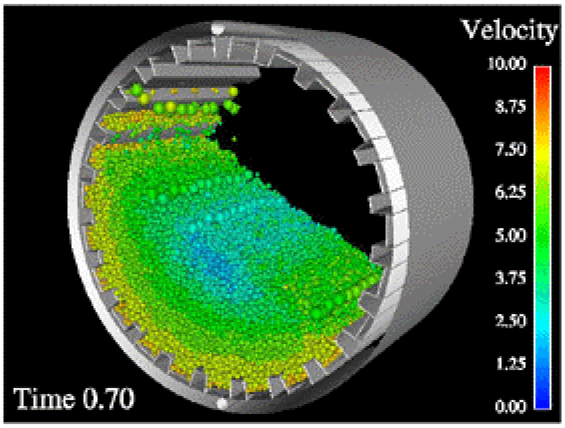
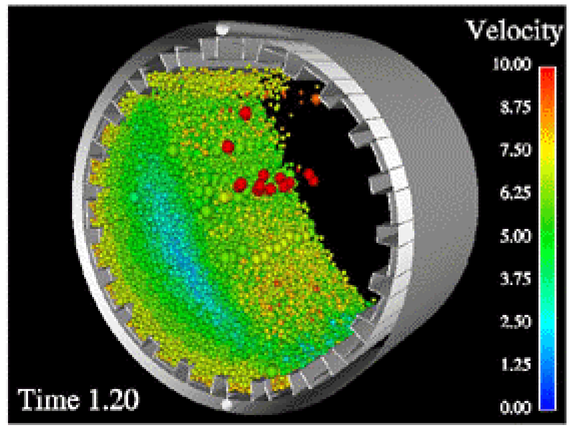
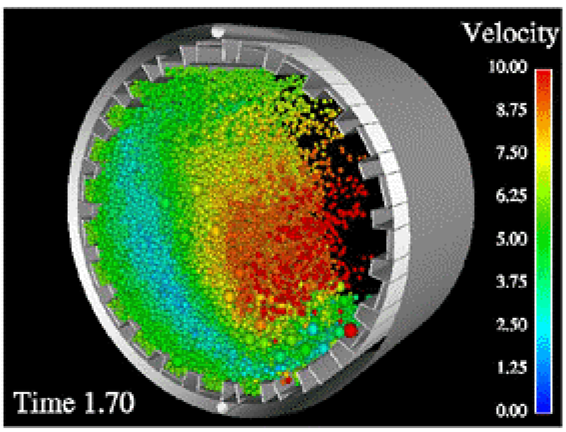
SOLID MECHANICS
FLOW OF POWDER AND BULK SOLIDS

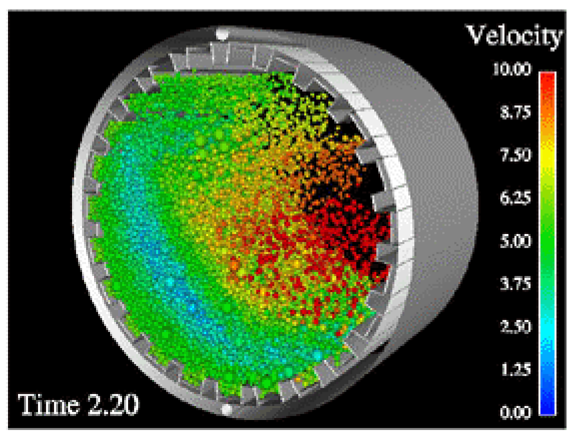
GeoEE 500

Ball Milling Example – Discrete Element Codes (http://www.cmis.csiro.au/cfd/dem/ballmill_3D/index.htm)









2.1.2.

SIMILARITIES BETWEEN FLUID & SOLID MECHANICS

Fluid Mechs. | **Solid Mechs.**

CONSERVATION:

(Eulerian Ref frame)

Momentum:

$$\frac{\partial}{\partial t}\rho v_x = -\left(\frac{\partial}{\partial x}\rho v_x v_x + \frac{\partial}{\partial y}\rho v_y v_x + \dots\right)$$

$$-\left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \dots\right) - \frac{\partial p}{\partial x} + \rho g_x$$

(Lagrangian Ref frame)

$$\rho\frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} - \left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right) + \rho g_x \quad (3)$$

Solid Mechs.:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = \rho g_x + \rho\frac{\partial^2 u_x}{\partial t^2} \quad (3)$$

Mass:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\rho\underline{v}$$

(Eulerian)

$$\frac{\partial\rho}{\partial t} + \left(v_x\frac{\partial\rho}{\partial x} + \dots\right) = -\rho\left(\frac{\partial v_x}{\partial x} + \dots\right)$$

(Lagrangian)

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot\underline{v}) \quad (1)$$

Solid Mechs.:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \quad (6)$$

CONSTITUTIVE:

Linear:

$$\sigma_x = -p + \lambda\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + 2\mu\frac{\partial v_x}{\partial x} \quad (6)$$

Poisson Eqn. $\lambda = -\frac{2}{3}\mu$

Solid Mechs.:

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \quad (6)$$

$$\left(G = \frac{E}{2(1+\nu)}\ ; \quad \lambda = \frac{2G\nu}{(1-2\nu)}\right.$$

Failure:

$$\sigma_1 = N\sigma_3 + (1-N)p + 2cN^{1/2}$$

$$\left(N = (1+\sin\phi)/(1-\sin\phi)\right)$$

Variables:

$v_x\ ;\ v_y\ ;\ v_z\ ;\ p$

$$\text{or} \equiv \underbrace{\dot{u}_x\ ;\ \dot{u}_y\ ;\ \dot{u}_z}_{\text{velocities}}\ ;\ p$$

Solid Mechs.:

$$\underbrace{u_x\ ;\ u_y\ ;\ u_z}_{\text{displacements}}\ ;\ \cancel{p}^{\to 0}$$

$\lambda = \frac{2}{3}\mu$

μ = dynamic viscosity

ν = Poisson ratio

G = shear modulus

λ = Lamé constant; c = cohesion

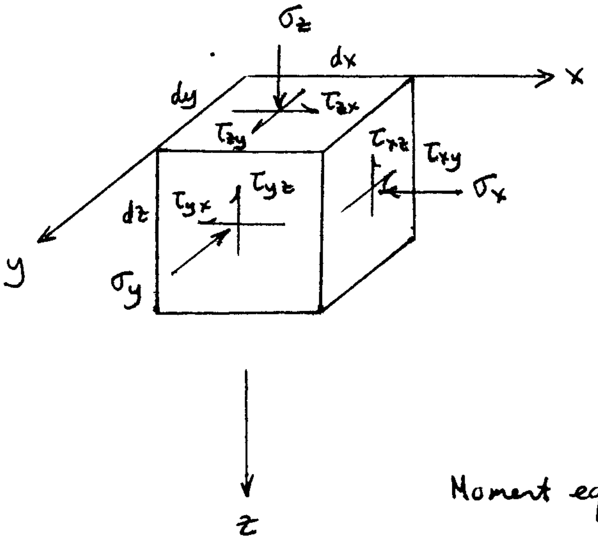
STRESSES

Different from fluid mechanics — stresses in a fluid $\equiv p \neq \sigma_x \neq \sigma_y \neq \sigma_z$

but rotations of stress not important since no failure

Newton's law relates pressures and τ_{xy} to $\partial u / \partial y$ etc.

Solid mechanics — $\sigma_x \neq \sigma_y \neq \sigma_z$



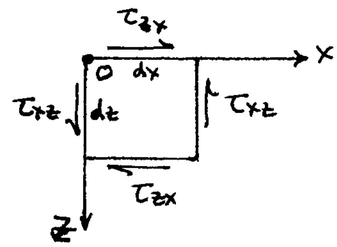
Conventions: Geosci/Geotech — Compression +ve

Mechanics — Tension +ve.

τ_{xz} ← in the z direction.

↑ face acting on ⊥ to x

Moment equilibrium:



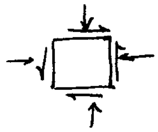
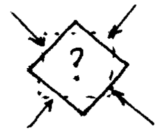
$$\tau_{xz}\,(dy.dz)\,dx = \tau_{zx}\,(dx.dy)\,dz$$

$$\tau_{xz} = \tau_{zx}$$

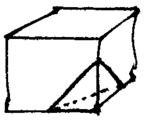
$$\tau_{zy} = \tau_{yz}$$

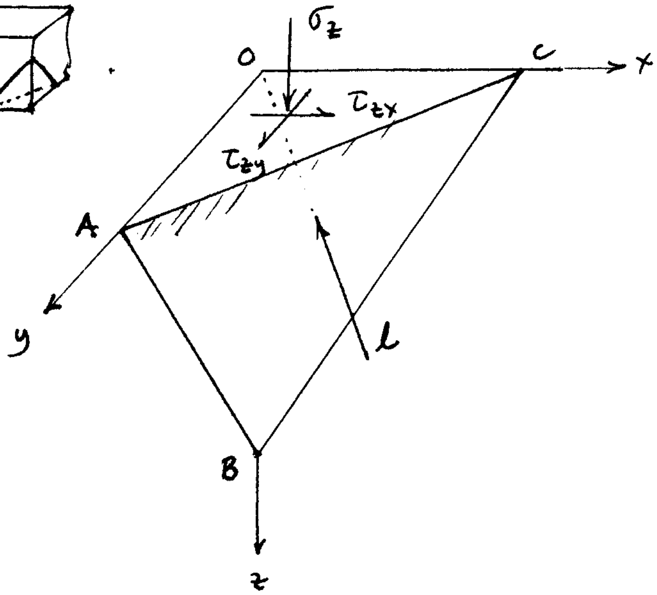
$$\tau_{yx} = \tau_{xy}$$

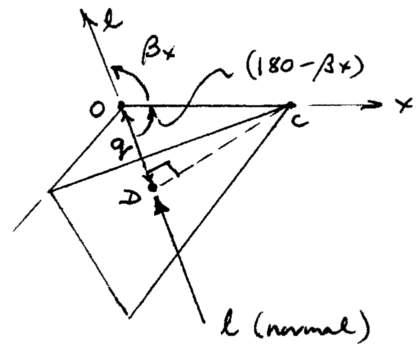
STRESS TRANSFORMATION EQUATIONS

From given:  evaluate: 

Cube:







ODC ≡ 90°

β_x ≡ angle between l and x axes.

Areas:

$ABC = a$

$AOB = a_x$

$OBC = a_y$

$OAC = a_z$

$$q = OC\cos(180^\circ - \beta_x) = -OC\cos\beta_x = -OC\, l_x$$

$$\text{Volume of tetrahedron} = \frac{1}{3}\text{ base area} \times \text{height} = \underbrace{\frac{1}{3}a_x \cdot OC = \frac{1}{3}a \cdot q}$$

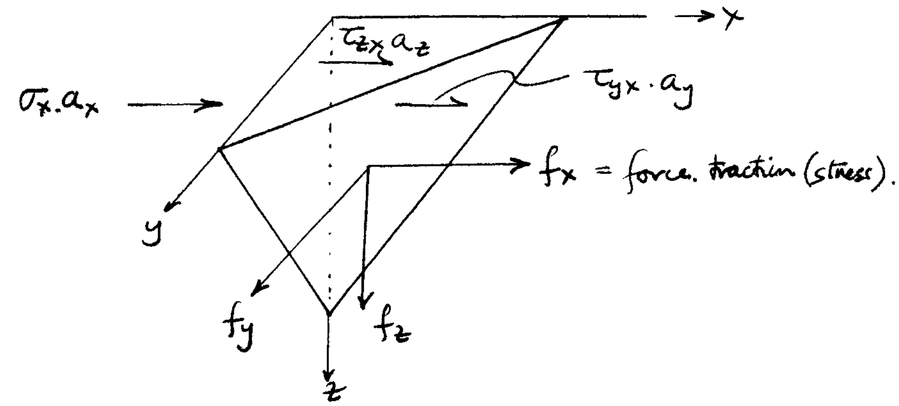
$$a_x = \frac{q}{OC}\cdot a = -l_x \cdot a$$

$$a_x = -l_x \cdot a$$

$$a_y = -l_y \cdot a$$

$$a_z = -l_z \cdot a$$

BALANCE STRESSES



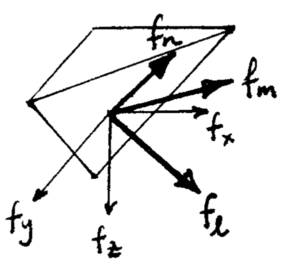
$$f_x . a + \sigma_x . a_x + \tau_{yx} . a_y + \tau_{zx} . a_z = 0$$

$$f_x = -\frac{a_x}{a} . \sigma_x - \frac{a_y}{a} . \tau_{yx} - \frac{a_z}{a} . \tau_{zx}$$

$$\left.\begin{aligned} f_x &= l_x . \sigma_x + l_y \tau_{yx} + l_z \tau_{zx} \\ f_y &= l_x . \tau_{xy} + l_y \sigma_y + l_z \tau_{yz} \\ f_z &= l_x . \tau_{zx} + l_y \tau_{yz} + l_z \sigma_z \end{aligned}\right\} (1) \qquad \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix}$$

Stress vector = Stress vector × Plane normal.

Relate to stress vectors in rotated form: $f_x ; f_y ; f_z \Rightarrow f_l ; f_m ; f_n$



Substitute from (1), above.

$$\left.\begin{aligned} f_l &= l_x f_x + l_y f_y + l_z f_z \equiv \sigma_l \\ f_m &= m_x f_x + m_y f_y + m_z f_z \equiv \tau_{lm} \\ f_n &= n_x f_x + n_y f_y + n_z f_z \equiv \tau_{ln} \end{aligned}\right\} (2)$$

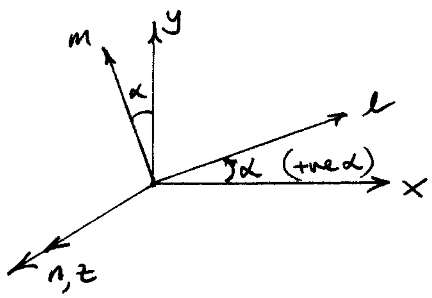
Substituting (1) into (2) and simplifying:

$$\sigma_l = l_x^2 \sigma_x + l_y^2 \sigma_y + l_z^2 \sigma_z + 2 l_x l_y \tau_{xy} + 2 l_y l_z \tau_{yz} + 2 l_z l_x \tau_{zx}$$

$$\tau_{lm} = l_x m_x \sigma_x + l_y m_y \sigma_y + l_z m_z \sigma_z + (l_x m_y + l_y m_x) \tau_{xy} + (l_y m_z + l_z m_y) \tau_{yz} + (l_z m_x + l_x m_z) \tau_{zx}$$

Cyclic permutation: $l \to m \to n \to l$

TWO-DIMENSIONAL EQUATIONS



n & z axes coincide.

$$l_x = \cos\alpha \qquad l_y = \sin\alpha \qquad l_z = 0$$
$$m_x = -\sin\alpha \qquad m_y = \cos\alpha \qquad m_z = 0$$
$$n_x = 0 \qquad n_y = 0 \qquad n_z = 1$$

From previous transformation equations:

$$\sigma_l = \cos^2\alpha\,\sigma_x + \sin^2\sigma_y + \cancelto{0}{0^2}\sigma_z + \dots + 2\cos\alpha\sin\alpha\,\tau_{xy} + 0$$

or
$$= \tfrac{1}{2}(1+\cos 2\alpha)\sigma_x + \tfrac{1}{2}(1-\cos 2\alpha)\sigma_y + 2.\tau_{xy}\tfrac{1}{2}\sin 2\alpha$$

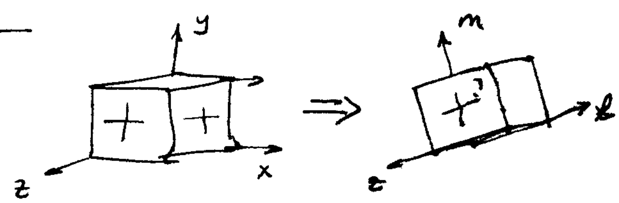
and
$$\sigma_l = \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

Repeating for σ_l and σ_m and τ_{lm}.

$$\left.\begin{aligned}
\sigma_l &= \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha \\
\sigma_m &= \tfrac{1}{2}(\sigma_x+\sigma_y) - \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\alpha - \tau_{xy}\sin 2\alpha \\
\tau_{lm} &= \tau_{xy}\cos 2\alpha - \tfrac{1}{2}(\sigma_x-\sigma_y)\sin 2\alpha
\end{aligned}\right\}$$

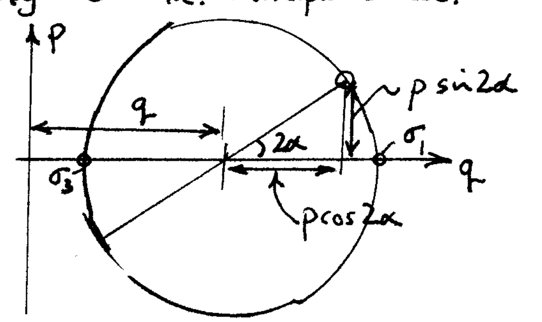
Stress transformation, +ve α ↻

Also:
$$\tau_{ml} = \tau_{zx}\cos\alpha + \tau_{zy}\sin\alpha$$
$$\tau_{nm} = -\tau_{zx}\sin\alpha + \tau_{zy}\cos\alpha$$



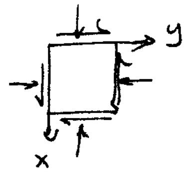
Note: If $q = \tfrac{1}{2}(\sigma_x+\sigma_y)$; $p = \tfrac{1}{2}(\sigma_x-\sigma_y)$ and $\tau_{xy} = 0$ i.e. Principal stresses.
then $q = \tfrac{1}{2}(\sigma_1+\sigma_3)$; $p = \tfrac{1}{2}(\sigma_1-\sigma_3)$

Then:
$$\sigma_l = q + p\cos 2\alpha$$
$$\sigma_m = q - p\cos 2\alpha$$
$$\tau_{lm} = -p\sin 2\alpha$$



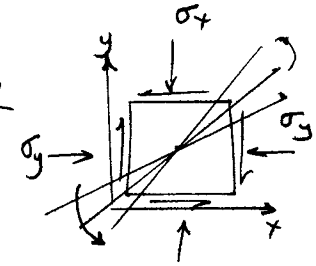
MOHR'S CIRCLE OF STRESS

Abandon existing shear stress convention, defined +ve.



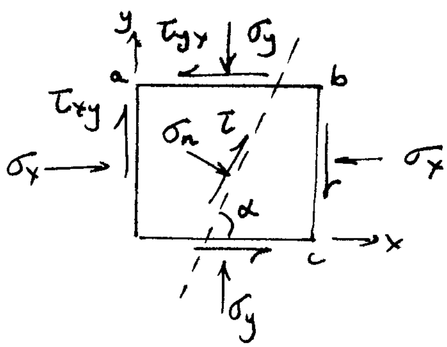
Use τ counter-clockwise, clockwise

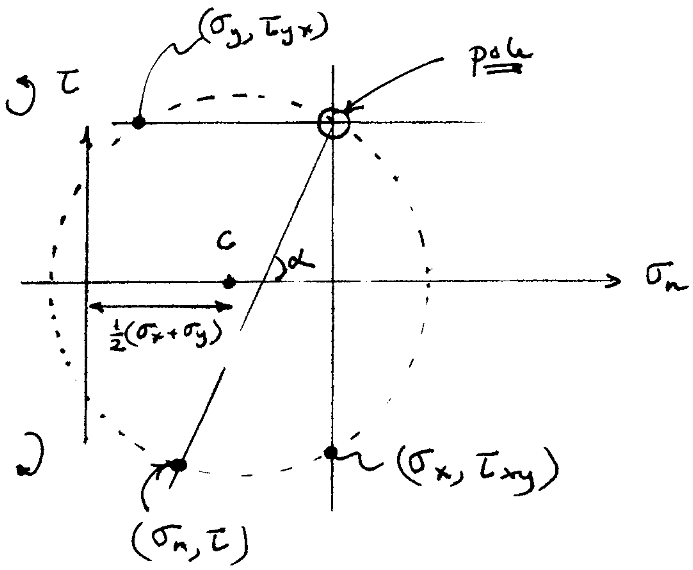
Mohr's circle – ① represents the possible state of stress on a fan of inclined planes passing through a 'stressed' point.



② We can locate one location, called the <u>POLE</u>, (and one location only) through which all the planes in the physical world pass. –

<u>Define the pole:</u>





Procedure:

① Plot τ_{yx}, σ_y on figure. τ_{yz} in this case ↺ above σ_n axis

② Draw orientation of plane 'ab' through ~~this~~ point

③ Plot the stress coordinates of another plane. Not necessarily ⊥ to 'ab'.

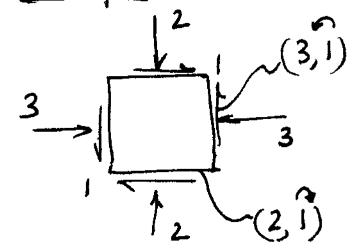
In this case use σ_x, τ_{xy}.

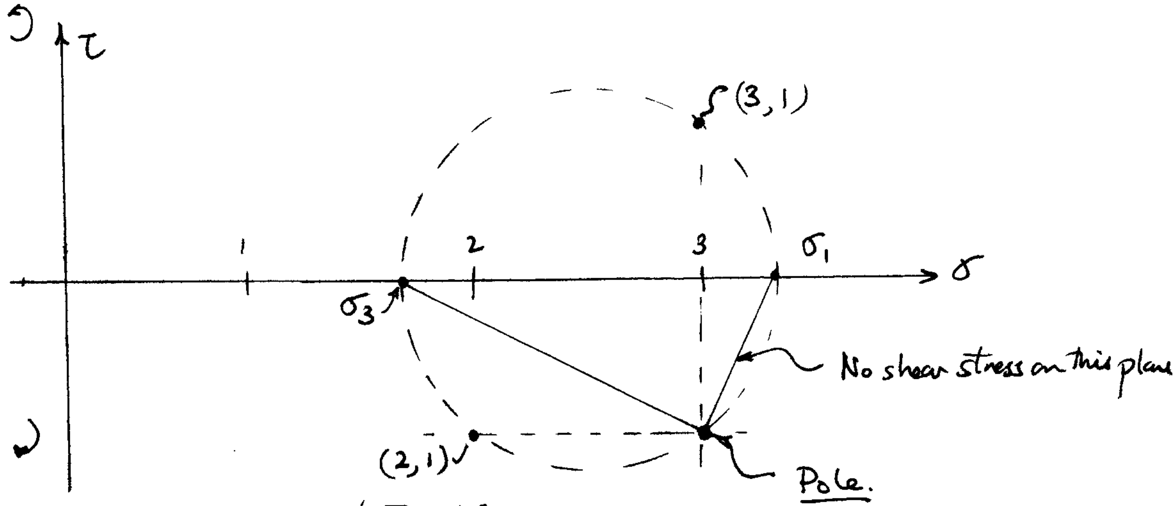
④ Draw orientation of plane 'bc' through this point

This defines the pole.

⑤ Use this to define stresses on interior 'planes' of cube. e.g. (σ_n, τ). shown.

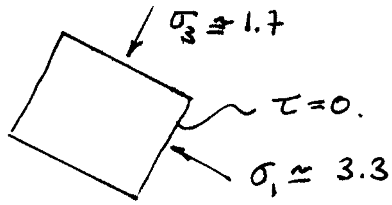
Example



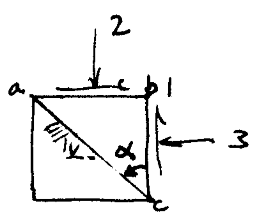


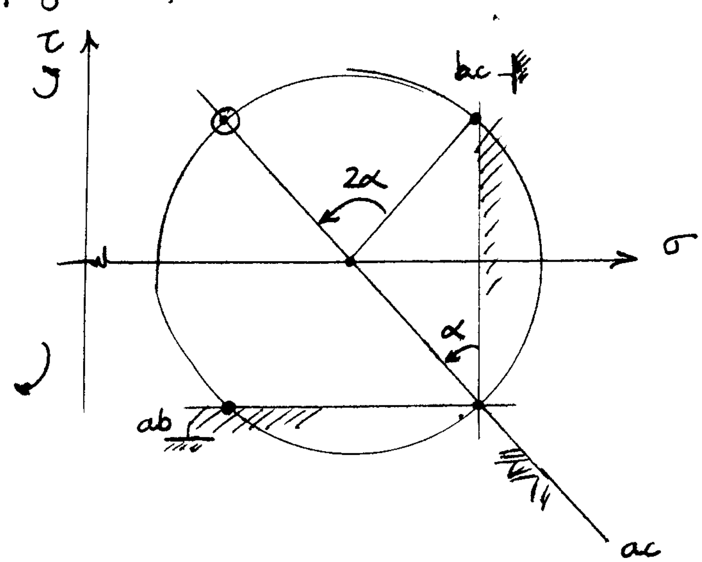
Principal stresses

i.e No shear stress.



Note that a rotation in the "Mohr circle plane" is double that in the physical plane.



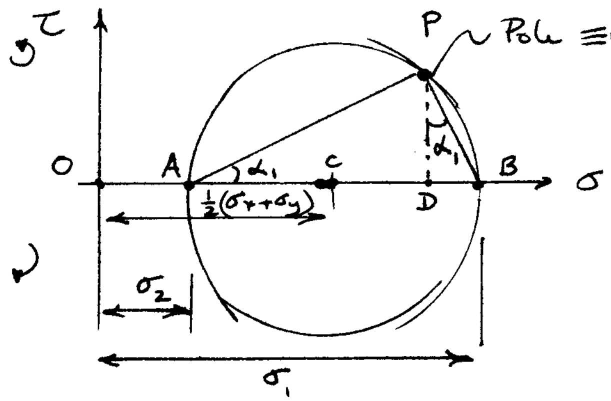


PRINCIPAL STRESSES (σ_1, σ_3) or (σ_1, σ_2).

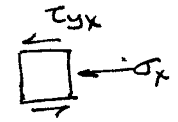
Major (most compressive) — σ_1

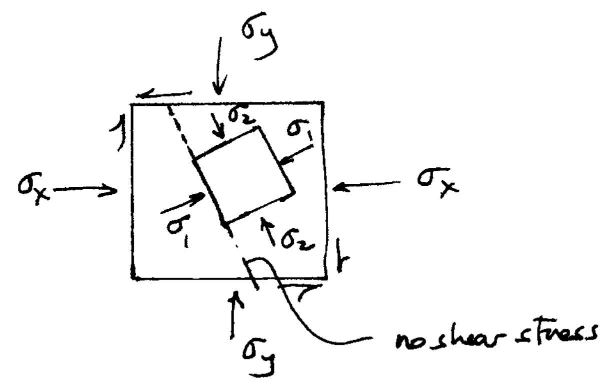
Minor principal stress — σ_2

Most failure criteria & yield criteria defined relative to (σ_1, σ_2) or $(\sigma_1 - \sigma_2)$.



Pole $\equiv (\sigma_x, \tau_{yx})$ — note that these are on different planes.





From geometry: Radius $= \sqrt{CD^2 + DP^2}$

$DP = \tau_{yx}$

$CD = \sigma_x - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y)$

$$R = \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{yx}^2}$$

$$\sigma_1 = OC + R = \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}$$

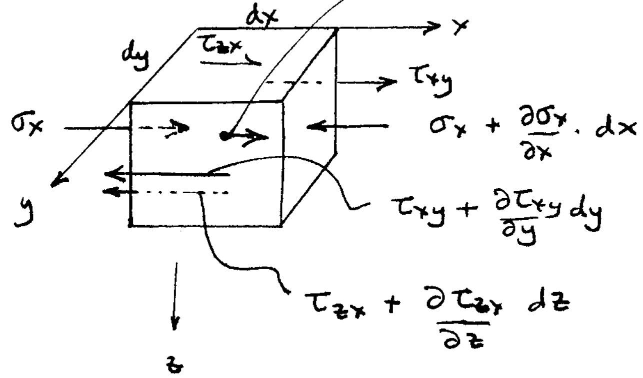
$$\sigma_2 = OC - R = \frac{1}{2}(\sigma_x + \sigma_y) - \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}$$

Inclination: (σ_1 Relative to x-axis)

$$\tan \alpha_1 = \frac{BD}{DP} = \frac{\sigma_1 - \sigma_x}{\tau_{yx}} \qquad \therefore \quad \alpha_1 = \tan^{-1}\left(\frac{\sigma_1 - \sigma_x}{\tau_{yx}}\right)$$

EQUILIBRIUM EQUATIONS (CARTESIANS)

$b_x\,dx\,dy\,dz \;=\; \frac{\text{body force}}{\text{volume}} \times \text{volume}.$



Equilibrium in x-direction:

$$\sigma_x.dy.dz - \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}.dx\right).dy.dz$$

$$+ \quad \tau_{xy}.dx.dz - \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}.dy\right)dx.dz$$

$$+ \quad \tau_{zx}.dx.dy - \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}.dz\right)dx.dy \quad + \quad b_x\,dx.dy.dz = 0$$

Rearrange $\rightarrow$ gives single equation + 2 others for y. and z.

$$\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= b_x \\[2ex]
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= b_y \\[2ex]
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= b_z
\end{aligned}\right\} \qquad \sigma_{ij,j} = b_i$$

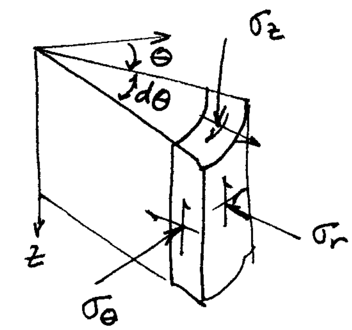
3 equations.

EQUILIBRIUM – CYLINDRICAL COORDS

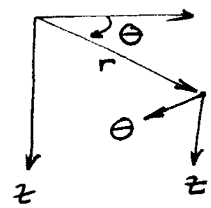
$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{(\sigma_r - \sigma_\theta)}{r} = b_r$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2\tau_{r\theta}}{r} = b_\theta$$

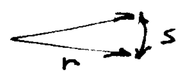
$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = b_z$$



No coaxiality of tangential components $\sigma_\theta, \tau_{r\theta}, \tau_{z\theta}$

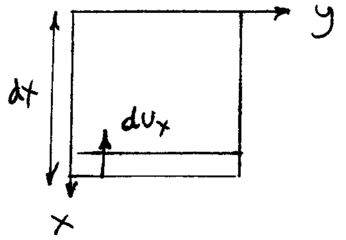


Note definition of θ



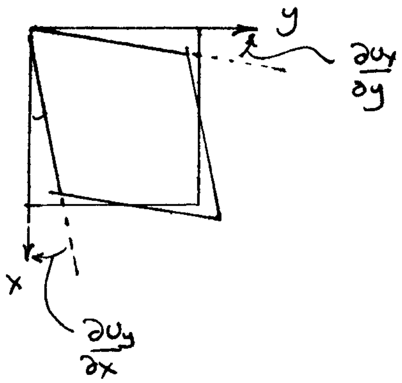
$$\theta = \frac{s}{r} = \text{non dimensional.}$$

STRAINS – & COMPATABILITY OF DISPLACEMENTS



Normal strains:

$$\varepsilon_x = -\frac{\partial U_x}{\partial x} \; ; \qquad \varepsilon_y = -\frac{\partial U_y}{\partial y} \; ; \qquad \varepsilon_z = -\frac{\partial U_z}{\partial z}$$



Shear strains:

$$\gamma_{xy} = -\left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y} \right)$$

$$\gamma_{yz} = -\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} \right)$$

$$\gamma_{zx} = -\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial y} \right)$$

Note strain tensor:
$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

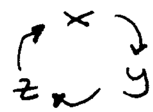
$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \varepsilon_{yx}$

(Symmetric)

Only 6 independent.

COMPATABILITY (6 EQUATIONS)

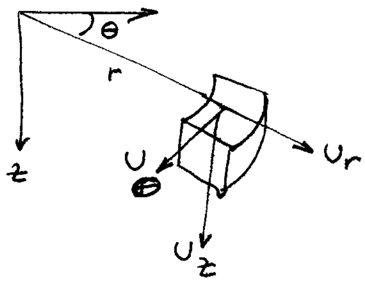
(3 like)
$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$



(3 like)
$$\frac{2\,\partial^2 \varepsilon_x}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

Volume strain, $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$ an invariant.

STRAINS IN CYLINDRICAL COORDINATES



$$\varepsilon_r = -\frac{\partial U_r}{\partial r}$$

$$\varepsilon_{\theta z} = -\left(\frac{1}{r}\frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial z}\right)$$

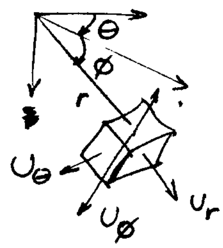
$$\varepsilon_\theta = -\frac{U_r}{r} - \frac{1}{r}\frac{\partial U_\theta}{\partial \theta}$$

$$\varepsilon_{zr} = -\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right)$$

$$\varepsilon_z = -\frac{\partial U_z}{\partial z}$$

$$\varepsilon_{r\theta} = -\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r} - \frac{1}{r}\frac{\partial U_r}{\partial \theta}$$

STRAINS IN SPHERICAL COORDINATES



$$\varepsilon_{rr} = -\frac{\partial U_r}{\partial r} \;; \qquad \varepsilon_{\theta\theta} = -\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} - \frac{U_r}{r} \;; \qquad \varepsilon_{\phi\phi} = -\frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{U_\theta}{r}\cot\theta - \frac{U_r}{r}$$

$$\varepsilon_{\theta\phi} = -\frac{1}{r}\left(\frac{\partial U_\phi}{\partial \theta} - U_\phi \cot\theta\right) - \frac{1}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi}$$

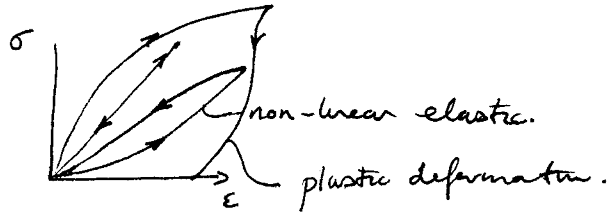
$$\varepsilon_{\phi r} = -\frac{1}{r\sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{\partial U_\phi}{\partial r} + \frac{U_\phi}{r} \;; \qquad \varepsilon_{r\theta} = -\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}$$

CONSTITUTIVE EQUATIONS

- Darcy's Law: $v = \frac{k}{\mu} \partial P/\partial x$
- Fourier's Law: $q_T = D\, \partial T/\partial x$
- Newton's Law: $\sigma = -p + 2\mu(\partial v/\partial x) + \lambda\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$
- Hooke's Law: $\sigma = f(E, \nu)\, \partial u/\partial x.$

Link stresses to strains: $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$

Linear elasticity



Two approaches to problems:

ELASTIC SOLN.

(1) Apply stresses and evaluate admissible stress state.

Must satisfy:
1) Equilibrium conditions (Momentum)
2) Compatability of displacements.
3) Boundary conditions.

Can only check (2) if we can link stresses & displacements.

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$$

PLASTIC SOLN.

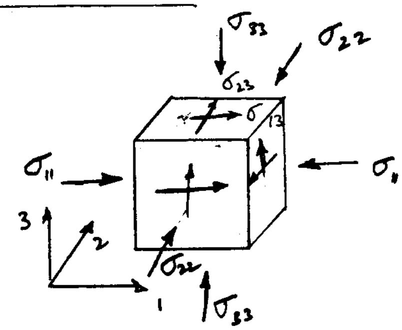
(2) Apply stresses and evaluate admissible stress state.

Must satisfy:
1) Equilibrium only.
(2) Failure state satisfied everywhere
3) Boundary conditions.

Strains ignored – statically determinate system.

e.g. $\sigma_1 = N\sigma_3 + (1-N)p + 2cN^{1/2}$

STRESS / STRAIN RELATIONSHIPS FOR 3-D ISOTROPIC, LINEAR ELASTICITY



General equations

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right] \quad (1)$$

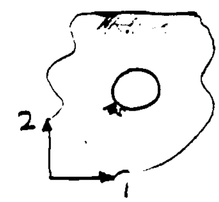
$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right] \quad (2)$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right] \quad (3)$$

$$\gamma_{12} = \sigma_{12}/G \; ; \quad \gamma_{13} = \sigma_{13}/G \; ; \quad \gamma_{23} = \sigma_{23}/G \quad (4) \qquad G = \frac{E}{2(1+\nu)}$$

σ_{13} ← orthogonal plane to x_3 axis
direction of stress

For 2-D representation, let (1,2) be the plane of interest with the 3 axis perpendicular to this plane. eg. Tunnel



Plane strain : By definition;

- The (1,2) plane is a 'principal' plane on which no shear stresses act

$$\therefore \quad \sigma_{13} = \sigma_{23} = 0$$
$$\longrightarrow \quad \gamma_{13} = \gamma_{23} = 0$$

- No displacement (strain) is allowed perpendicular to the (1,2) plane

$$\therefore \quad \varepsilon_{33} = 0$$

Setting $\varepsilon_{33} = 0$ in equation (3)

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

Substituting into (1) and (2)

$$\varepsilon_{11} = \frac{1}{E}\left[(1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22}\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[(1-\nu^2)\sigma_{22} - \nu(1+\nu)\sigma_{11}\right]$$

or $\underline{\varepsilon} = \underline{A}\,\underline{\sigma}$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} (1-\nu^2) & -\nu(1+\nu) & 0 \\ -\nu(1+\nu) & (1-\nu^2) & 0 \\ 0 & 0 & E/G \end{bmatrix}\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Since for $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$, $\underline{D} = \underline{A}^{-1}$

The third equation of the matrix identity is independent of the other 2 therefore $\rightarrow$ $\sigma_{12} = G\,\gamma_{12}$. The remaining 2×2 matrix may be inverted to give.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$

N.B. since $\underline{D} = \underline{A}^{-1}$; $\underline{A}^{-1}\underline{D} = \underline{I}$ as a check. ✓

Plane stress

Definition ; • (1, 2) plane is principal plane $\therefore$ $\sigma_{13} = \sigma_{23} = 0$

• No stress perpendicular to (1, 2) plane $\sigma_{33} = 0$

Substituting $\sigma_{13} = \sigma_{23} = 0$ into equn (4) and $\sigma_{33} = 0$ into equns (1, 2, 3) and rearranging terms:

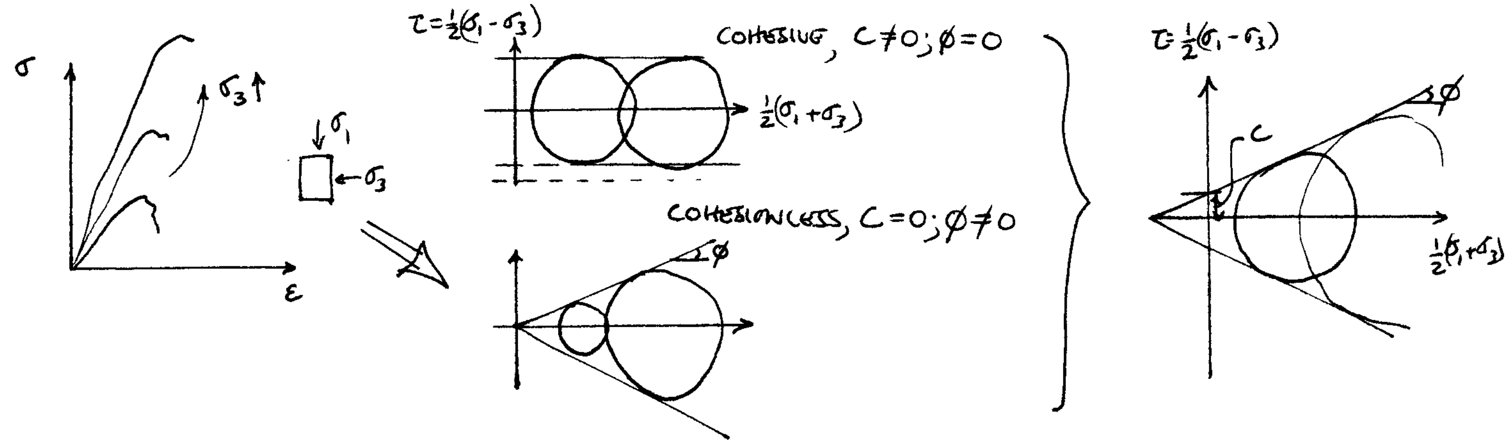
$$\underline{\varepsilon} = \underline{A}\,\underline{\sigma} \qquad \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

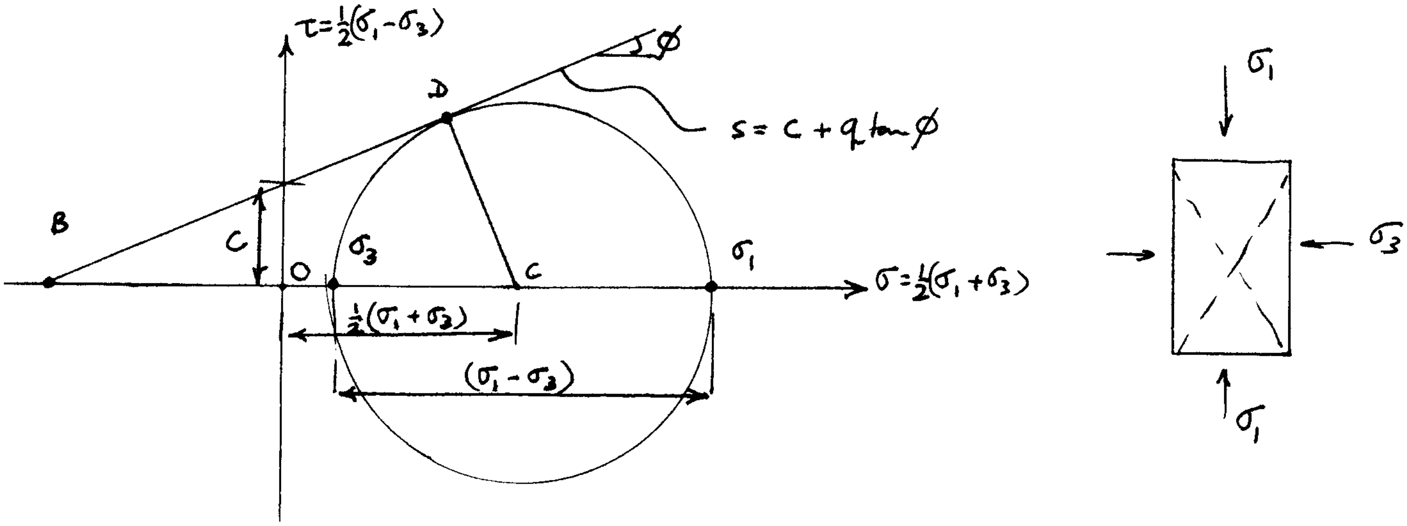
$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon} \qquad \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$

<u>Plane strain</u> – most useful in geotechnical, geological situations eg – slice through a dam, a tunnel – confined problems

<u>Plane stress</u> – many uses in structural mechanics – ie plate bending fracture mechanics ie small specimens.

FAILURE – OF PARTICULATE (& OTHER) MEDIA





From figure: $\sin\phi = \dfrac{CD}{BC}$

$$\begin{cases} CD = \frac{1}{2}(\sigma_1 - \sigma_3) \\ BC = BO + OC \begin{cases} BO = c/\tan\phi \\ OC = \frac{1}{2}(\sigma_1 + \sigma_3) \end{cases} \end{cases}$$

$$\therefore \quad \sin\phi = \frac{CD}{BC} = \frac{\frac{1}{2}(\sigma_1 - \sigma_3)}{\frac{1}{2}(\sigma_1 + \sigma_3) + c/\tan\phi} \equiv \frac{(\sigma_1 - \sigma_3)}{(\sigma_1 + \sigma_3) + 2c/\tan\phi}$$

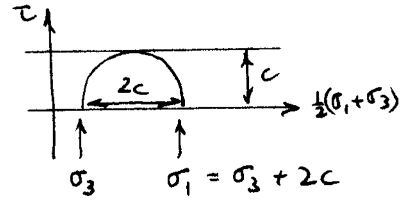
$$\sigma_1 \sin\phi + \sigma_3 \sin\phi + 2c\sin\phi/\tan\phi = \sigma_1 - \sigma_3$$

$$\sigma_1(1 - \sin\phi) = \sigma_3(1 + \sin\phi) + 2c\,\cancel{\sin\phi}\,\frac{\cos\phi}{\cancel{\sin\phi}}$$

$$\underline{\sigma_1 = N\sigma_3 + 2c\sqrt{N}} \qquad N = \frac{(1 + \sin\phi)}{(1 - \sin\phi)} \equiv \tan^2\left(45 + \frac{\phi}{2}\right)$$

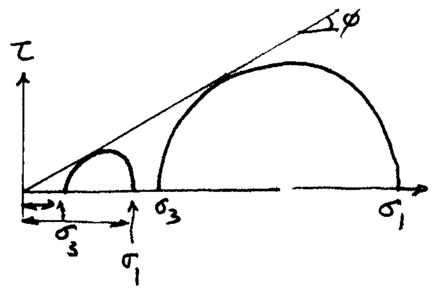
SPECIAL CONDITIONS

COHESIVE SOIL: $\phi = 0$ $\quad N = \left(\dfrac{1+\sin\phi}{1-\sin\phi}\right) = 0$ $\quad \sigma_1 = \cancelto{1}{N}\sigma_3 + 2c\sqrt{\cancelto{1}{N}}$

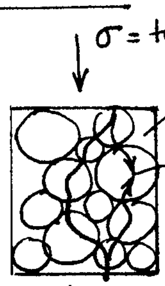


COHESIONLESS SOIL: $c = 0$ $\quad \sigma_1 = N\sigma_3 + 2\cancelto{0}{c}\sqrt{N}$

$\sigma_1/\sigma_3 = N$ = constant.



FLUIDS PRESENT:



σ = total stress

p = fluid pressure (static)

σ' = intergranular stress.

σ

Terzaghi's Law of Effective Stress.

$$\sigma = \sigma' + \cancelto{1}{\alpha} p$$

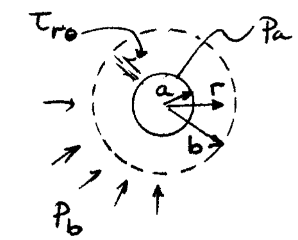
Importance: 1. Equilibrium equations $\quad \sigma_{ij,j} = b_j$

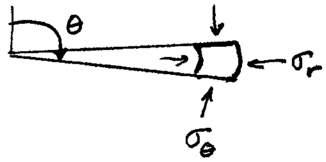
2. Failure - effective stress.

Strength: $\quad \sigma_1' = N\sigma_3' + 2c\sqrt{N}$ $\quad \begin{cases} \sigma_1' = \sigma_1 - p \\ \sigma_3' = \sigma_3 - p \end{cases}$

$$\underline{\sigma_1 = N\sigma_3 + (1-N)p + 2c\sqrt{N}}$$

THICK-WALLED CYLINDER





Assumptions:

1. Linear Elastic
2. Plane strain $\rightarrow$ $\varepsilon_y = \gamma_{yx} = \gamma_{yz} = 0$

   by cylindrical $\varepsilon_y = \gamma_{ry} = \gamma_{\theta y} = 0$

3. Axially symmetric $\therefore$ $\tau_{r\theta} = 0$

A. Strain - Displacement Relations

$$\varepsilon_r = -\frac{\partial u_r}{\partial r} = -\frac{du_r}{dr} \leftarrow \text{only one variable.} \qquad (1)$$

$$\varepsilon_\theta = -\frac{u_r}{r} - \frac{1}{r}\cancelto{0}{\frac{\partial u_\theta}{\partial \theta}} \qquad (2)$$

B. Stress - strain Relationships

$$\sigma_r = \lambda \overbrace{\Delta}^{\varepsilon_r + \varepsilon_\theta + \cancelto{0}{\varepsilon_y}} + 2G\varepsilon_r$$

$$\sigma_\theta = \lambda\Delta + 2G\varepsilon_\theta \qquad \lambda = \frac{2G\nu}{(1-2\nu)} \qquad (3)$$

Expanding:

$$\sigma_r = -\frac{2G\nu}{1-2\nu}\left(\frac{du_r}{dr} + \frac{u}{r}\right) - 2G\frac{du_r}{dr}$$

$$= -\frac{2G}{1-2\nu}\left[\left(\frac{du_r}{dr} + \frac{u_r}{r}\right)\nu + (1-2\nu)\cdot\frac{du_r}{dr}\right] \qquad (4)$$

Resulting in:

$$\sigma_r = -\frac{2G}{1-2\nu}\left[(1-\nu)\frac{du_r}{dr} + \nu\frac{u_r}{r}\right] \qquad (5)$$

$$\sigma_\theta = -\frac{2G}{1-2\nu}\left[(1-\nu)\frac{u_r}{r} + \nu\frac{du_r}{dr}\right] \qquad (6)$$

c. <u>Equilibrium Equations</u>

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\cancelto{0}{\frac{\partial \tau_{r\theta}}{\partial \theta}} + \cancelto{0}{\frac{\partial \tau_{ry}}{\partial y}} + \frac{(\sigma_r - \sigma_\theta)}{r} = \cancelto{0}{b_r} \qquad (7)$$

Substitute constitutive relations:

$$\frac{\partial}{\partial r}\left[(1-\nu)\frac{du_r}{dr} + \nu\frac{u_r}{r}\right] + \frac{1}{r}\left[(1-2\nu)\left(\frac{du_r}{dr} - \frac{u_r}{r}\right)\right] = 0$$

$$(1-\nu)\frac{d^2u_r}{dr^2} + \frac{\nu}{r}\frac{du_r}{dr} - \frac{\nu u_r}{r^2} + (1-2\nu)\left[\frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right] = 0 \qquad (8)$$

Gather terms:
$$(1-\nu)\frac{d^2u_r}{dr^2} + \frac{(1-\nu)}{r}\frac{du_r}{dr} - (1-\nu)\frac{u}{r^2} = \frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u}{r^2} = 0 \qquad (9)$$

$$\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) = 0 \qquad (10)$$

Integrate once:
$$\frac{du_r}{dr} + \frac{u_r}{r} = A \quad \Rightarrow \quad \cancel{\frac{1}{r}}\frac{d}{dr}(ru_r) = A.r \qquad (11)$$

Integrate twice:
$$u_r.r = \frac{1}{2}Ar^2 + B \quad \Rightarrow \quad \underline{\underline{u_r = \frac{1}{2}A.r + \frac{B}{r}}} \qquad (12)$$

Substitute (12) into constitutive relations, (5) and (6)

$$\sigma_r = \frac{-2G}{1-2\nu}\left[(1-\nu)\frac{du_r}{dr} + \nu\frac{u_r}{r}\right] \equiv \frac{-2G}{1-2\nu}\left[\frac{1}{2}A - (1-2\nu)\frac{B}{r^2}\right] \equiv C - \frac{D}{r^2} \qquad (13)$$

$$\sigma_\theta = -\frac{2G}{1-2\nu}\left[(1-\nu)\frac{u_r}{r} + \nu\frac{du_r}{dr}\right] \equiv \frac{-2G}{1-2\nu}\left[\frac{1}{2}A + (1-2\nu)\frac{B}{r^2}\right] \equiv C + \frac{D}{r^2} \qquad (14)$$

$$C = \frac{-GA}{(1-2\nu)} \; ; \quad D = -2GB$$

<u>Match Boundary Conditions</u>

$$\begin{cases} \sigma_r = p_a & @ \; r=a \quad \therefore \quad p_a = C - D/a^2 \\ \sigma_r = p_b & @ \; r=b \quad \therefore \quad p_b = C - D/b^2 \end{cases}$$

$$C = \frac{(b^2 p_b - a^2 p_a)}{b^2 - a^2} \quad ; \quad D = \frac{(p_b - p_a)\, a^2 b^2}{b^2 - a^2}$$

<u>Summarizing:</u>

$$\sigma_r = C - D/r^2$$

$$\sigma_\theta = C + D/r^2$$

$$\tau_{r\theta} = 0$$

$$u_r = -\frac{1}{2G}\left[(1-2\nu)\, C.r + D/r\right]$$

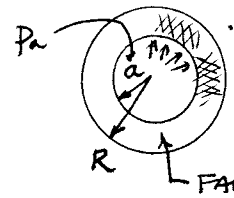
Where, $b \gg a$. $\quad \therefore \quad C \doteq p_b \quad ; \quad D \doteq (p_b - p_a)\, a^2$

$$\sigma_r = p_b - (p_b - p_a)\, a^2/r^2$$

$$\sigma_\theta = p_b + (p_b - p_a)\, a^2/r^2$$

$$u_r = -\frac{1}{2G}\left[(1-2\nu)\, p_b . r + (p_b - p_a)\, a^2/r\right]$$

EXPANSION OF A CYLINDRICAL CAVITY



FAILED ZONE

Equilibrium: $$\frac{d\sigma_r}{dr} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0 \quad (1)$$

Strength: $$\sigma_1 = N\sigma_3 + 2c\sqrt{N} \quad \begin{cases} \sigma_1 = \sigma_r \\ \sigma_3 = \sigma_\theta \end{cases}$$

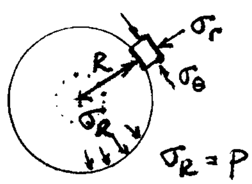
$$\therefore \ \sigma_\theta = \frac{1}{N}\sigma_r - 2c\frac{\sqrt{N}}{N} \quad (2)$$

$$\frac{d\sigma_r}{dr} + \frac{(1-\frac{1}{N})}{r}\sigma_r + \frac{2c}{\sqrt{N}}\frac{1}{r} = 0 \quad (3)$$

with $$\sigma_r\big|_{r=a} = p_a \quad (4)$$

Solving (3) & (4) $$\sigma_r = \frac{1}{(N-1)}\left\{\left[2c\sqrt{N} + (N-1)p_a\right]\left(\frac{a}{r}\right)^{(1-\frac{1}{N})} - 2c\sqrt{N}\right\} \quad (5)$$

ELASTIC ZONE



$\Leftarrow \sigma_\infty = p_b$

Lamé solution: $$\sigma_\theta = 2\sigma_\infty - \sigma_r$$
$$= 2\sigma_\infty - \sigma_R$$
$$\sigma_R = 2\sigma_\infty - \sigma_\theta \quad (6)$$

Equating radial stresses at $r = R$: Eq.(5)$\big|_{r=R}$ = (6)$\big|_{r=R}$

$$\sigma_R = 2\sigma_\infty - \sigma_\theta = 2\sigma_\infty - \frac{1}{N}\sigma_R - \frac{2c}{\sqrt{N}} \quad \text{or} \quad \sigma_R = \frac{1}{(1+\frac{1}{N})}\left(2\sigma_\infty - \frac{2c}{\sqrt{N}}\right) \quad (7)$$

$$\sigma_R = \frac{2N}{(1+N)}\left(\sigma_\infty - \frac{c}{\sqrt{N}}\right) \quad (7)$$

$$\sigma_R = \frac{1}{(N-1)}\left\{\left[2c\sqrt{N} + (N-1)p_a\right]\left(\frac{a}{R}\right)^{(1-\frac{1}{N})} - 2c\sqrt{N}\right\} \quad (8)$$

$$\left(\frac{R}{a}\right)^\beta = \frac{(1+N)\left[2c\sqrt{N} + (N-1)p_a\right]}{2\left[(N-1)N\sigma_\infty + 2c\sqrt{N}\right]} \quad (9)$$

$$\sigma_R = \frac{1}{(N-1)}\left\{\frac{2\left[(N-1)N\sigma_\infty + 2c\sqrt{N}\right]}{(1+N)} - 2c\sqrt{N}\right\} \quad (10)$$

$$\beta = (1 - \tfrac{1}{N})$$

FINAL SOLUTION:

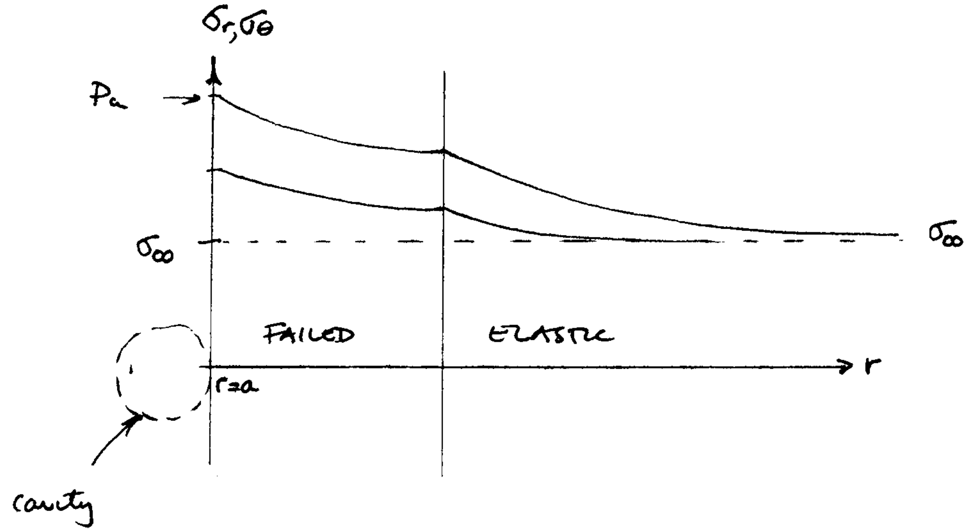
FRACTURED ZONE: $(r \leq R)$

$$\sigma_r = \frac{1}{(N-1)} \left\{ \left[2c\sqrt{N} + (N-1)p_a \right] \left(\frac{a}{r} \right)^{\beta} - 2c\sqrt{N} \right\} \quad (11)$$

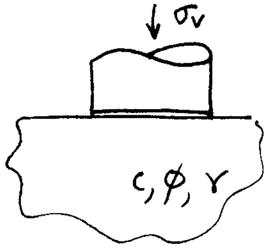
$$\sigma_\theta = \frac{1}{N} \sigma_r - \frac{2c}{\sqrt{N}} \quad (12)$$

ELASTIC ZONE: $(r \geq R)$

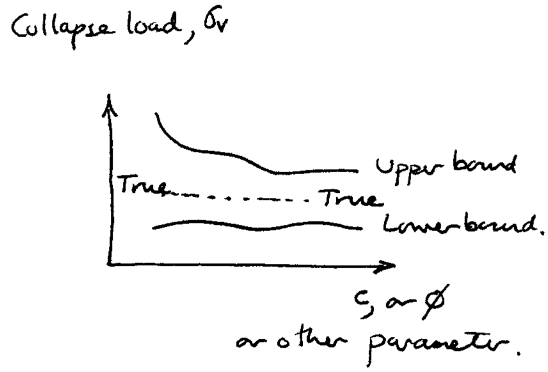
$$\left.\begin{aligned} \sigma_r &= \sigma_\infty - (\sigma_\infty - \sigma_R)\, R^2/r^2 \\ \sigma_\theta &= \sigma_\infty + (\sigma_\infty - \sigma_R)\, R^2/r^2 \end{aligned}\right\} (13)$$

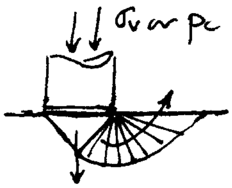


PLASTICITY SOLUTIONS — RIGID PLASTIC

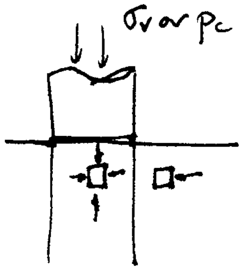


Desire to predict collapse load





<u>Upper bound</u>:
- Choose a failure mode
- Evaluate $\sigma_{collapse}$ for that failure mode.
- Upper-bound is <u>always</u> an <u>overprediction</u> of collapse load. — depends on how good the failure mechanism is.
- <u>Unconservative</u> estimate

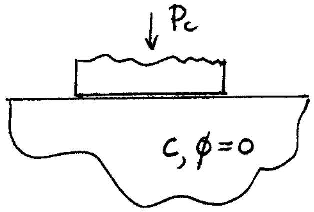


<u>Lower bound</u>:
- Choose a statically admissable stress state.
- Assume soil is <u>everywhere</u> in a state of failure.
- Lower bound will always under-estimate the collapse load.
- Conservative estimate.

$$P_{lower\ bound} \leq \text{true } p_c \leq P_{upper\ bound}$$

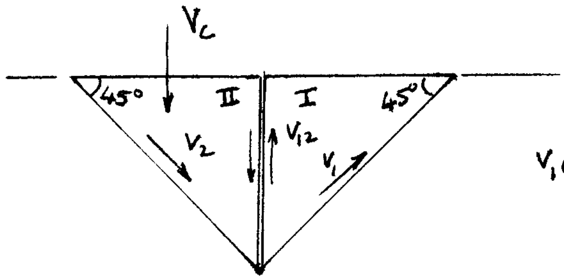
Desire to reduce difference (gap) between P_{upper} and $P_{lower} \rightarrow P_{true}$.

EXAMPLE

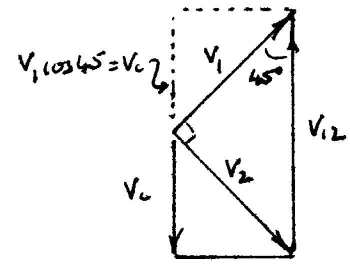


For the soil, $\phi = 0$, $c \neq 0$, determine the collapse load.

Upper bound:



HODOGRAPH



Let $V_c \equiv 1$
$$\begin{cases} V_2 = \sqrt{2} \\ V_1 = \sqrt{2} \\ V_{12} = 2 \end{cases}$$

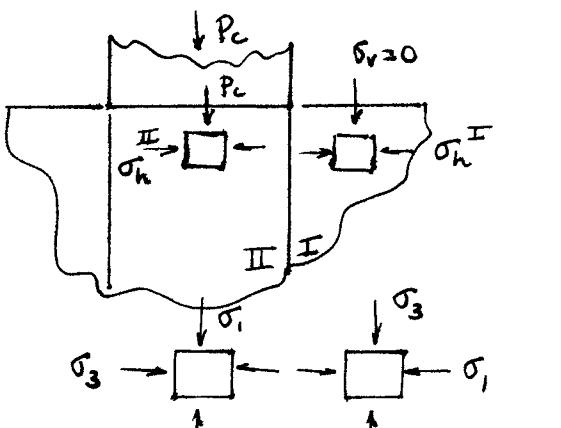
Load × despl (or velocity)

$$P_c V_c + \cancel{W_2}^{\,0} V_c = \sqrt{2}.c.V_2 + \sqrt{2}.c.V_1 + c.V_{12} + \cancel{W_1}^{\,0} V_1 \cos 45^\circ$$

External = Internal work.

$$P_c = \sqrt{2}\sqrt{2}\,c + \sqrt{2}\sqrt{2}.c + 2c \equiv \underline{6c = P_U.}$$

Lower bound:



Strength: $\sigma_1 = \cancel{N}^{1}\sigma_3 + 2c\sqrt{\cancel{N}^{1}}$

$\sigma_1 = \sigma_3 + 2c$

or $\sigma_3 = \sigma_1 - 2c$

$$\underbrace{\underset{\underset{\sigma_h^{II}}{\uparrow}}{\sigma_3} = \underset{\underset{P_c}{\uparrow}}{\sigma_1} - 2c \qquad \underset{\underset{\sigma_h^{I}}{\uparrow}}{\sigma_1} = \underset{\underset{0}{\uparrow}}{\sigma_3} + 2c}$$

$$P_c - 2c = 2c$$

$$\underline{P_c = 4c = P_L}$$

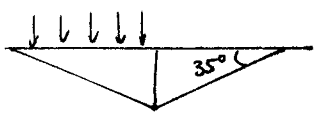
Note that:

$P_L = 4c \leq P_{TRUE} \leq P_U = 6c$

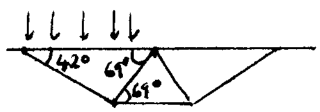
To narrow range, choose ① More realistic stress distribution $P_L \uparrow$

② More realistic failure mode $P_U \downarrow$

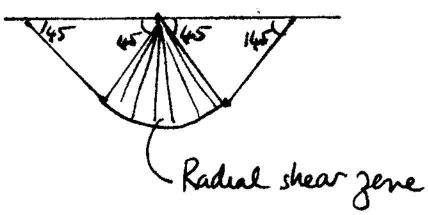
Upper Bound



$P_c = 5\frac{2}{3}c$

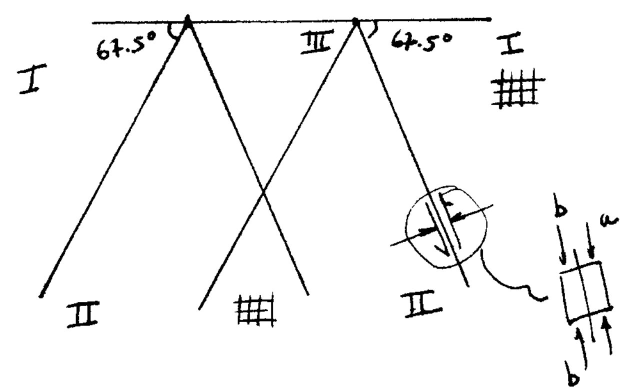


$P_c = 5.29c$



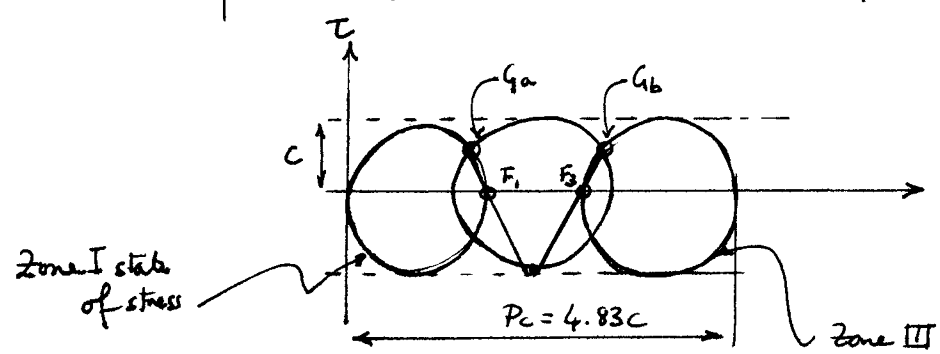
$\underline{P_c = 5.14c}$

Lower Bound



$\underline{P_c = 4.83c}$

a & b different.



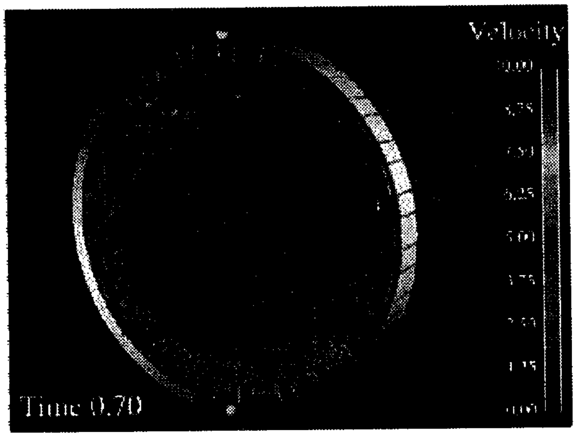
G_a = state of stress on inclined plane, at a

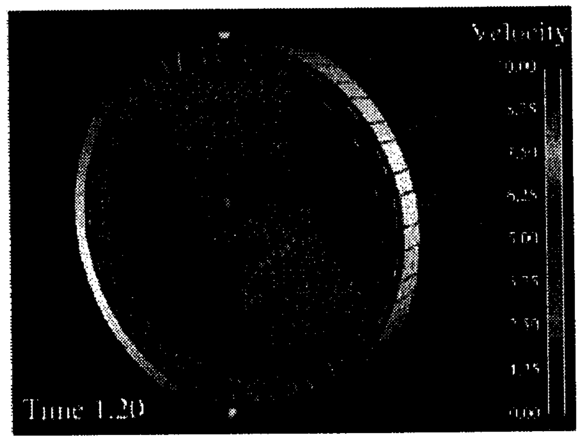
G_b = " " b

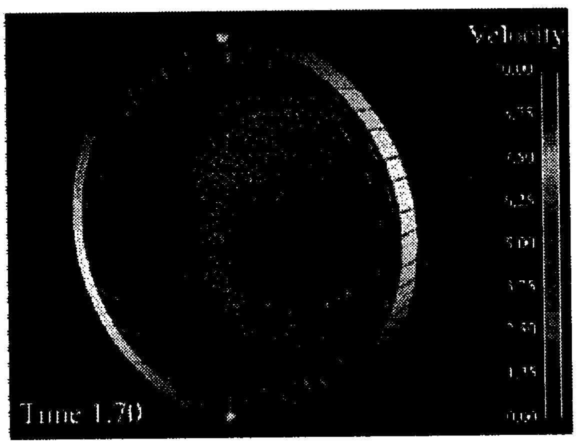
True collapse load is $P_c = 5.14c$

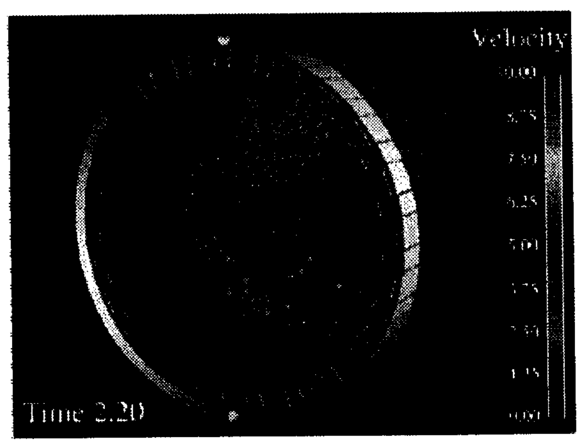
Ball Milling Example – Discrete Element Codes

(http://www.cmis.csiro.au/cfd/dem/ballmill_3D/index.htm)

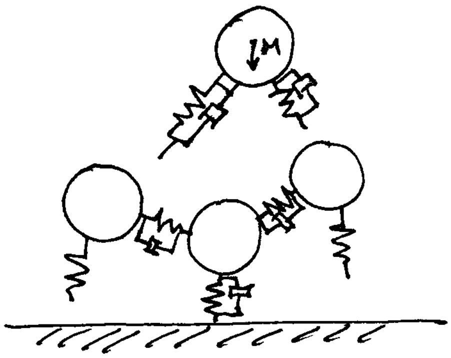








DISCONTINUUM APPROACHES TO PARTICULATE MECHANICS



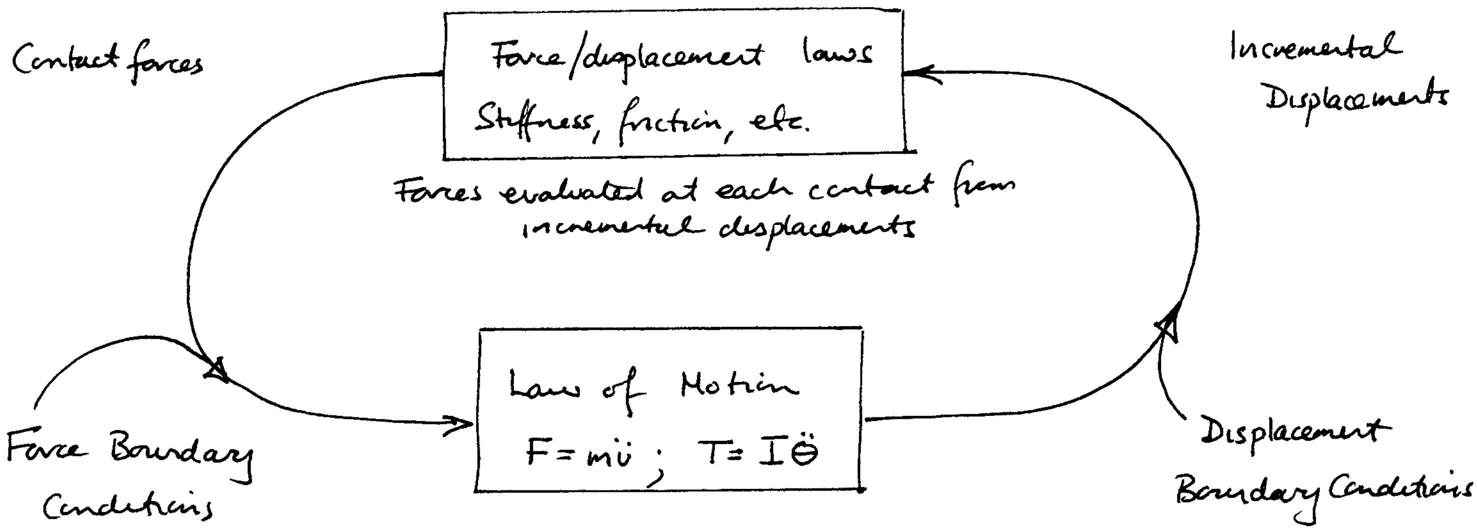
Solve for system of interacting particles.

Momentum balance $\Rightarrow$ $\sum F = m\ddot{u}$; $\sum T = I\ddot{\theta}$

Compatability $\Rightarrow$ ok: Not ok:

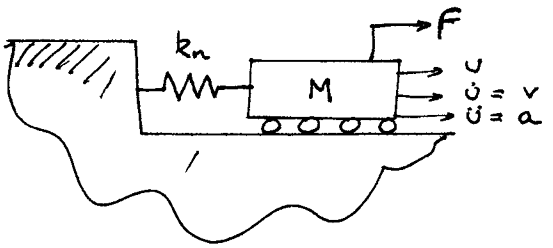
Constitutive $\Rightarrow$ $f = Ku$

Boundary conditions $\Rightarrow$



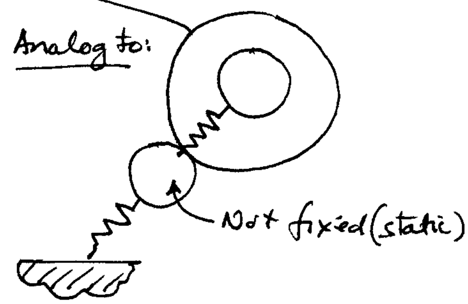
Net forces on each block cause accelerations that are integrated twice ($\iint \ddot{u} \rightarrow u$) to give incremental displacements over one time step, Δt.

1-D Example

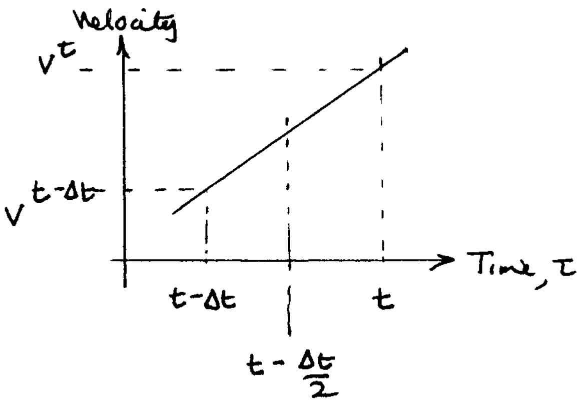


Force, f
Displacement, u
Velocity, $\dot{u}$
Acceleration, $\ddot{u}$

Analog to:



Fundamental Relations



$$v = \frac{\partial u}{\partial t}$$

$$a = \frac{\partial v}{\partial t} \doteq \frac{\Delta v}{\Delta t}$$

Velocities in terms of accelerations:

$$\left(v^t - v^{t-\Delta t}\right) = a^{t-\frac{1}{2}\Delta t} \cdot \Delta t$$

$$\boxed{v^t = a^{t-\frac{1}{2}\Delta t} \cdot \Delta t + v^{t-\Delta t}} \qquad (1)$$

Displacements in terms of velocities:

$$v = \partial u / \partial t \doteq \Delta u / \Delta t$$

$$\boxed{u^t = v^{t-\frac{1}{2}\Delta t} \cdot \Delta t + u^{t-\Delta t}} \qquad (2)$$

Conservation of Momentum:

Force – displacement relation: $f_x = k u$ (3)

Newton's Second law: $f_x = m a$ (4)

$\Sigma f_x = 0$ (5)

(3) & (4) into (5)

$$m a^t + k u^t = 0$$

$$\boxed{a^t = -\frac{k u^t}{m}} \qquad (6)$$

Solve: Equations (1), (2) and (6) are a complete set required:

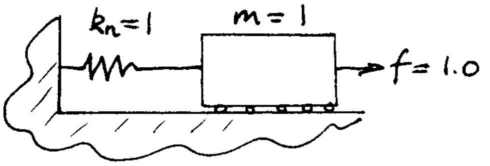
3 unknowns: u; $\dot{u}$; $\ddot{u}$ $\therefore$ 3 equations.

Simplify by writing (1) and (2) relative to next time step and previous time step.

(7) $$v^t = a^{t-\Delta t} \cdot \Delta t + v^{t-\Delta t}$$

(8) $$u^t = v^t \cdot \Delta t + u^{t-\Delta t}$$

(9) $$a^t = -\frac{ku^t}{m}$$



$\Delta t = 0.1$
$f = 1.0$ @ $t = 0^+$ and then released.

Since @ rest initially. – from (3) $f = ku$; $u^0 = 1.0$
from (6) $a = -ku/m$; $a^0 = -1.0$

Time, $t = 0.1$

(7) $v = -1.0 \times 0.1 + 0 = -0.1$

(8) $u = (-0.1 \times 0.1) + 1.0 = 0.99$

(9) $a = -1.0 \times 0.99 = -0.99$

Time, $t = 0.2$

(7) $v = (-0.99 \times 0.1) + (-0.1) = -0.199$

(8) $u = (-0.199 \times 0.1) + (0.99) = 0.970$

(9) $a = -1.0(0.970 = -0.970$

Time, $t = 0.3$ etc.

PROCEDURE:

1. Apply boundary conditions to blocks. (including self weight

2. Use Newton's 2nd Law $\left. \begin{aligned} f &= m\ddot{u} \\ T &= I\ddot{\theta} \end{aligned} \right\}$ known, m & I $\rightarrow \left\{ \begin{aligned} &\ddot{u} \\ &\ddot{\theta} \end{aligned} \right.$

3. Over time step, Δt use accelerations to obtain:

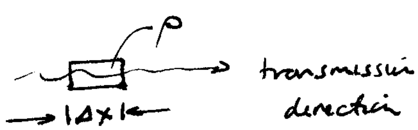
linear velocities: $V = V_0 + a\,\Delta t$

linear displacements: $u = u_0 + V\,\Delta t$

rotational velocities: $\dot{\theta} = \dot{\theta}_0 + \ddot{\theta}\,\Delta t$

displacements: $\theta = \theta_0 + \dot{\theta}\,\Delta t$

Note, for stability $\Delta t \ll \Delta x \sqrt{\dfrac{\rho}{E}}$



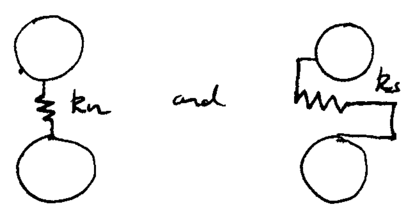
$E = \dfrac{\sigma}{\varepsilon}$ etc.

gives $\Delta t \ll \sqrt{m/k}$

4. Update forces

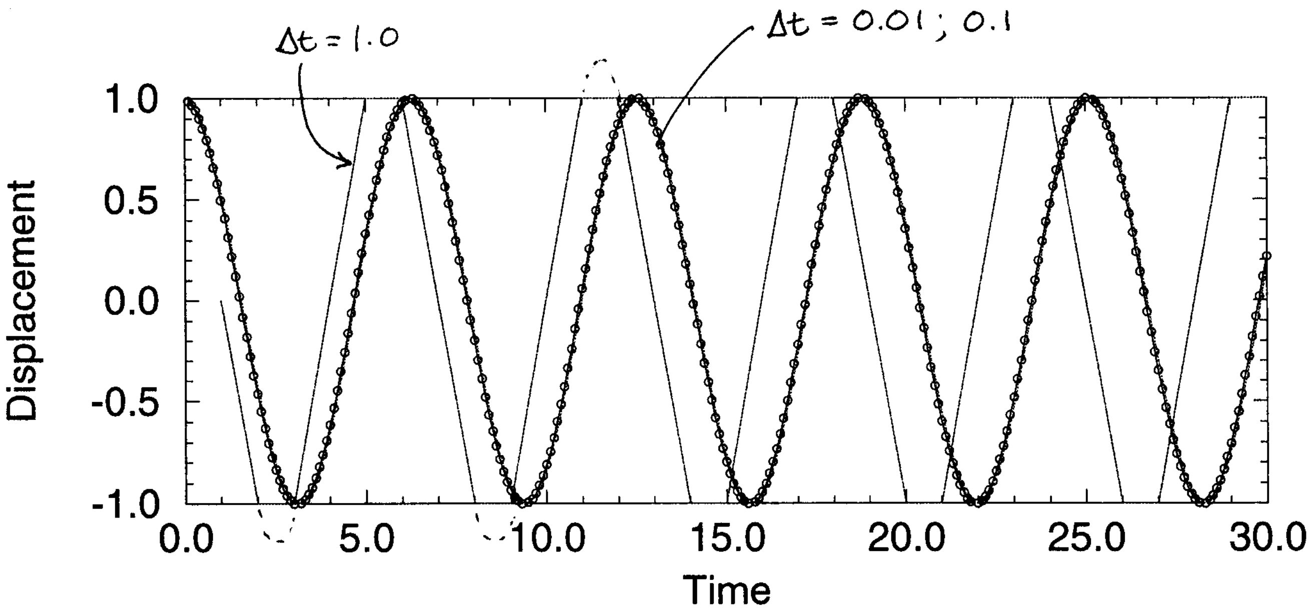
$f_n = k_n\, u_n$

$f_s = k_s\, u_s$

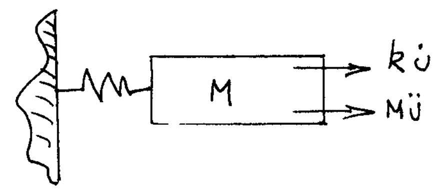


5. Σf to define $f^{t+\Delta t}$ and return to 2 to define accelerations

$\Delta t \ll \sqrt{M/k}$ i.e. $\Delta t \ll \sqrt{1}$



<u>Analytical Solution: Undamped System</u>



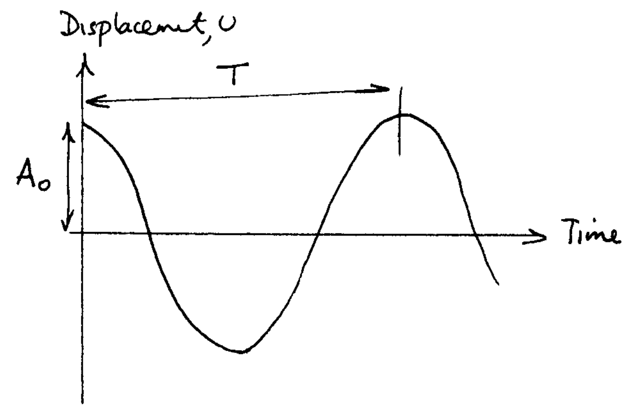
$$M \frac{d^2 u}{dt^2} + k u = 0$$

Solution: $u = A_0 \cos \omega_0 t$

$$\omega_0 = \sqrt{\frac{k}{M}}$$

$$T = \frac{2\pi}{\omega_0}$$

$$f = \frac{1}{T}$$



In this case $\omega_0 = \sqrt{\frac{1}{1}} = 1$

$\therefore T = 2\pi$ (secs).

QED.

& $A_0 = 1$