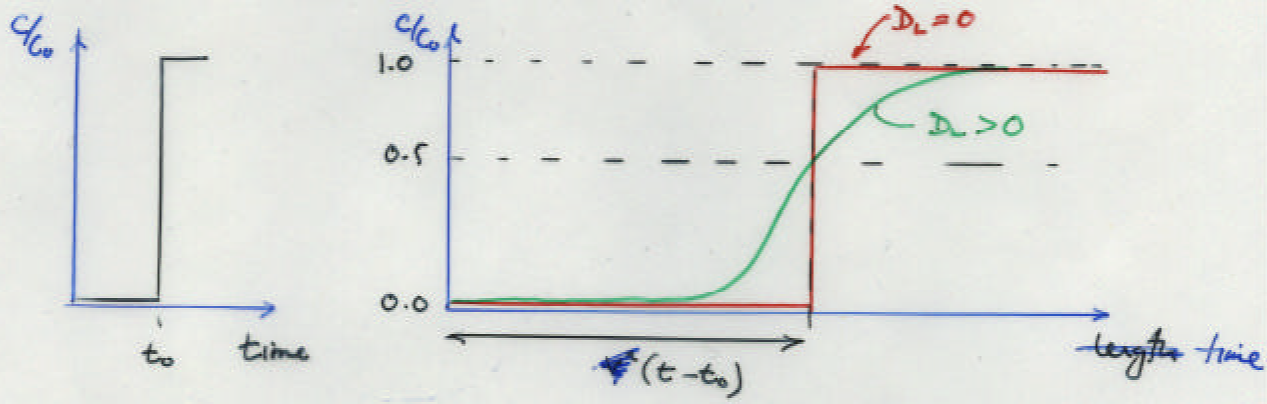
1. Enable complex 'real' geometries

2. Complex heterogeneity incorporated
3. Computer solutions - may include "numerical" dispersion.

Philosophical Trade:

- Data accuracy / reliability
- Sensitivity studies.

LIMITS OF APPLICABILITY OF APPROXIMATIONS



This view of the RTD curve is true only for large Peclet Nos (P_e)

$$P_e \geqslant 10$$

$$P_e = \frac{V^a L}{D_L} \longleftarrow D_L = \cancel{D^*}^{\,0} + \alpha_L V^a$$

$$\alpha_L \simeq \frac{1}{10} L$$

$$\therefore \; D_L \cong \frac{L V^a}{10}$$

Resubstituting then

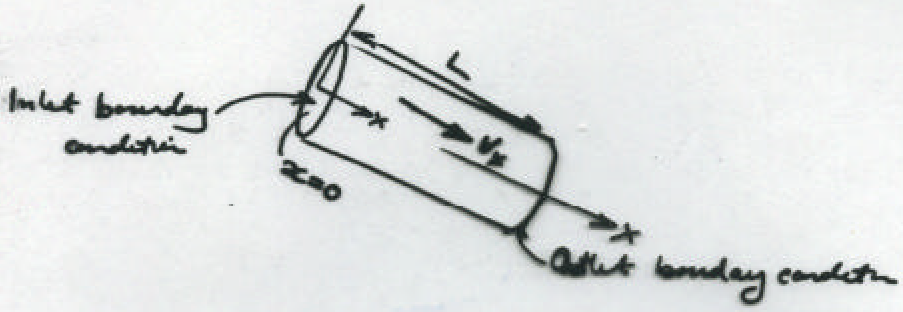
$$P_e = \frac{V^a L}{L V^a} \cdot 10 \quad = 10$$

$\therefore$ ok to use approximations.

ANALYTICAL SOLUTIONS (1-D)

or Temperature, T.

Equation:
$$D\frac{\partial^2 c}{\partial x^2} - v_x\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t}$$



Boundary Condition Types:

First type: Fixed concentration: $C = C_0$

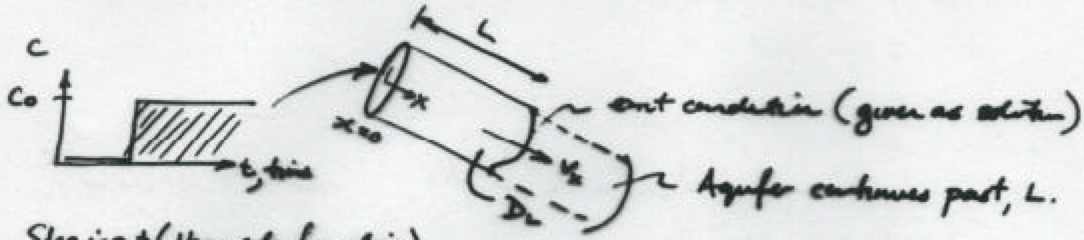
Second type: Fixed gradient $\left.\frac{\partial c}{\partial x}\right|_{x=?} = \text{constant}$

includes zero flux $\partial c/\partial x = 0$
(accumulating mass).

Third type: Variable flux $-D\frac{\partial c}{\partial x} + v_x c = v_x c(t)$

i.e. input flux of constant
concentration as VC_0 = prescribed m

ONE-DIMENSIONAL STEP CHANGE IN CONCENTRATION (Ogata & Banks, 1961)



Step input (Heaviside function)

Initial conditions: $\quad C(x,0) = 0 \qquad x \geq 0$

Boundary conditions: $\quad C(0,t) = C_0 \qquad t \geq 0^+$

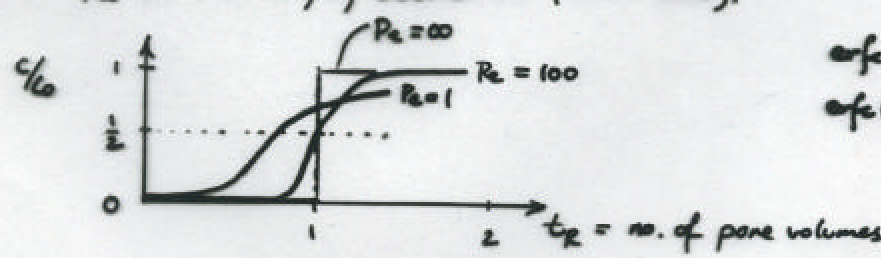
$\quad C(\infty,t) = 0 \qquad t \geq 0^+$

$$C = \frac{1}{2} C_0 \left[\operatorname{erfc}\left(\frac{L - V_x t}{2\sqrt{D_L t}}\right) + \exp\left(\frac{V_x L}{D_L}\right) \operatorname{erfc}\left(\frac{L + V_x t}{2\sqrt{D_L t}}\right) \right]$$

Non-dimensionalize: $\quad P_e = \frac{V_x L}{D_L} \quad ; \quad t_R = \frac{V_x t}{L} \equiv$ 'pore' volumes. as $V_x =$

$$\frac{C}{C_0} = \frac{1}{2}\left[\operatorname{erfc}\left[\left(\frac{P_e}{4t_R}\right)^{1/2}(1 - t_R)\right] + \exp(P_e)\operatorname{erfc}\left[\left(\frac{P_e}{4t_R}\right)^{1/2}(1 + t_R)\right]\right]$$

Observe change in C/C_0 at location, L, downstream (with time).

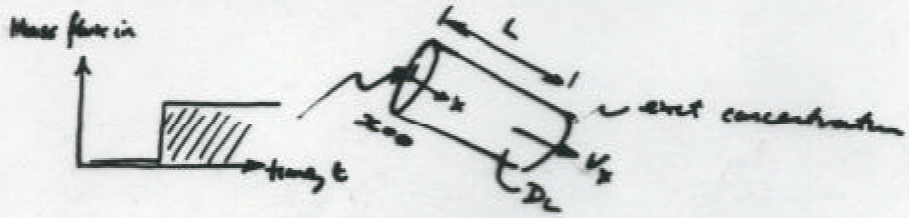


$\operatorname{erfc}(0) = 1$

$\operatorname{erfc}(\infty) = 0$

ONE - DIMENSIONAL CONTINUOUS INJECTION AT CONSTANT CONCENTRATION, C_0.

(Sauty, 1980)



Initial conditions: $C(x,0) = 0 \qquad -\infty < x < +\infty$

Boundary conditions: $\int_{-\infty}^{+\infty} n_e C(x,t)\,dx = C_0 n_e v_x t \qquad t > 0$

Dispersion occurs both downstream and upstream

$C(\infty,t) = 0 \qquad t \geq 0$

$$C = \frac{1}{2} C_0 \left[\operatorname{erfc}\left(\frac{L - v_x t}{2\sqrt{D_L t}}\right) - \exp\left(\frac{v_x L}{D_L}\right) \operatorname{erfc}\left(\frac{L + v_x t}{2\sqrt{D_L t}}\right)\right]$$

Non-dimensional:

Same as step injection but -ve.

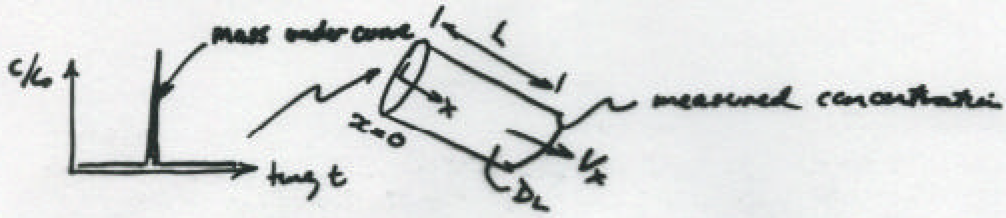
$$\frac{C}{C_0} = \frac{1}{2}\left[\operatorname{erfc}\left\{\left(\frac{P_e}{4t_R}\right)^{1/2}(1-t_R)\right\} - \exp(P_e)\operatorname{erfc}\left\{\left(\frac{P_e}{4t_R}\right)^{1/2}(1+t_R)\right\}\right]$$

For $P_e \to \infty$
$$\frac{C}{C_0} \simeq \frac{1}{2}\operatorname{erfc}\left\{\left(\frac{P_e}{4t_R}\right)^{1/2}(1-t_R)\right\}$$

Solution identical for both $\begin{cases} C_0 = \text{constant} \\ VC_0 = \text{constant} \end{cases}$

i.e. @ $P_e \to \infty$ Dispersive effects $\to 0$.

ONE – DIMENSIONAL – SLUG INJECTION (Sauty, 1980)



Mass slug injected: At some later time, a maximum concentration, C_{max}, results at time, t_{max}.

$$C_R = \frac{E}{(t_R)^{1/2}} \exp\left\{ -\frac{P_e}{4t_R}(1-t_R)^2 \right\}$$

$$E = (t_{R_{max}})^{1/2} \cdot \exp\left\{ \frac{P_e}{4t_{R_{max}}}(1-t_{R_{max}})^2 \right\}$$

$t_{R_{max}}$ = time of peak concentration

$= (1 + P_e^{-2})^{1/2} - P_e^{-1}$

$C_R = C/C_{max}$