3. Elementary Fluid Mechanics [4-5]

$$\frac{dp}{ds}+\frac{1}{2}\rho\frac{d(V^2)}{ds}+\gamma\frac{dz}{ds}=0 \text{ (along streamline)}$$

$$\int\frac{dp}{\rho}+\frac{1}{2}V^2+gz=\text{constant (along streamline)}$$

$$\gamma\frac{dz}{dn}+\frac{\partial p}{\partial n}+\frac{\rho V^2}{R}=0 \text{ (normal to streamline)}$$

$$p+\rho\int\frac{V^2}{R}dn+\gamma z=\text{constant (normal to streamline)}$$

$$V=\sqrt{2gh} \text{ Free jets.}$$

$$A_1V_1=A_2V_2 \text{ Conservation of mass.}$$

$$Q=z_2b\sqrt{\frac{2g(z_1-z_2)}{1-(z_2/z_1)^2}} \text{ Sluice.} \quad Q=C_1b\sqrt{2g}h^{\frac{3}{2}} \text{ Sharp crested weir.}$$

4. Reynolds' Transport Theorem [6]

$$\text{Material derivative: } \frac{D()}{Dt}=\frac{\partial()}{\partial t}+(\mathbf{V}\cdot\nabla)()$$

$$\mathbf{V}\cdot\nabla()=u\frac{\partial()}{\partial x}+v\frac{\partial()}{\partial y}+w\frac{\partial()}{\partial z}$$

$$\text{Streamline acceleration: } \mathbf{a}=V\frac{\partial V}{\partial s}\hat{s}+\frac{V^2}{R}\hat{n}$$

$$\text{Transport Theorem: } \frac{DB_{sys}}{Dt}=\frac{\partial}{\partial t}\int_{cv}\rho b\,d\forall+\int_{cs}\rho b\mathbf{V}\cdot\hat{n}dA \text{ for } b=\frac{B}{m}$$

5. Conservation Laws [7,8]

$$\text{Relative velocities: } \mathbf{V}=\mathbf{W}+\mathbf{V_{cs}}$$

$$\text{Mass (continuity): } b=1 \text{ and } \frac{D}{Dt}M_{sys}=\frac{D}{Dt}\int_{sys}\rho\,d\forall=\frac{\partial}{\partial t}\int_{cv}\rho\,d\forall+\int_{cs}\rho\mathbf{W}\cdot\hat{n}dA=0$$

Linear Momentum:

$$\text{Static: } b=\mathbf{V} \text{ and } \frac{D}{Dt}F_{sys}=\frac{D}{Dt}\int_{sys}\mathbf{V}\rho\,d\forall=\frac{\partial}{\partial t}\int_{cv}\mathbf{V}\rho\,d\forall+\int_{cs}\mathbf{V}\rho\mathbf{V}\cdot\hat{n}dA=\sum\mathbf{F}$$

Moving and steady: $\int_{cs} (\mathbf{W} + \mathbf{V_{cs}}) \rho \mathbf{W} \cdot \hat{n} dA = \sum \mathbf{F}$

Moment-of-Momentum:

Steady: $b = (\mathbf{r} \times \mathbf{V})$ and $\int_{cs} (\mathbf{r} \times \mathbf{V}) \rho \mathbf{V} \cdot \hat{n} dA = \sum (\mathbf{r} \times \mathbf{F})$

$$T_{shaft} = \pm r V_\theta \dot{m}; \quad \dot{W}_{shaft} = T_{shaft} \omega; \quad w_{shaft} = \frac{\dot{W}_{shaft}}{\dot{m}}$$

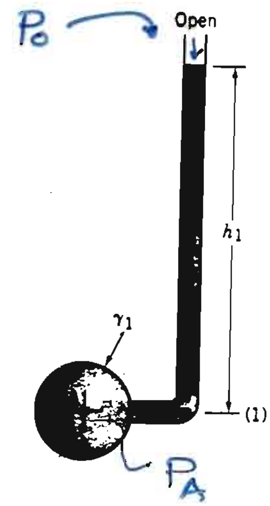
MANOMETRY – Static columns of liquids to measure pressures (gage).

PIEZOMETER

$$p = \gamma h + \cancel{p_o} \quad \text{gage} \equiv 0$$

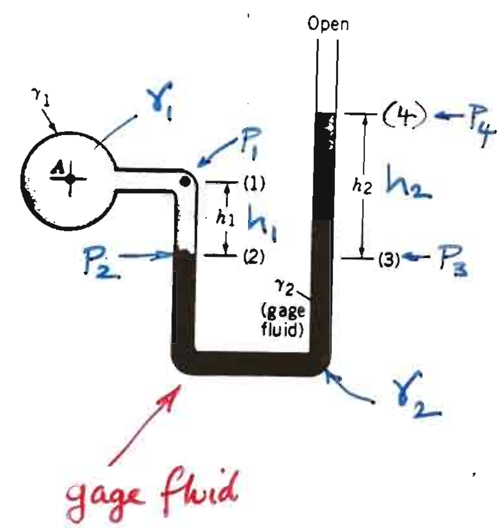
$$\boxed{p_A = \gamma h_1}$$

- Cannot measure tension/suction
- Limited to low pressures (groundwater)
- Container fluid cannot be gas



U-TUBE MANOMETER:

$$p = \gamma h + p_o$$



AT POINT (1)

$$p_A = p_1 = -\gamma_1 h_1 + p_2 \quad (\mathrm{I})$$

AT POINT (2)

$$p_2 = p_3 = +\gamma_2 h_2 + \cancel{p_4}^{\,0} \quad (\mathrm{II})$$

Combining equations (I) and (II)

$$\boxed{p_1 = -\gamma_1 h_1 + \gamma_2 h_2} \equiv p_A$$

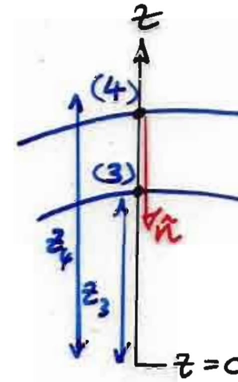
- Gage fluid may be different from measured fluid/gas

If fluid (1) is gas then $\gamma_1 \ll \gamma_2$ and $\boxed{p_1 = \gamma_2 h_2}$

Choose gage fluid for γ_2 → Sensitivity/readability.

Writing Bernoulli $\perp$ to streamline.

1. $\bar{n}$ direction always points to center of revolution.
$\therefore$ in this case $dn = -dz$



$$p_4 + \rho \int_{z_0}^{z_4} \frac{V^2}{R}\, dn + \gamma z_4 = p_3 + \rho \int_{z_0}^{z_3} \frac{V^2}{R}\, dn + \gamma z_3$$

$$\rho \int_{z_0}^{z_4} \frac{V^2}{R}(-dz) - \rho \int_{z_0}^{z_3} \frac{V}{R}(-dz) = \rho \int_{z_3}^{z_4} \frac{V^2}{R}(-dz)$$

Resubstitute:

$$p_4 - \rho \int_{z_3}^{z_4} \frac{V^2}{R}\, dz + \gamma z_4 = p_3 + \gamma z_3$$

Boundary conditions: $p_4 = 0$ $\qquad z_4 - z_3 = h_{3-4}$

$$p_3 = \gamma h_{3-4} - \rho \int_{z_3}^{z_4} \frac{V^2}{R}\, dz$$

Integral (+ve) $\therefore$ reduced pressure over hydrostatic.

$$p_3 = \gamma h_{3-4} + \rho \int_{z_3}^{z_4} \frac{V^2}{R}\, dz$$ since $dn = +dz$

PHYSICAL INTERPRETATION

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{const.} \qquad \text{along streamline}$$

$$p + \rho \int \frac{V^2}{R}\, dn + \gamma z = \text{const.} \qquad \text{across streamline}$$

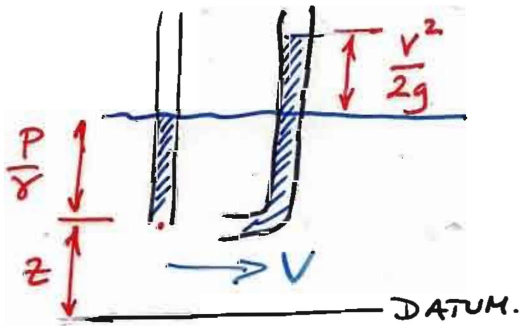
Requirements: Incompressible (liquids)
Inviscid (not porous media/pipes)
Steady

Each term represents "force" needed to provide an acceleration of a fluid particle.

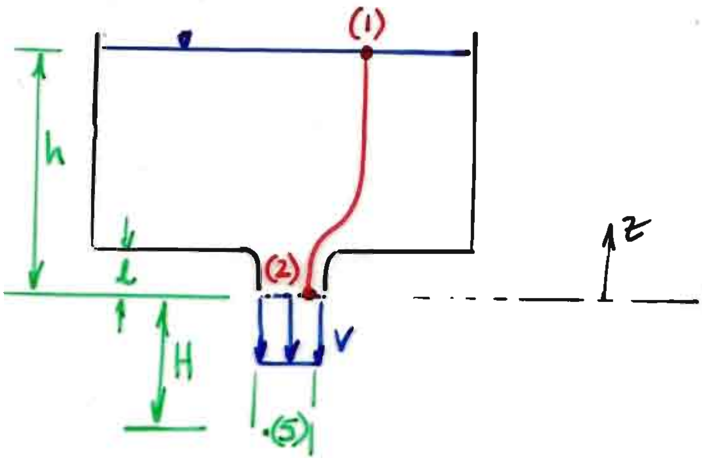
i.e Forces due to: pressure, p
body force/gravity, γz
kinetic energy, $\frac{1}{2}\rho V^2$; $\rho \int \frac{V^2}{R} dn$

In terms of heads: $$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{const.}$$

$\frac{V^2}{2g} \equiv$ vertical distance for freefalling body to reach velocity, V.



FREE JETS



Meets Bernoulli Requirements?

1. Tank is 'large' compared to jet outflow ∴ steady
2. Fluid is water ∴ incompressible
3. Inviscid — V is large compared to τ and ∴ τ.

(1) & (2) on streamline ∴ $$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

Point (1) $\quad p_1 = 0 \; ; \; V_1 = 0 \quad$ and $\quad \gamma z_1 = \gamma h$

Point (2) $\quad p_2 = 0 \; ; \; V_2 \neq 0 \quad$ and $\quad \gamma z_2 = 0$

$$\therefore \quad \gamma h = \tfrac{1}{2}\rho V_2^2$$

Free jet, implies $p_2 = 0$.

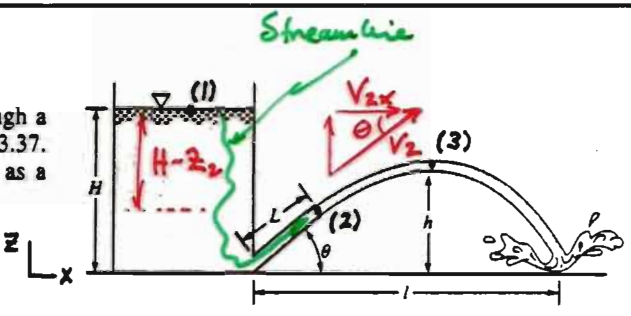
$$V = \sqrt{\frac{2\gamma h}{\rho}} = \sqrt{2gh}$$

If $p = 0$ @ point (2) and $p = 0$ @ point (5) then fluid falls as a "free" jet. With zero pressure throughout (along streamline).

$$p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2 = p_5 + \tfrac{1}{2}\rho V_5^2 + \gamma z_5$$

3.37

3.37 Water flows from a large tank of depth H, through a pipe of length L, and strikes the ground as shown in Fig. P3.37. Viscous effects are negligible. Determine the distance h as a function of θ.



■ FIGURE P3.37

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2\,,$$ where $p_1 = p_2 = 0$, $z_1 = H$, $z_2 = L\sin\theta$, and $V_1 = 0$

Hence, $H=\frac{V_2^2}{2g}+L\sin\theta$

or $\quad V_2=\sqrt{2g(H-L\sin\theta)}$ = Same as free jet $(H-z_2)$ (1)

Also since from (2) to (3) the only acceleration the particle feels is that of gravity, it follows that $a_x = 0$. Thus, $V_3 = V_{2x} = V_2\cos\theta$ (2)

From the Bernoulli equation between (1) and (3),

$$\frac{p_3}{\gamma}+\frac{V_3^2}{2g}+z_3=\cancelto{0}{\frac{p_1}{\gamma}}+\cancelto{0}{\frac{V_1^2}{2g}}+\cancelto{H}{z_1}\,,$$ where $p_1 = p_3 = 0$, $V_1 = 0$, $z_1 = H$, and $z_3 = h$

or

$$H=\frac{V_3^2}{2g}+h$$

By using Eqs. (1) and (2) this gives

$$H=\frac{V_2^2\cos^2\theta}{2g}+h=\frac{\overbrace{2g(H-L\sin\theta)}^{V_2}\cos^2\theta}{2g}+h$$

Thus,

$$h=H(1-\cos^2\theta)+L\sin\theta\cos^2\theta$$

or since $1-\cos^2\theta=\sin^2\theta$, $\quad \underline{h = H\sin^2\theta + L\sin\theta\cos^2\theta}$

Note: 1) If $\theta = 0$, then $h = 0$

2) If $\theta = 90°$, then $h = H$

3) If $L\sin\theta > H$, then the above is not valid since $V_2 = \sqrt{\text{negative number}}$ (see Eq. 1), which is not possible. Why is this so?

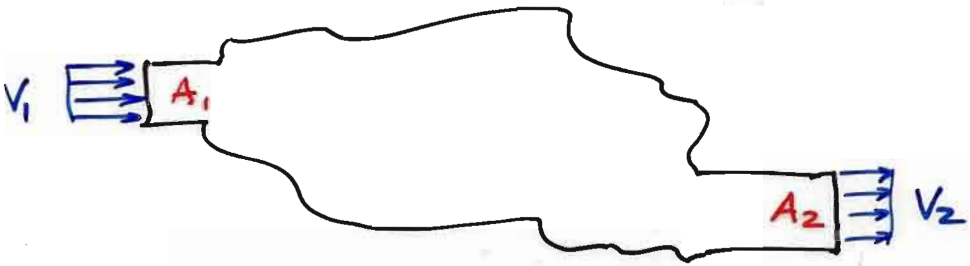
CONFINED FLOWS

- Sometimes extra "constraint" is needed to give an extra "equation" to the Bernoulli expression.

- Typically apply "conservation of mass"

  Mass flowrate in = Mass flowrate out

  If constant density $\longrightarrow$ Volume in = volume out.



$$\rho_1 A_1 V_1 \cancel{dt} = \rho_2 A_2 V_2 \cancel{dt}$$

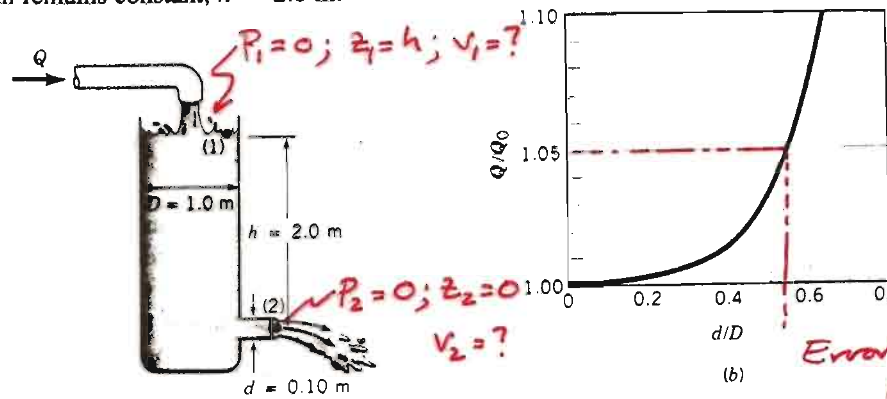
$$Q_1 = Q_2$$

EXAMPLE 3.7

A stream of water of diameter $d = 0.1$ m flows steadily from a tank of diameter $D = 1.0$ m as shown in Fig. E3.7*a*. Determine the flowrate, Q, needed from the inflow pipe if the water depth remains constant, $h = 2.0$ m.

FLOW INTO TOP OF TANK $\therefore V_1 \neq 0$

Need to apply continuity, $\dot{m}_{in} = \dot{m}_{out}$



(*a*) (*b*)

Error < 5% if $d/D < .52$

SOLUTION

For steady, inviscid, incompressible flow the Bernoulli equation applied between points (1) and (2) is

$$\cancel{p_1}^{\,0} + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = \cancel{p_2}^{\,0} + \tfrac{1}{2}\rho V_2^2 + \cancel{\gamma z_2}^{\,0} \tag{1}$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2 \tag{2}$$

Although the water level remains constant (h = constant), there is an average velocity, V_1, across section (1) because of the flow from the tank. From Eq. 3.19 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\boxed{\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2}$$

CONTINUITY.

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2 \tag{3}$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1 - (d/D)^4}}$$

Thus, with the given data

$$V_2 = \sqrt{\frac{2(9.81\ \text{m/s}^2)(2.0\ \text{m})}{1 - (0.1\text{m}/1\text{m})^4}} = 6.26\ \text{m/s}$$

and

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1\ \text{m})^2(6.26\ \text{m/s}) = 0.0492\ \text{m}^3/\text{s} \qquad \textbf{(Ans)}$$

If Q_0 = flow rate if V_1 set to zero.

Then Q/Q_0 is ratio of complex and simplified calculations.

In this example we have not neglected the kinetic energy of the water in the tank ($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D \gg d$), Eq. 3 indicates that $V_1 \ll V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error associated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq 0$, denoted Q, to that assuming $V_1 = 0$, denoted Q_0. This ratio, written as

$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{D=\infty}} = \frac{\sqrt{2gh/[1 - (d/D)^4]}}{\sqrt{2gh}} = \frac{1}{\sqrt{1 - (d/D)^4}}$$

is plotted in Fig. E3.7*b*. With $0 < d/D < 0.4$ it follows that $1 < Q/Q_0 \lesssim 1.01$, and the error in assuming $V_1 = 0$ is less than 1%. Thus, it is often reasonable to assume $V_1 = 0$.

3.53

3.53 Oil of specific gravity 0.83 flows in the pipe shown in Fig. P3.53. If viscous effects are neglected, what is the flowrate?

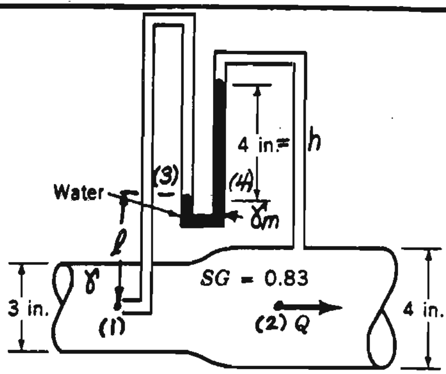


FIGURE P3.53

WRITE BERNOULLI @ 1 & 2

$$\frac{p_1}{\gamma}+\frac{\cancel{V_1^2}^{\,0}}{2g}+\cancel{z_1}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+\cancel{z_2} \quad \text{where } z_1=z_2 \text{ and } V_1=0$$

Thus,

$$\boxed{\frac{V_2^2}{2g}=\frac{p_1-p_2}{\gamma}} \qquad (1)$$

but,

$$p_1=p_3+\gamma\ell=p_4+\gamma\ell$$

MANOMETER EQUATION from (3) or (4).

and

$$p_2=\gamma(\ell+h)-\gamma_m h+p_4$$

MANOMETER EQUATION from (4)

Thus,

$$\boxed{p_1-p_2=(\gamma_m-\gamma)h} \qquad (2)$$

Combine Eqs. (1) and (2) to obtain

$$V_2=\sqrt{2g\left(\frac{p_1-p_2}{\gamma}\right)}=\sqrt{2g\left(\frac{\gamma_m}{\gamma}-1\right)h}=\sqrt{2\left(32.2\,\frac{ft}{s^2}\right)\left(\frac{62.4\,\frac{lb}{ft^3}}{0.83\left(62.4\,\frac{lb}{ft^3}\right)}-1\right)\left(\frac{4}{12}\,ft\right)}$$

or

$$V_2=2.10\,\frac{ft}{s}$$

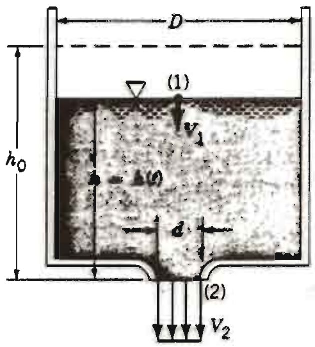
Thus,

$$Q=A_2V_2=\frac{\pi}{4}\left(\frac{4}{12}\,ft\right)^2\left(2.10\,\frac{ft}{s}\right)=\underline{\underline{0.183\,\frac{ft^3}{s}}}$$

UNSTEADY FLOWS (Contd)

EXAMPLE 3.18

A stream of liquid of diameter d drains from a circular tank of diameter D as is shown in Fig. E3.18. The depth of the water was h_0 at time $t = 0$. Determine the water depth as a function of time, $h = h(t)$.



■ FIGURE E3.18

SOLUTION

Clearly this is an unsteady flow—the deeper the water, the faster it flows from the tank. However, if the hole in the tank is not too big, the water will drain slowly, and the unsteady effect, $\partial V/\partial t$, at any point in the flow will be smaller than the steady effect, $V\,\partial V/\partial s$. Under these conditions it is reasonable to consider the flow as "quasisteady" and to apply the steady Bernoulli equation as follows.

As was shown in Example 3.7, the velocity of the water leaving the tank can be written as

$$\boxed{V_2 = \sqrt{\frac{2gh}{1 - (d/D)^4}}}$$

Flowrate assuming $V_1 \neq 0$.

Hence, by equating the flowrate from the tank, $V_2 A_2$, and the rate at which the amount of water in the tank changes with time, $-(dh/dt)A_1$, we obtain

Apply continuity again.

$$\frac{dh}{dt} = -V_1$$

$$V_1 A_1 = V_2 A_2$$

$$\text{or} \rightarrow \quad V_1 = \frac{V_2 A_2}{A_1}$$

$$\therefore \quad \frac{dh}{dt} = -\frac{V_2 A_2}{A_1}$$

$$\frac{dh}{dt} = -\frac{V_2 A_2}{A_1} = -\left(\frac{d}{D}\right)^2 \sqrt{\frac{2gh}{1 - (d/D)^4}} \tag{1}$$

This result can be integrated from the initial time and depth, $t = 0$ when $h = h_0$, to an arbitrary time and depth as follows.

$$\int_{h_0}^{h} \frac{1}{\sqrt{h}}\,dh = -\left(\frac{d}{D}\right)^2 \sqrt{\frac{2g}{1 - (d/D)^4}} \int_0^t dt$$

or

$$2(\sqrt{h} - \sqrt{h_0}) = -\left(\frac{d}{D}\right)^2 \sqrt{\frac{2g}{1 - (d/D)^4}}\; t$$

This can be arranged into the form

Correlates OK with expt.

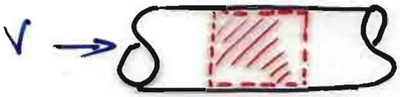
Also $d \ll D$

$h/h_0 \rightarrow 1$

$$\frac{h}{h_0} = \left[1 - \frac{t\sqrt{g/2h_0}}{\sqrt{(D/d)^4 - 1}}\right]^2 \qquad \textbf{(2)} \quad \textbf{(Ans)}$$

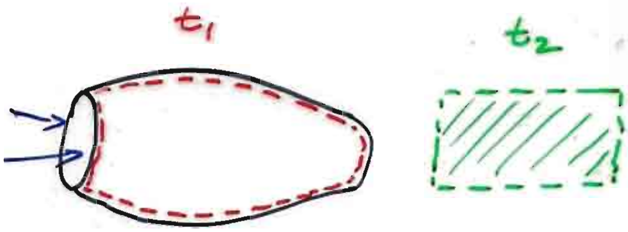
The results of Eq. 2 correlate quite well with experiments, provided d/D is not too large, even though we have used a steady flow analysis for an unsteady flow. This is another way of saying $\partial V/\partial t \ll V\,\partial V/\partial s$. For larger values of d/D the unsteady Bernoulli equation gives a nonlinear, second-order differential equation that, unlike Eq. 1, is not easy to integrate.

CONTROL VOLUMES



FIXED

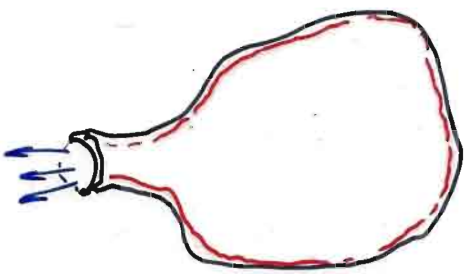
- Static
- Constant volume
- Extensive quantity (m, mv etc.) tracked through the volume.



FIXED w.r.t aircraft

MOVING w.r.t. ground

- Static or moving depends on ref. frame.
- Constant volume (does not deform).

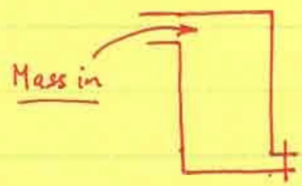


DEFORMING

- Static or moving in space
- Control volume changes with time.

Reynold's Transport Theorem provides a means of unifying these concepts in a single form.

For Simplified System



Mass in; $\Delta \forall = 0$

$\therefore$ $\Delta \rho$ with time

Assume V is constant on area segments $\perp$

Assume ρ is uniform in space!

~~Control volume~~

$$\frac{\partial}{\partial t} \int_{cv} \rho \, d\forall \; + \; \int_{cs} \rho \, \underline{V} \cdot \underline{n} \, dA = 0$$

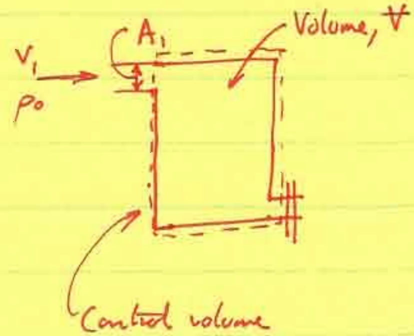
$\underline{V} \cdot \bar{n} = V n \cos \theta$

$$\frac{\partial}{\partial t}(\rho \forall) \; + \; (\pm) \rho V A = 0$$

+ve out
−ve in

$$\forall \frac{\partial \rho}{\partial t} \; + \; \rho \frac{\partial \forall}{\partial t} \; + \; \Sigma \dot{m} \; = \; 0$$

Eg 1: Compressor pumps water into closed rigid vessel at constant velocity, V_1 and area, A.



$$\forall \frac{d\rho}{dt} + \rho \cancel{\frac{\partial \forall}{\partial t}}^{\,0} + \Sigma \dot{m} = 0$$

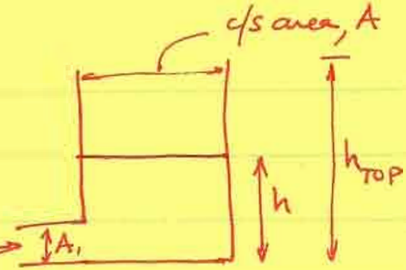
$$\forall \frac{d\rho}{dt} - \rho_0 V_1 A_1 = 0$$

$$\forall \int_{\rho_0}^{\rho_f} d\rho = \rho_0 V_1 A \int_0^t dt$$

$$\forall (\rho_f - \rho_0) = \rho_0 V_1 A \, t$$

Solve for ρ_f.

Eg 2: Water flows into tank of constant cross sectional area, A, at flowrate $V_1 A_1$



$$\forall \cancel{\frac{d\rho}{dt}} + \rho \frac{d\forall}{dt} + \Sigma \dot{m} = 0$$

$$\rho_0 \frac{d\forall}{dt} - \rho_0 V_1 A_1 = 0$$

$$d\forall = V_1 A_1 \, dt$$

and $\forall = Ah$; $d\forall = A\,dh$

$$A \int_0^{h_{TOP}} dh = V_1 A_1 \int_0^{t} dt$$

$$A\, h_{TOP} = V_1 A_1 \, t$$

Recap

Reynolds' transport theorem $\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\forall + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} dA \text{ for } b = \frac{B}{m}$

Outline

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho W \cdot n \, dA = 0$$

$$\frac{\partial}{\partial t}(\rho \forall) + \sum \rho W A = 0$$

	Vcs	dVol/dt	
Static – Non-deforming	0	0	Vstatic=W+Vcs W=Vstatic-Vcs
Moving – Non-deforming	Vcv	0	
Moving - Deforming	Vcs	Not 0	

CONSERVATION OF LINEAR MOMENTUM

ALTERNATIVE APPROACH (REPRESENTATION).

In any of the coordinate directions, say x:

$$\frac{\partial}{\partial t}\int_{cv} u\rho \, d\forall + \int_{cs} u\rho \, \underline{V}\cdot\hat{n}\, dA = \Sigma F_x$$

$$\dot{m} = \rho \int_{cs} \underline{V}\cdot\hat{n}\, dA$$

$$\dot{m} = \rho V A$$

$$\boxed{u\dot{m} = \Sigma F_x}$$

mass rate of flow $\Rightarrow$ $\dot{m}$ $\begin{cases} \text{+ve for outflow from body} \\ \text{-ve for inflow into body} \end{cases}$

flow velocity $\Rightarrow$ u positive/negative in coordinate directions.

Note; $\underline{V}\cdot\hat{n} = V n \cos\theta$

5.34

5.34 Determine the magnitude and direction of the x and y components of the anchoring force required to hold in place the horizontal 180° elbow and nozzle combination shown in Fig. P5.34. Also determine the magnitude and direction of the x and y components of the reaction force exerted by the 180° elbow and nozzle on the flowing water.

$$\left.\begin{array}{l} V\cdot n = U_2 \\ V V\cdot n = -U_2^2 \end{array}\right\}$$

$$\left.\begin{array}{l} V\cdot n = -U_1 \\ V V\cdot n = -U_1^2 \end{array}\right\}$$

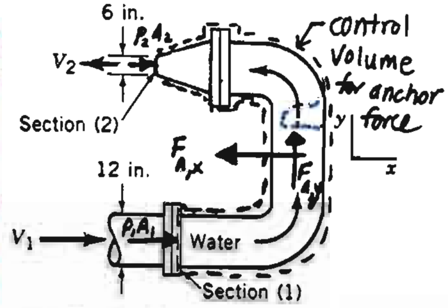


FIGURE P5.34 $p_1 = 15$ psi, $V_1 = 5$ ft/s

$p_2 = 0 =$ Atmospheric!!

For determining the x and y direction components of the anchoring force a control volume that contains the elbow, nozzle and water between sections (1) and (2) is used. Th control volume and the forces involved are shown in the sketch above. Application of the y direction component of the linear momentum equation (Eq. 5.22) leads to

$$F_{A,y} = \underline{\underline{0}}$$

$\rho \int_{cs} V(V\cdot n)\,dA$

Application of the x direction component of the linear momentum equation yields

CONS of MOMENTUM

$$\boxed{-u_1 \rho u_1 A_1 - u_2 \rho u_2 A_2 = p_1 A_1 - F_{A,x} + p_2 A_2} \quad (1)$$

From the conservation of mass equation

CONS of MASS

$$\boxed{\dot{m} = \rho u_1 A_1 = \rho u_2 A_2} \quad \therefore \text{ use to relate } U_1 \text{ to } U_2 \quad (2)$$

Thus Eq. 1 may be expressed as

$$-\rho u_1 A_1 (u_1 + u_2) = p_1 A_1 - F_{A,x} + p_2 A_2$$

and

$$F_{A,x} = \rho u_1 A_1 (u_1 + u_2) + p_1 A_1 + p_2 A_2 = \rho u_1 \frac{\pi D_1^2}{4}(u_1 + u_2) + p_1 \frac{\pi D_1^2}{4} + (0) A_2$$

Also from Eq. 2

$$u_2 = \frac{A_1}{A_2} u_1 = \frac{D_1^2}{D_2^2} u_1$$

Thus

$$F_{A,x} = \rho u_1 \frac{\pi D_1^2}{4}\left(u_1 + \frac{D_1^2}{D_2^2} u_1\right) + p_1 \frac{\pi D_1^2}{4}$$

(con't)

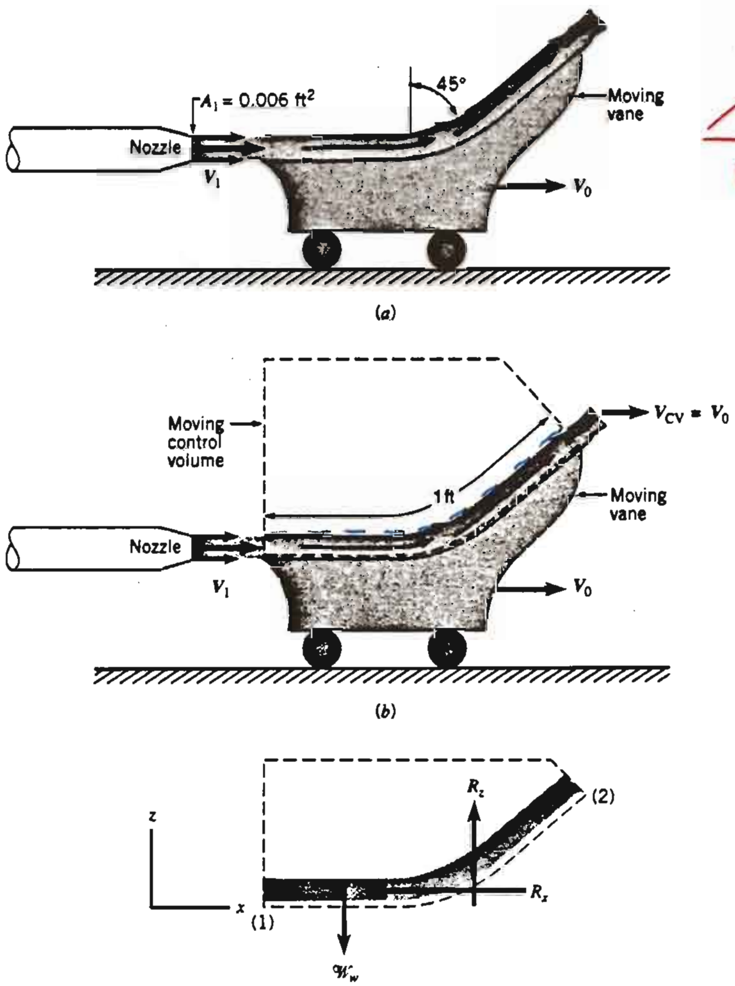
EXAMPLE 5.16

A vane on wheels moves with constant velocity $\mathbf{V}_0$ when a stream of water having a nozzle exit velocity of $\mathbf{V}_1$ is turned 45° by the vane as indicated in Fig. E5.16*a*. Note that this is the same moving vane considered in Section 4.4.6 earlier. Determine the magnitude and direction of the force, **F**, exerted by the stream of water on the vane surface. The speed of the water jet leaving the nozzle is 100 ft/s and the vane is moving to the right with a constant speed of 20 ft/s.

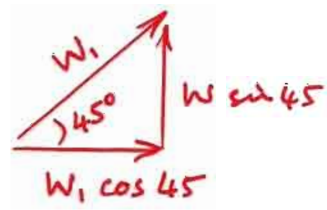
Relative velocities :

$\underline{V} = \underline{W} + \underline{V}_{cs}$

$\underline{W} = \underline{V} - \underline{V}_{cs}$



(*a*) (*b*) (*c*)



$$\boxed{\int_{cs} \mathbf{W}\rho\mathbf{W} \cdot \hat{\mathbf{n}}\, dA = \sum \mathbf{F}_{\text{contents of the control volume}}}$$

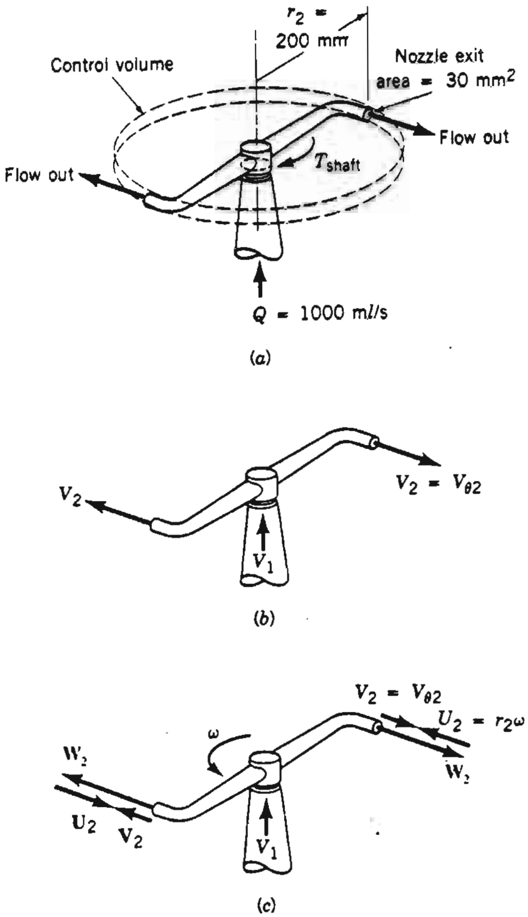
x direction:

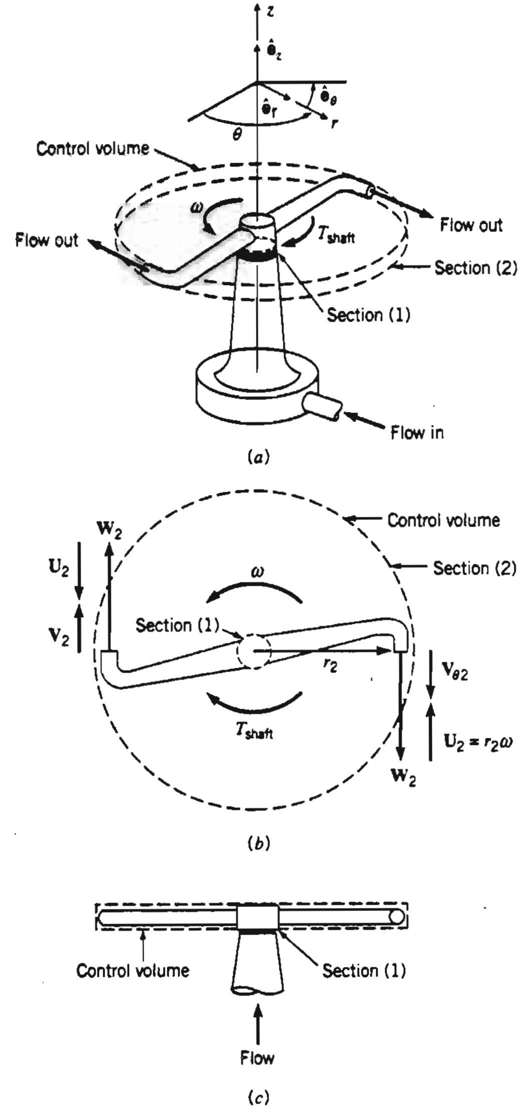
$$(W_1)\rho(-W_1)A_1 + (W_1 \cos 45)\rho(W_1)A_1 = -R_x$$

direction :

$$0\rho(-W_1)A_1 + (W_1 \sin 45)\rho(W_1)A_1 = R_z - W_w$$

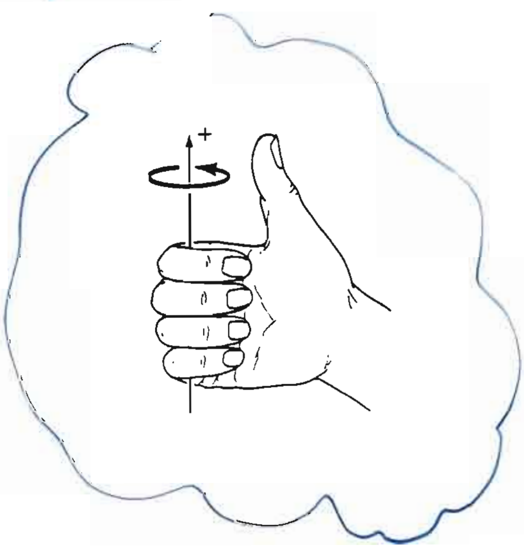
$$W_w = \rho g A_1 \ell$$





W_2 = relative velocity
V_2 = velocity relative to fixed frame of ref.
U_2 = velocity of control surface

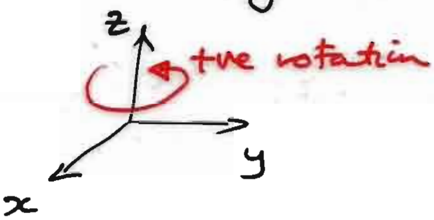
$$\boxed{V_2 = W_2 + U_2}$$



RIGHT-HAND RULE

PHYSICAL MEANING OF M-o-M EQUATION

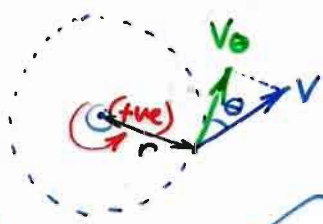
- Consider only rotation around a single axis. z.



- Steady behavior $\therefore \frac{\partial}{\partial t} \to 0$
- Fixed control surface.

$$\int_{cs} (\underline{r}\times\underline{V})\,\rho\,\underline{V}\cdot\hat{n}\,dA \;=\; \overbrace{\Sigma(\underline{r}\times\underline{F})}^{\text{Torque}}$$

$\underline{r}\times\underline{V} = r\,V_\theta$



Mass rate of flow $\dot{m} \approx VA\rho$ leaving the control volume (+ve) or entering (−ve).

Note also that V is relative to the static control volume, and as previous

$$\underline{V} = \underline{W} + \underline{U}$$

Relative to static = Relative to control volume + Velocity of control volume

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Rewriting

$$\boxed{T_{shaft} = r\,V_\theta\,\dot{m}}$$

Shaft power, $\dot{W}_{shaft} = T_{shaft}\,\omega$ $\qquad \omega$ = rotational speed.

Shaft work per unit mass, $w_{shaft} = \dfrac{T_{shaft}\,\omega}{\dot{m}}$

5.67

5.67 Five liters/s of water enters the rotor shown in Fig. P5.67 along the axis of rotation. The cross section area of each of the three nozzle exits normal to the relative velocity is 18 mm². How large is the resisting torque required to hold the rotor stationary? How fast will the rotor spin steadily if the resisting torque is reduced to zero and: (a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 60°$?

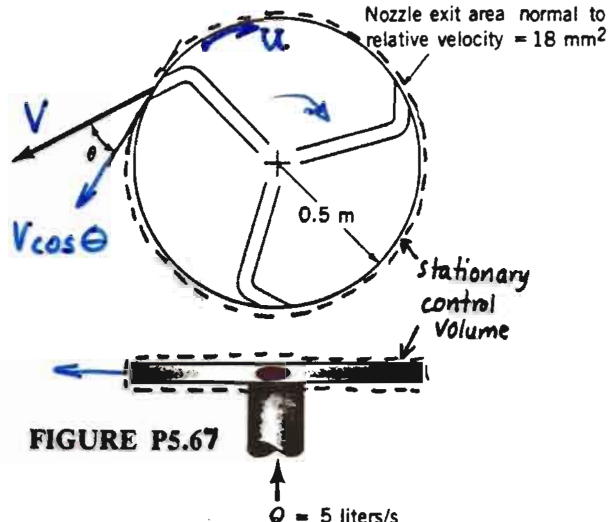

FIGURE P5.67

Hold the rotor static $V = W + \cancel{U}^{0}$

To determine the torque required to hold the rotor stationary we use the moment-of-momentum torque equation (Eq. 5.50) to obtain

$$T_{shaft} = \dot{m}\, r_{out} V_{out} \cos\theta \qquad (1)$$

We note that

$$\dot{m} = \rho Q \qquad (2)$$

and

$$V_{out} = \frac{Q}{3A_{\substack{nozzle\\exit}}} \qquad (3)$$

Combining Eqs. 1, 2 and 3 we get

$$T_{shaft} = \frac{\rho Q^2 r_{out} \cos\theta}{3\, A_{\substack{nozzle\\exit}}} \qquad (4)$$

Allow rotor to spin

To determine the rotor angular velocity associated with zero shaft torque we again use the moment-of-momentum torque equation (Eq. 5.50) to obtain, this time with rotation,

i.e. T=0

$$T_{shaft} = \dot{m}\, r_{out} (\underbrace{W_{out} \cos\theta}_{} - U_{out}) \qquad (5)$$

Relative flow velocity in direction of circumference.

We note that

$$U_{out} = r_{out}\, \omega \qquad (6)$$

and

$$W_{out} = \frac{Q}{3A_{\substack{nozzle\\exit}}} \qquad (7)$$

Solve for ω

(con't)

Summary: <u>MASS</u> $\frac{d}{dt}\int_{CV} \rho \, d\forall \quad + \quad \int_{CS} \rho \, \underline{W} \cdot \underline{n} \, dA \quad = 0$

Includes $\quad \underline{V} = \underline{W} + \underline{V}_{cs}$

$$\rho_0 \frac{d\forall}{dt} + \forall \frac{d\rho}{dt} + \dot{m} = 0$$

<u>LM</u> $\quad \int_{CS} \underline{W} \rho (\underline{W} \cdot \underline{n}) \, dA = \Sigma \underline{F}$

$$\underline{W} \dot{m} = \Sigma \underline{F}$$

<u>Ang Momentum</u> $\quad \int (\underline{r} \times \underline{V}) \rho \, \underline{V} \cdot \underline{n} \, dA = \Sigma (\underline{r} \times \underline{F})$

$$\pm r V_\theta \dot{m} = T_{shaft}$$