[1:1] Introduction

Outline

Syllabus
Presentations
Relevance

http://en.wikipedia.org/wiki/Supercritical_fluid

Assignment

Review past exams

EME 303 - FLUID MECHANICS

SUMMARIZED EQUATIONS AND CONCEPTS

Some Useful Conversion Factors

	SI	BGS	EE
Temperature:	$K = {}^oC + 273.15$	${}^oR = {}^oF + 459.67$	${}^oR = {}^oF + 459.67$
Force:	$1N = (1kg)(1m/s^2)$	$1lb = (1slug)(1ft/s^2)$	$1lbf = (1lbm)(32.2ft/s^2)$
Mass:	kg	$slug$	lbm
Density:	$1\,kg/m^3$	$0.00194\,slug/ft^3$	$0.06243\,lbm/ft^3$
ρ_{water}	$1000kg/m^3$	$1.94slugs/ft^3$	$62.4lbm/ft^3$
Pressure:	$1Pa = 1N/m^2$	$0.0209\,lb/ft^2$	$0.0209\,lbf/ft^2$
Work, energy:	$1J = 1N.m$	$1ft.lb = 778.2Btu$	—
Power:	$1W = 1N.m/s$	$1hp = 550ft.lb/s$	—

1. General [Topic:1]

$$\mathbf{F} = m\mathbf{a} \text{ or } \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = m \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

$$p = \rho RT \begin{cases} Isothermal: \dfrac{p}{\rho} = const. \\ Isentropic: \dfrac{p}{\rho^k} = const. \end{cases}$$

$$\tau = \mu \frac{du}{dy}; \quad \nu = \frac{\mu}{\rho}$$

$$E_v = -\frac{dp}{d\forall/\forall} = \frac{dp}{d\rho/\rho}$$

$$h = \frac{2\sigma\cos\theta}{\gamma R}$$

2. Fluid Statics [2,3]

$$p_x = p_y = p_z = p$$

$$-\nabla p - \gamma\hat{\mathbf{k}} = \rho\mathbf{a} \text{ or } -\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} p - \gamma \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \rho \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

Simplifies when: $a_x = a_y = a_z = 0$ to $\dfrac{dp}{dx} = \dfrac{dp}{dy} = 0$ and $\dfrac{dp}{dz} = -\gamma; \dfrac{dz}{dx} = -\dfrac{a_x}{g + a_z}$

Rigid body rotation: $\dfrac{\partial p}{\partial r} = \rho r\omega^2; \dfrac{\partial p}{\partial \theta} = 0; \dfrac{\partial p}{\partial z} = -\gamma; \begin{cases} z = \frac{\omega r^2}{2g} + c \\ p = \frac{p\omega^2 r^2}{2} - \gamma z + c \end{cases}$

Incompressible fluid: $p = \gamma h + p_0$

Compressible fluid: $\dfrac{dp}{dz} = -\dfrac{gp}{RT}$ and integrate w.r.t (p, z).

Manometer rules: $(\uparrow -ve)(\downarrow +ve); \dfrac{dp}{dx} = \dfrac{dp}{dy} = 0; p_v$ if evacuated; $\gamma_{gas} \to 0$.

$$F_R = \gamma h_c A; \quad F_R = \sqrt{F_H^2 + F_V^2}$$

F_R acts through center of pressure $\begin{cases} y_R = \dfrac{I_{xc}}{y_c A} + y_c \\ x_R = \dfrac{I_{xyc}}{y_c A} + x_c \end{cases}$

$$F_B = \gamma \cancel{V}$$

3. Elementary Fluid Mechanics [4-5]

$\dfrac{dp}{ds} + \dfrac{1}{2}\rho\dfrac{d(V^2)}{ds} + \gamma\dfrac{dz}{ds} = 0$ (along streamline)

$\displaystyle\int \dfrac{dp}{\rho} + \dfrac{1}{2}V^2 + gz =$ constant (along streamline)

$\gamma\dfrac{dz}{dn} + \dfrac{\partial p}{\partial n} + \dfrac{\rho V^2}{R} = 0$ (normal to streamline)

$p + \rho\displaystyle\int \dfrac{V^2}{R}dn + \gamma z =$ constant (normal to streamline)

$V = \sqrt{2gh}$ Free jets.

$A_1 V_1 = A_2 V_2$ Conservation of mass.

$Q = z_2 b\sqrt{\dfrac{2g(z_1 - z_2)}{1 - (z_2 / z_1)^2}}$ Sluice. $Q = C_1 b\sqrt{2g}h^{\frac{3}{2}}$ Sharp crested weir.

4. Reynolds' Transport Theorem [6]

Material derivative: $\dfrac{D()}{Dt} = \dfrac{\partial()}{\partial t} + (\mathbf{V}\cdot\nabla)()$

$$\mathbf{V}\cdot\nabla() = u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + w\frac{\partial()}{\partial z}$$

Streamline acceleration: $\mathbf{a} = V\dfrac{\partial V}{\partial s}\hat{s} + \dfrac{V^2}{R}\hat{n}$

Transport Theorem: $\dfrac{DB_{sys}}{Dt} = \dfrac{\partial}{\partial t}\displaystyle\int_{cv}\rho b d\not{V} + \int_{cs}\rho b\mathbf{V}\cdot\hat{n}dA$ for $b = \dfrac{B}{m}$

5. Conservation Laws [7,8]

Relative velocities: $\mathbf{V} = \mathbf{W} + \mathbf{V_{cs}}$

Mass (continuity): $b = 1$ and $\dfrac{D}{Dt}M_{sys} = \dfrac{D}{Dt}\displaystyle\int_{sys}\rho d\not{V} = \dfrac{\partial}{\partial t}\int_{cv}\rho d\not{V} + \int_{cs}\rho\mathbf{W}\cdot\hat{n}dA = 0$

Linear Momentum:

Static: $b = \mathbf{V}$ and $\dfrac{D}{Dt}F_{sys} = \dfrac{D}{Dt}\displaystyle\int_{sys}\mathbf{V}\rho d\not{V} = \dfrac{\partial}{\partial t}\int_{cv}\mathbf{V}\rho d\not{V} + \int_{cs}\mathbf{V}\rho\mathbf{V}\cdot\hat{n}dA = \sum\mathbf{F}$

Moving and steady: $\displaystyle\int_{cs}(\mathbf{W} + \mathbf{V_{cs}})\rho\mathbf{W}\cdot\hat{n}dA = \sum\mathbf{F}$

Moment-of-Momentum:

Steady: $b = (\mathbf{r}\times\mathbf{V})$ and $\displaystyle\int_{cs}(\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{n}dA = \sum(\mathbf{r}\times\mathbf{F})$

$$T_{shaft} = \pm rV_\theta\dot{m};\quad \dot{W}_{shaft} = T_{shaft}\omega;\quad w_{shaft} = \frac{\dot{W}_{shaft}}{\dot{m}}$$

First Law of Thermodynamics: $b = e$ and

$$\frac{\partial}{\partial t}\int_{cv}e\rho d\not{V} + \int_{cs}e\rho\mathbf{V}\cdot\hat{n}dA = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$\frac{p_{out}}{\gamma} + \frac{\alpha_o V_{out}^2}{2g} + z_{out} + h_L = \frac{p_{in}}{\gamma} + \frac{\alpha_i V_{in}^2}{2g} + z_{in} + h_P$$

$$\dot{m}[(\breve{h}_{out} - \breve{h}_{in}) + \tfrac{1}{2}(v_{out}^2 - v_{in}^2) + g(z_{out} - z_{in})] = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$h_P = \frac{w_{shaftin}}{g};\quad w_{shaftin} = \frac{\dot{W}_{shaftin}}{\dot{m}}$$

$\alpha = 1$ for uniform flow.

6. Differential Analysis of Fluid Flow [7,8]

$$\text{Euler's Equation: } \rho\mathbf{g} - \nabla p = [\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}]$$

$$\text{Parallel plate flow: } q = -\frac{(2b)^3}{12\mu}\frac{\partial p}{\partial x}; \quad \hat{U} = -\frac{(2b)^2}{12\mu}\frac{\partial p}{\partial x}$$

$$\text{Circular pipe flow: } q = -\pi\frac{(2R)^4}{128\mu}\frac{\partial p}{\partial x}; \quad \hat{U} = -\frac{(2R)^2}{32\mu}\frac{\partial p}{\partial x}$$

7. Dimensional Analysis [9]

$$\mathbf{Re} = \frac{\rho V l}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{gl}}; \quad \mathbf{Eu} = \frac{p}{\rho V^2}$$

8. Pipe Flow [10-11]

$$\tau_w = \frac{\rho V^2}{8} f; \quad h_L^{major} = f(\frac{l}{D})\frac{V^2}{2g}; \quad h_P = \frac{Power}{\gamma Q}$$

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 + h_P = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 + \sum h_L^{major} + \sum h_L^{minor}$$

$$h_L^{minor} = K_L\frac{V^2}{2g}; \quad l_{eq}^{minor} = \frac{K_L D}{f}; \quad K_L = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

$$\text{Non-circular: Laminar: } [f = \frac{C}{\mathbf{Re}_h}; D_h = \frac{4A}{P}] \qquad \text{Turbulent: [Use Moody; } f = \varphi(\frac{\varepsilon}{D_h})]$$

$$\text{Series: } h_L = h_{L_1} + h_{L_2} + \ldots + h_{L_n}; \qquad \text{Parallel: } h_{L_1} = h_{L_2} = \ldots = h_{L_n}$$

$$\text{Flow meters: } Q = CA\sqrt{\frac{2(p_1 - p_2)}{\rho(1-\beta^4)}}; \qquad \beta = \frac{D_2}{D_1}$$

9. External Flows [12]

$$D = \int dF_x = \int p\cos\theta dA + \int \tau_w \sin\theta dA$$

$$L = \int dF_y = -\int p\sin\theta dA + \int \tau_w \cos\theta dA$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}; \qquad C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

10. Open Channel Flow [13-14]

$$R_h = \frac{A}{P}; \quad \mathbf{Re} = \frac{V R_h \rho}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{gy}}; \quad c = \sqrt{gy}$$

Specific Energy: $E = y + \frac{q^2}{2gy^2}$; Specific Momentum: $M = \frac{y^2}{2} + \frac{q^2}{gy}$

Energy Equation: $y_1 + \frac{q_1^2}{2gy_1^2} + z_1 = y_2 + \frac{q_2^2}{2gy_2^2} + z_2 + S_f l \quad \rightarrow \quad E_1 = E_2 + (S_f - S_0) l$

$$E_{min} = \frac{3y_c}{2} \text{ at } \mathbf{Fr} = 1$$

$$\frac{dy}{dx} = \frac{S_f - S_0}{1 - \mathbf{Fr}^2}$$

Uniform Flow: $V = \frac{\kappa}{n} R_h^{\frac{2}{3}} S_0^{\frac{1}{2}} Q = \frac{\kappa}{n} A R_h^{\frac{2}{3}} S_0^{\frac{1}{2}} \kappa = 1(SI) \kappa = 1.49(BGS)$

Hydraulic Jump: $\frac{y_2}{y_1} = \frac{1}{2}(-1 + \sqrt{1 + 8Fr_1^2}) \; \frac{h_L}{y_1} = 1 - \frac{y_2}{y_1} + \frac{1}{2} Fr_1^2 [1 - (\frac{y_1}{y_2})^2]$

Sharp-Crested Weir: $Q = C_{rectangular} \frac{2}{3} \sqrt{2gh^3} b; Q = C_{triangular} \frac{8}{15} \tan \frac{\theta}{2} \sqrt{2g} H^{5/2}.$

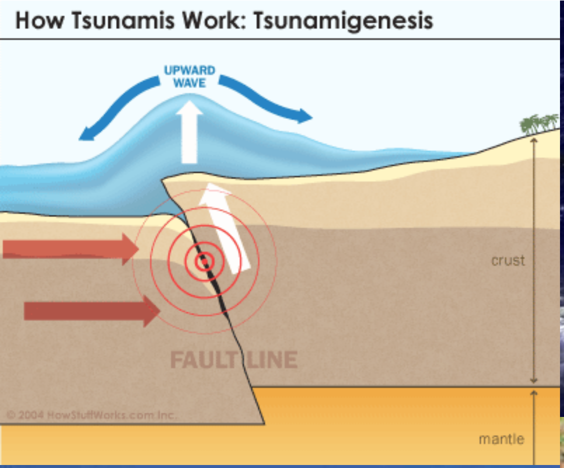
Broad-Crested Weir: $Q = C_{wb} b \sqrt{g} (\frac{2}{3})^{3/2} H^{3/2} : C_{wb} = \frac{0.65}{(1 + H / P_w)^{1/2}}.$

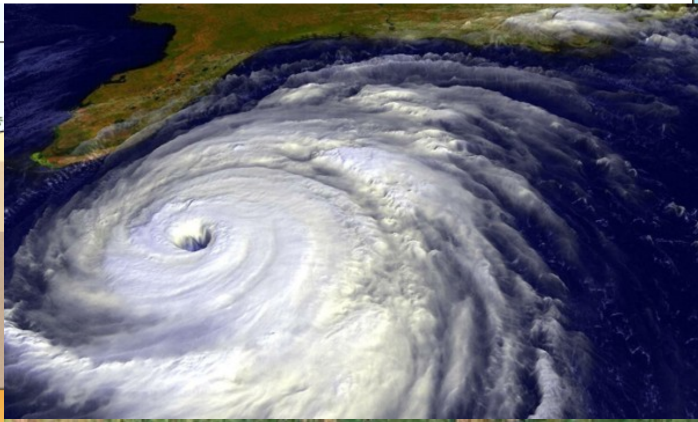
Underflow Gates: $q = C_d a \sqrt{2gy_1}$

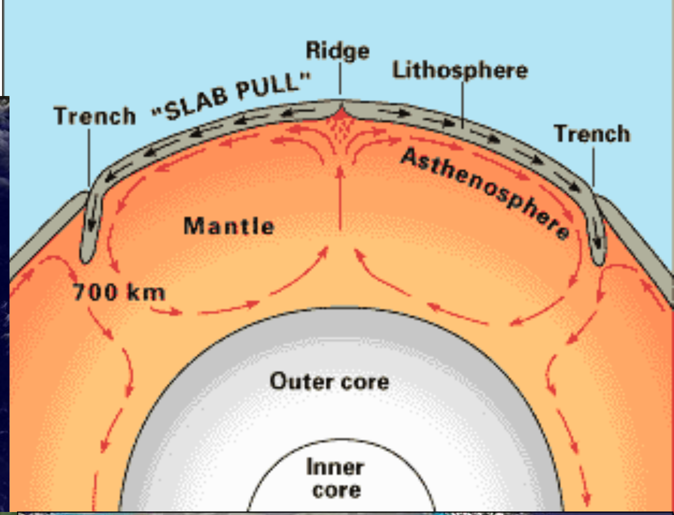
[1]

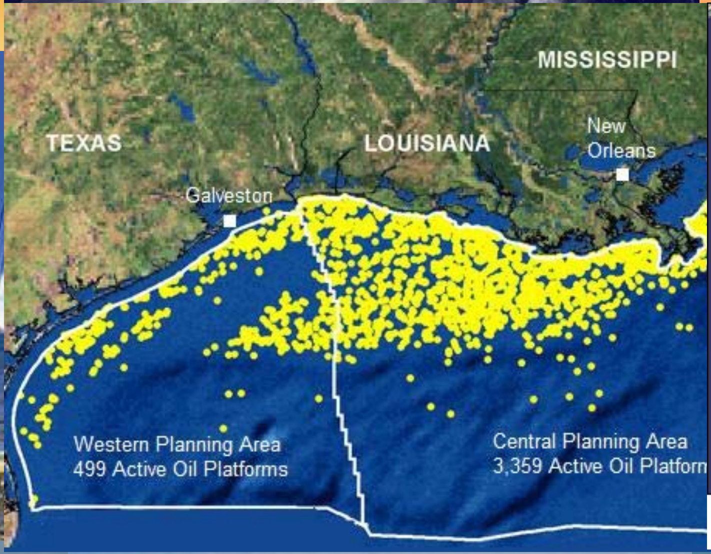
Fluid Properties

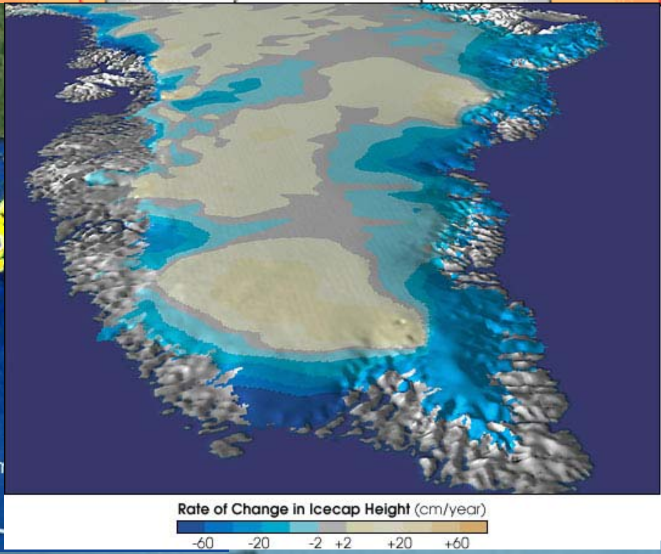
Fluid Mechanics in Natural Systems



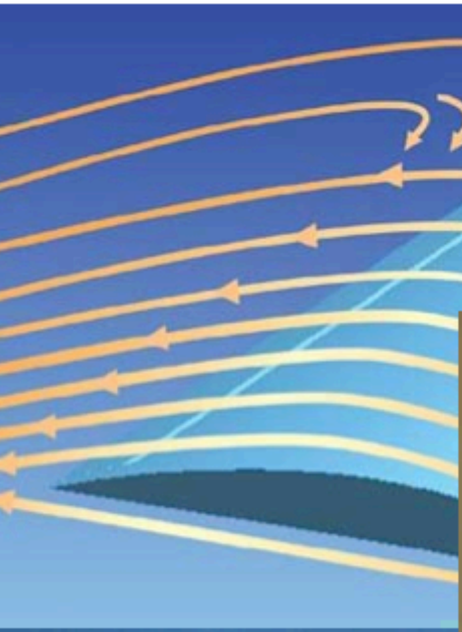


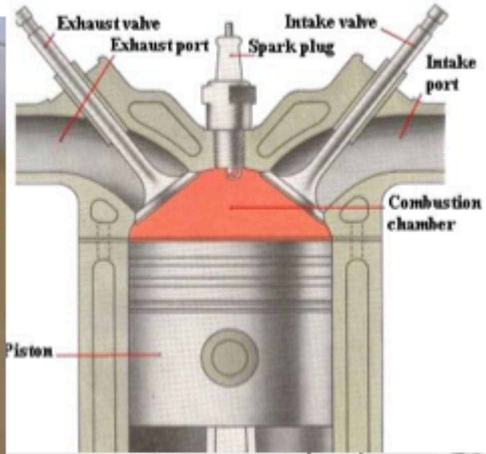


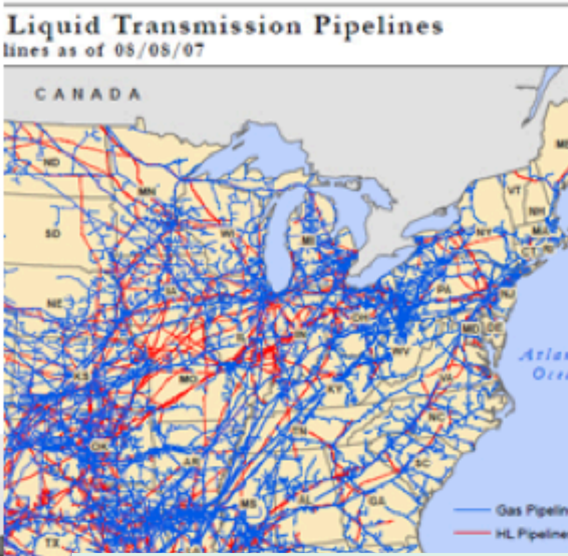


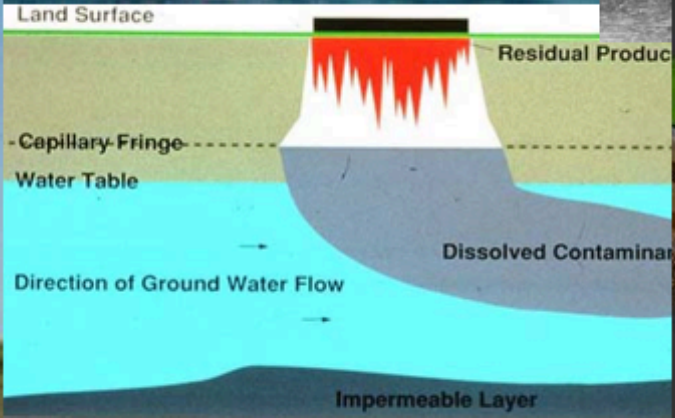


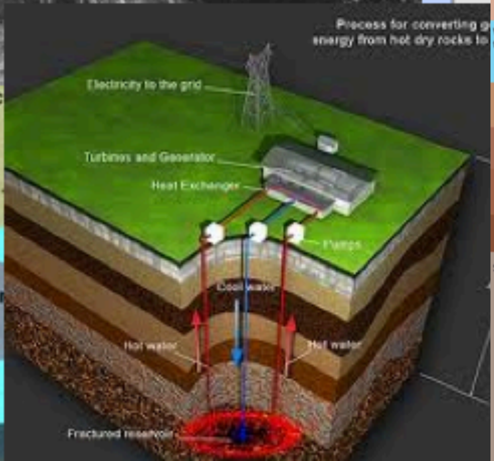
Fluid Mechanics in Engineered Systems

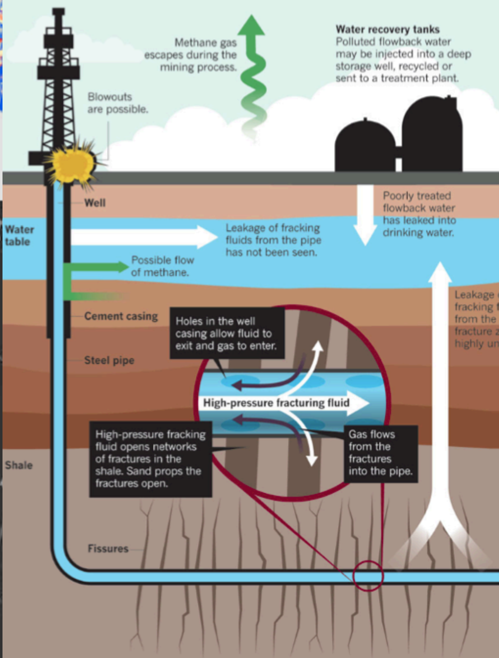










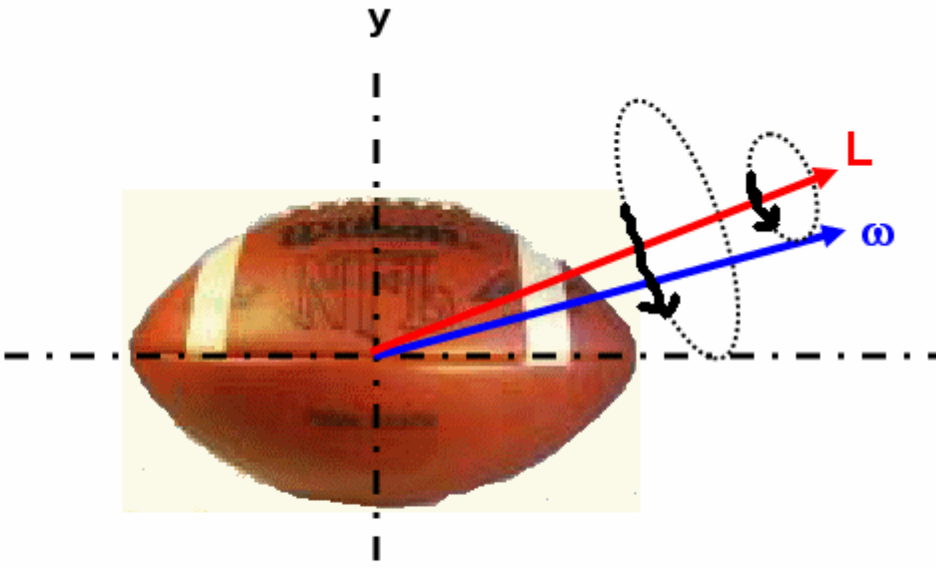


Fluid Mechanics in Recreation







EME 303 - FLUID MECHANICS

SUMMARIZED EQUATIONS AND CONCEPTS

Some Useful Conversion Factors

	SI	BGS	EE
Temperature:	$K = {}^{o}C + 273.15$	${}^{o}R = {}^{o}F + 459.67$	${}^{o}R = {}^{o}F + 459.67$
Force:	$1N = (1kg)(1m/s^2)$	$1lb = (1slug)(1ft/s^2)$	$1lbf = (1lbm)(32.2ft/s^2)$
Mass:	kg	$slug$	lbm
Density:	$1\,kg/m^3$	$0.00194\,slug/ft^3$	$0.06243\,lbm/ft^3$
ρ_{water}	$1000kg/m^3$	$1.94slugs/ft^3$	$62.4lbm/ft^3$
Pressure:	$1Pa = 1N/m^2$	$0.0209\,lb/ft^2$	$0.0209\,lbf/ft^2$
Work, energy:	$1J = 1N.m$	$1\,ft.lb = 778.2Btu$	—
Power:	$1W = 1N.m/s$	$1hp = 550\,ft.lb/s$	—

General [Topic:1]

$$\mathbf{F} = m\mathbf{a} \text{ or } \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = m \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

$$p = \rho RT \begin{cases} Isothermal : \dfrac{p}{\rho} = const. \\[2ex] Isentropic : \dfrac{p}{\rho^k} = const. \end{cases}$$

$$\tau = \mu \frac{du}{dy}; \quad \nu = \frac{\mu}{\rho}$$

$$E_v = -\frac{dp}{d\cancel{V}/\cancel{V}} = \frac{dp}{d\rho/\rho}$$

$$h = \frac{2\sigma\cos\theta}{\gamma R}$$

[1:2] Fluid Properties

Outline

Fluid=liquid vs. gas

Dimensional homogeneity

Fluid Properties

Mass and Weight $M = \rho \cancel{V}; W = Mg$

Equations of State $p = \rho R T$

Compressibility $E_v = -\dfrac{dp}{d\cancel{V} \,/\, \cancel{V}_0} = +\dfrac{dp}{d\rho \,/\, \rho_0}$

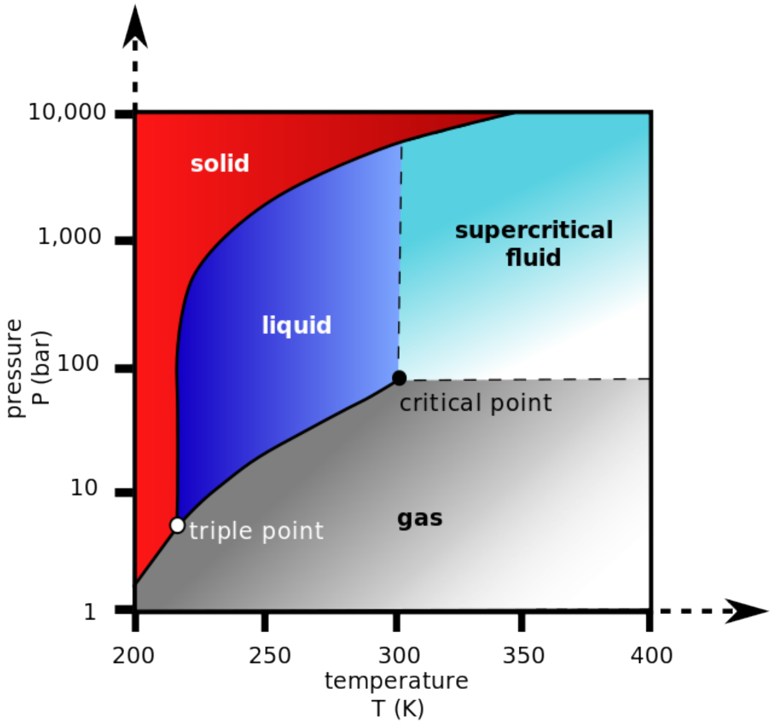
Wave Speeds $c = \sqrt{gy}\ or\ \sqrt{E_v \,/\, \rho}$

http://www.youtube.com/watch?v=629em0mPpUY

Viscosity $\tau = \mu \dfrac{\partial v_x}{\partial y}$

Vapor pressure

Surface tension (vinometer)



http://en.wikipedia.org/wiki/Supercritical_fluid

1.4 MEASURES OF FLUID MASS AND WEIGHT

1.4.1) **Density** (ρ) of a fluid is its mass per unit volume.

1.4.2 **Specific weight** (γ) of a fluid is its weight per unit volume.

f(gravity).

$$\underbrace{\rho = \gamma/g} \quad or \quad \gamma = \rho g$$

$$\boxed{* \gamma_{water} = \frac{62.4 \text{ lbs}}{ft^3} = \frac{9.81 \text{ kN}}{m^3}}$$

Note: g (acceleration of gravity) = 32.2 ft/sec^2 = 9.81 m/sec^2

English Units:

$$\rho = mass/volume = \boxed{\gamma/g} = \frac{lb/ft^3}{ft/sec^2} = \frac{lb \bullet s^2}{ft^4} = slug/ft^3$$

$$\boxed{slug = lb \bullet sec^2/ft}$$

$$\boxed{* \text{ WATER} \Rightarrow \frac{1.94 \text{ slug}}{ft^3}}$$

SI Units:

$$\boxed{1N = 1 kg\, m/s^2}$$

$$\rho = mass/volume = \gamma/g = \frac{N/m^3}{m/s^2} = \frac{N \bullet s^2}{m^4} = kg/m^3$$

$$\boxed{* \text{ WATER} \Rightarrow 1000 \frac{kg}{m^3}}$$

Specific Volume (v) is the reciprocal of the density.

$$v = 1/\rho \text{ (ft}^3\text{/slug or m}^3\text{/kg)}$$

1.4.3 **Specific Gravity** (s) of a liquid is the ratio of its density to that of pure water at a standard temperature. Physics: $\underbrace{39.2°F\ (4°C)}_{\text{Max density}}$; Engineering: 60°F

$$s_{fluid} = \gamma_{fluid}/\gamma_{water}$$

SOLIDS VS. FLUIDS (SECTION 1.1)

Since the molecular attraction of a solid is greater than that of a fluid, solids tend to retain their shape and deform when acted upon by an external force (strain). Fluids have smaller attractive forces, thus flow when pushed.

However, there are certain "plastic solids" that will flow under the proper circumstances - AND - there are certain very viscous liquids which do not flow readily.

The distinction:

> A fluid (no matter how viscous) will yield in time to the slightest stress.
>
> A solid (no matter how plastic) requires a certain magnitude of stress to be exerted before it will flow.

(Section 1.1 Cont'd)

Fluid Mechanics is the science of the mechanics of liquids and gases and is based on the same fundamental principles that are employed in the mechanics of solids.

FLUID MECHANICS

1.) Fluid Statics - the study of fluids at rest.

2.) Kinematics - velocities and streamlines (not considering forces or energy)

3.) Fluid Dynamics - the study of fluids in motion (velocities and accelerations)

Classical HYDRODYNAMICS is a subject of mathematics and deals with imaginary ideal frictionless fluids, thus limited in practical application. Empirical formulas, developed through experimentation, supplied answers to practical problems involving real fluids - the subject of HYDRAULICS.

A combination of the classical hydrodynamic approach and the study of real fluids (hydraulics) led to a new science termed "fluid mechanics" which can be applied to solve fluid flow problems of engineering significance.

DEFINITIONS

A fluid may be either a GAS or a LIQUID

A gas is very compressible and when external pressures are removed will tend to expand indefinitely.

A liquid is relatively incompressible and does not expand indefinitely. A liquid has a free surface.

A vapor is a gas whose temperature and pressure are such that it is very near the liquid phase. St~~r~~eam is considered a vapor since its state is normally not far from that of water.

Gas may be defined as a highly superheated vapor. Air is a gas because its state is normally very far from that of liquid air.

The volume of a gas or vapor is greatly affected by temperature and pressure, thus the relationship between fluid mechanics and thermodynamics.

UNITS (Section 1.2)

PRIMARY QUANTITIES (Combinations of these describe all secondary quantities)

TABLE 1.1

Dimensions Associated with Common Physical Quantities

	FLT System	MLT System
Acceleration	LT^{-2}	LT^{-2}
Angle	$F^0L^0T^0$	$M^0L^0T^0$
Angular acceleration	T^{-2}	T^{-2}
Angular velocity	T^{-1}	T^{-1}
Area	L^2	L^2
Density	$FL^{-4}T^2$	ML^{-3}
Energy	FL	ML^2T^{-2}
Force	F	$\boxed{MLT^{-2}}$
Frequency	T^{-1}	T^{-1}
Heat	FL	ML^2T^{-2}
Length	L	L
Mass	$FL^{-1}T^2$	M
Modulus of elasticity	FL^{-2}	$ML^{-1}T^{-2}$
Moment of a force	FL	ML^2T^{-2}
Moment of inertia (area)	L^4	L^4
Moment of inertia (mass)	FLT^2	ML^2
Momentum	FT	MLT^{-1}
Power	FLT^{-1}	ML^2T^{-3}
Pressure	FL^{-2}	$ML^{-1}T^{-2}$
Specific heat	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	FL^{-3}	$ML^{-2}T^{-2}$
Strain	$F^0L^0T^0$	$M^0L^0T^0$
Stress	FL^{-2}	$ML^{-1}T^{-2}$
Surface tension	FL^{-1}	MT^{-2}
Temperature	$\boxed{\Theta}$	$\boxed{\Theta}$
Time	T	T
Torque	FL	ML^2T^{-2}
Velocity	LT^{-1}	LT^{-1}
Viscosity (dynamic)	$FL^{-2}T$	$ML^{-1}T^{-1}$
Viscosity (kinematic)	L^2T^{-1}	L^2T^{-1}
Volume	L^3	L^3
Work	FL	ML^2T^{-2}

SECONDARY QUANTITIES

$$\text{Area} \doteq L^2$$

$$\text{Velocity} \doteq LT^{-1}$$

Newton's 2nd Law : $F = Ma \longrightarrow M = F/a$

$$\boxed{M = \frac{F}{a} \doteq \frac{N}{L/T^2} \doteq \frac{NT^2}{L} = (MLT^{-2}) \cdot T^2 \cdot L^{-1} = M}$$

$$1N = \boxed{1kg\,m/s^2} = MLT^{-2}$$

DIMENSIONAL HOMOGENEITY (Section 1.2)

From text: $d = 16.1\, t^2$

simplified from $d = \frac{1}{2} g t^2$

HOMOGENEOUS: $L \doteq (LT^{-2})\, T^2$

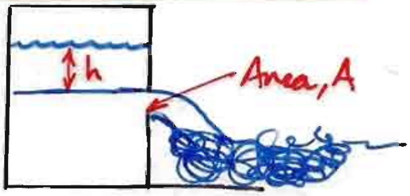
$L \doteq L$ ✓

Check units: $L \doteq 16.1\, T^2$

∴ $LT^{-2} \doteq 16.1$

This particular equation only works for ENGLISH System of units. where $\frac{1}{2} g = 16.1$ ft/s².

Also: $Q = 0.61\, A\sqrt{2gh}$



Check units: $\frac{L^3}{T} \doteq 0.61\; L^2 \sqrt{2\,\frac{L}{T^2}\,L}$

$$\frac{L^3}{T} \doteq (0.61\sqrt{2})\; L^2\sqrt{\frac{L^2}{T^2}} = \underbrace{(0.61\sqrt{2})}_{\text{Dimensionless}}\; \frac{L^3}{T}$$

Utility?

1. Check veracity (truth) of equations
2. Experimental analysis and physical modeling.

DIMENSIONAL HOMOGENEITY (Section 1.2)

Bernoulli Theorem

$$dp + \frac{1}{2}\rho\, d(V^2) + \gamma\, dz = 0 \qquad (3.4)$$

ignore derivatives

$$ML^{-1}T^{-2} + \underbrace{ML^{-3}\,(LT^{-1})^2}_{ML^{-1}T^{-2}} + \underbrace{ML^{-2}T^{-2}(L)}_{ML^{-1}T^{-2}} = 0$$

Integrated form of Bernoulli Theorem

– Integration/diff. changes dimensions.

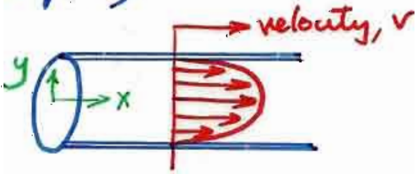
$$\int \frac{1}{\rho}\, dp + \frac{1}{2}V^2 + gz = C$$

$$\underbrace{(ML^{-3})^{-1} \quad ML^{-1}T^{-2}}_{L^2T^{-2}} \quad \underbrace{+ \quad L^2T^{-2}}_{L^2T^{-2}} + \underbrace{LT^{-2} \quad L}_{L^2T^{-2}}$$

Differential equations (1-D Navier-Stokes Equns).

(Ignore operators)

$$\frac{\boxed{d^2}V}{\boxed{d}y^2} = \frac{1}{\mu}\,\frac{\boxed{d}p}{\boxed{d}x}$$



$$\frac{LT^{-1}}{L^2} = \frac{1}{\cancel{M}L^{-1}T^{-1}}\;\frac{\cancel{M}L^{-1}T^{-2}}{L}$$

QED.

1.4.2. Specific Weight of Liquids

The specific weight ($\gamma = \rho g$) of a liquid varies only slightly with pressure (depending on the bulk modulus). Temperature can vary the specific weight considerably.

Dissolved air, salts, and suspended matter in a liquid will increase the value a slight amount. Ocean water may ordinarily be assumed to weigh 64.0 lb/ft^3 (10.1 kN/m^3).

Unless otherwise specified by some specific temperature being given, the specific weight of water in the text and in the problems is:
γ = 62.4 lb/ft^3 (9.81 kN/m^3).

Also known as "Unit Weight" γ.

Illustrative Example 1.1 The specific weight of water at ordinary pressure and temperature is 62.4 lb/ft³ (9.81 kN/m³). The specific gravity of mercury is 13.55. Compute the density of water and the specific weight and density of mercury.

$$\rho_{\text{water}} = \frac{\gamma_{\text{water}}}{g} = \frac{62.4\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} = 1.94\ \text{slugs/ft}^3 = \frac{9.81\ \text{kN/m}^3}{9.81\ \text{m/s}^2} = 1.00\ \text{Mg/m}^3 = 1.00\ \text{g/mL}$$

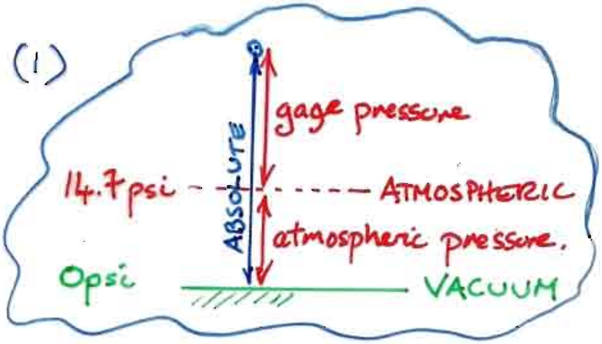
$$\gamma_{\text{mercury}} = s_{\text{mercury}}\gamma_{\text{water}} = 13.55(62.4) = 846\ \text{lb/ft}^3 = 13.55(9.81) = 133\ \text{kN/m}^3$$

$$\rho_{\text{mercury}} = s_{\text{mercury}}\rho_{\text{water}} = 13.55(1.94) = 26.3\ \text{slugs/ft}^3 = 13.55(1.00) = 13.55\ \text{Mg/m}^3$$

Equations of State for Gases (SECTION 1.5)

There is no such thing as a perfect gas, but air and other gases far removed from the liquid phase may be considered as ideal gases.

Ideal Gas Law: $\frac{p}{\rho} = pv = RT$ (1)



p = absolute pressure
ρ = density
v = specific volume ($1/\rho$)
R = gas constant
T = absolute temperature (Rankine or Kelvin)

For Air: R = 1715 ft • lb/(slug • °R) or 287 N • m/(kg • °K)

Since R is not a true constant (it is related to molecular weight or mass) the Ideal gas law may be expressed as:

$$p = \rho \frac{\bar{R}}{m} T \qquad (2) \qquad \text{From (1)} \quad R = \frac{\bar{R}}{m}$$

$\bar{R}$ = universal gas constant (8.313 $m^2/(sec^2$ • °K)
m = mass (kg) of 6.02 x 10^{23} molecules of gas (mole)
m = atomic "weight"/1000 to convert grams to kg.

Before we do an example problem a review of temperatures and conversions is in order.

Temperature

SI Units: K = Kelvin; C = Celsius

$^\circ K = {}^\circ C + 273.16$

English Units: R = Rankine; F = Fahrenheit

$^\circ R = {}^\circ F + 459.69$

Conversions: $^\circ C = ({}^\circ F - 32)\ 5/9$

$^\circ F = 9/5\ {}^\circ C + 32$

Temperature

SI Units: K = Kelvin; C = Celsius

$°K = °C + 273.16$

English Units: R = Rankine; F = Fahrenheit

$°R = °F + 459.69$

Conversions: $°C = (°F - 32)\ 5/9$

$°F = 9/5\ °C + 32$

Also: $p_2 v_2^n = p_1 v_1{}^n = \text{constant}$

p is absolute pressure

$v = 1/\rho$

If: isothermal (const. temp.); $n = 1$

isentropic (frictionless & adiabatic - no heat transfer); $n = k$

k is the ratio of specific heat at constant pressure to that at a constant volume.

1.7.2 COMPRESSION/EXPANSION OF GASES

Ideal gas law $p = \rho R T$

Compression @ constant temperature (ISOTHERMAL) $(dT=0)$

$$\frac{p}{\rho} = \text{constant} \qquad (1.14)$$

Compression – Frictionless (ISENTROPIC)

Adiabatic – no heat exchange to surroundings. – no heat loss

$$\frac{p}{\rho^k} = \text{constant} \qquad (1.15)$$

$$k = \frac{\text{Specific heat @ const. pressure}}{\text{Specific heat @ const volume}}$$

Tables 1.7 & 1.8.

Modulii of gases are given as, E_v

since $m = \underset{\downarrow}{\rho}\ \overset{\uparrow}{\forall}$ ∴

$$E_v = -\frac{dp}{d\forall/\forall_0} = +\frac{dp}{d\rho/\rho_0} = \frac{dp}{d\rho}\cdot\rho_0$$

Isothermal: $p = \rho\cdot\text{const.}$ $\therefore \frac{dp}{d\rho} = \text{const.}$ $\therefore E_v = \text{const.}\,\rho$

Also $\rho = \frac{p}{\text{const}}$ $\boxed{\therefore E_v = p}$

Isentropic: Similar procedure $\boxed{E_v = kp}$

MODULUS, E_v, IS DIRECTLY PROP. TO PRESSURE ∴ COMPRESSIBLE !!

$$\frac{p}{\rho^k} = \text{constant} \quad \text{(ISENTROPIC)}$$

EXAMPLE 1.6

A cubic foot of helium at an absolute pressure of 14.7 psi is compressed isentropically to $\frac{1}{2}$ ft^3. What is the final pressure?

SOLUTION

For an isentropic compression

$$\text{INITIAL}(i) \qquad \frac{p_i}{\rho_i^k} = \frac{p_f}{\rho_f^k} \qquad \text{FINAL}(f)$$

where the subscripts i and f refer to initial and final states, respectively. Since we are interested in the final pressure, p_f, it follows that

$$p_f = \left(\frac{\rho_f}{\rho_i}\right)^k p_i$$

As the volume is reduced by one half, the density must double, since the mass of the gas remains constant. Thus,

$k = 1.66$
See Table 1.8

$$p_f = (2)^{1.66}(14.7 \text{ psi}) = 46.5 \text{ psi (abs)} \qquad \textbf{(Ans)}$$

$\frac{\rho_f}{\rho_i} = 2$ since compressed to $\frac{1}{2}$ volume with no mass exchange.

NOTE: ABSOLUTE pressures used!!

[1:3] Fluid Properties

Recap

Definitions

Dimensional homogeneity

Fluid properties

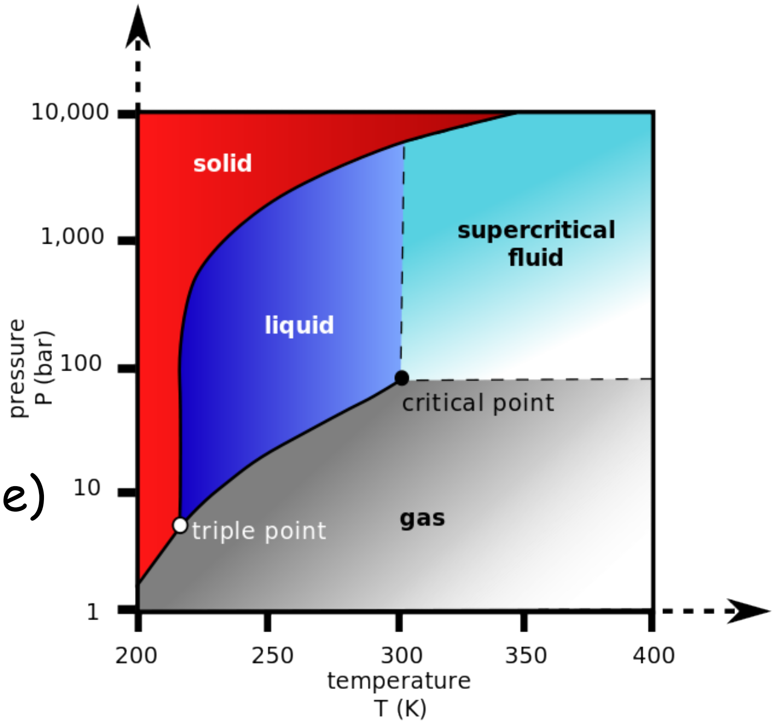
- Mass and Weight $M = \rho \cancel{V}; W = Mg$
- Equations of State $p = \rho R T$
- Compressibility $E_v = -\dfrac{dp}{d\cancel{V} \,/\, \cancel{V}_0} = +\dfrac{dp}{d\rho \,/\, \rho_0}$

Outline

- Wave Speeds $c = \sqrt{gy} \; or \; \sqrt{E_v \,/\, \rho}$ http://www.youtube.com/watch?v=629em0mPpUY
- Viscosity $\tau = \mu \dfrac{\partial v_x}{\partial y}$
- Vapor pressure (airfoil)
- Surface tension (vinometer)

Topic 2 Pressure

- Fluid pressure at a point
  - Incompressible (water)
  - Compressible (atmosphere)

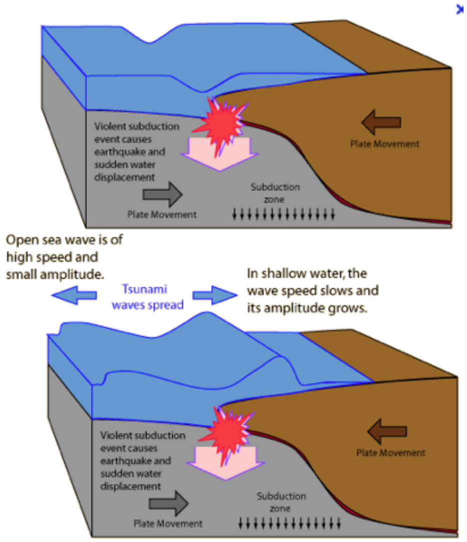


http://en.wikipedia.org/wiki/Supercritical_fluid

Wave Speeds

Compressional wave

$$c_p = \sqrt{\frac{K + \frac{4}{3}G}{\rho}}$$



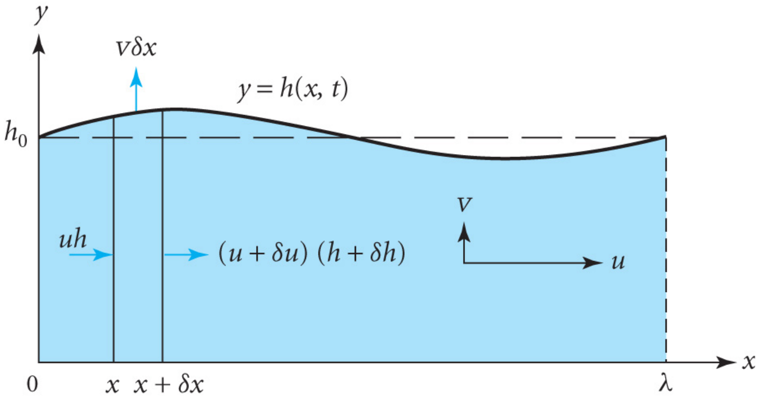
Shear wave

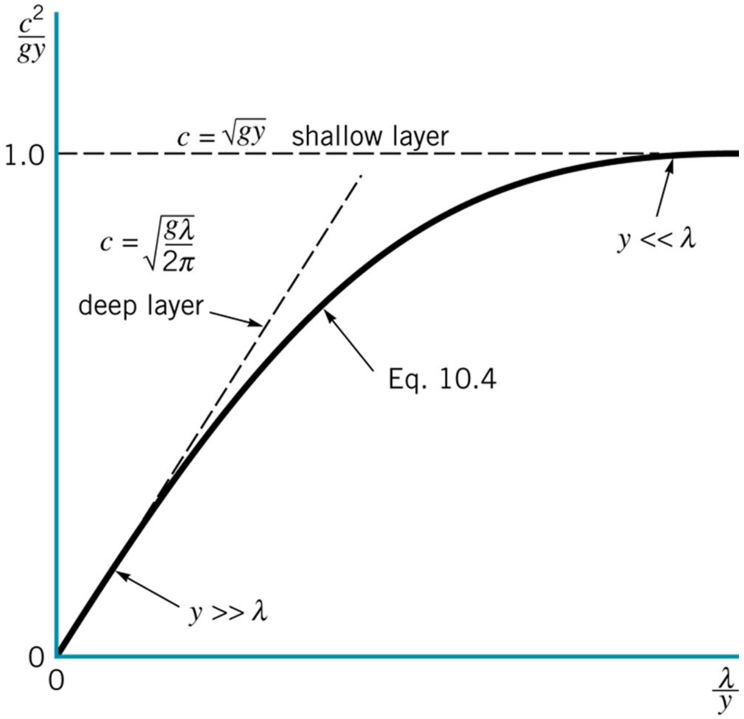
$$c_s = \sqrt{\frac{G}{\rho}}$$

Celerity wave

$$c = \sqrt{g h_0} \quad (shallow)$$

$$c = \sqrt{\frac{g\lambda}{2\pi}} \quad (deep)$$





[Munson, Young, Okishi et al, 2012]

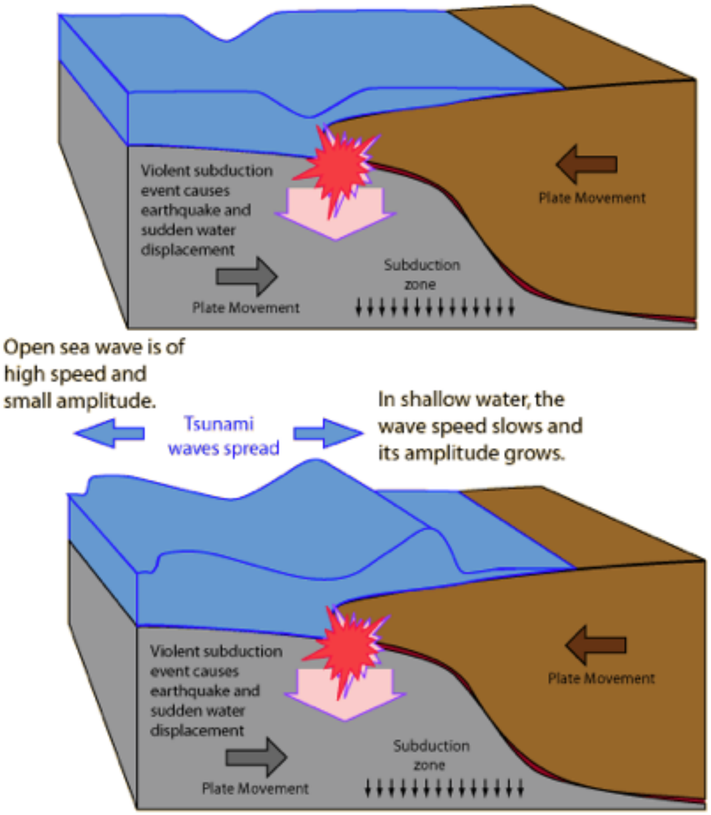
Tohoku Earthquake – March 9, 2011

3 Effects

- Earthquake
- Tsunami
- Fukushima meltdown

Order of arrivals and civil defense

- P-wave arrivals ~50 s delay
- S-wave arrivals ~100 s delay
- Tsunami arrival ~ 20 min delay



http://www.youtube.com/watch?v=629em0mPpUY

Ideal Fluids (Section 1.6)

An ideal fluid may be defined as one in which there is no friction. Such fluids do not exist in reality. These friction forces within the fluid are due to a property of the fluid called viscosity.

The action of a fluid may be described as a series of very thin sheets each of which slip relative to the next.

Through experimentation it has been shown that the velocity gradient (du/dy) times the viscosity (μ) is equal to the shearing stress (τ) between the thin sheets.

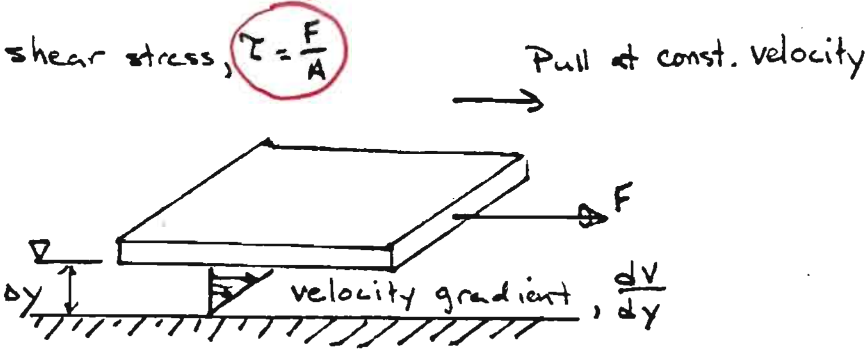
Experimentally determined $\longrightarrow$ $\tau = \mu \frac{du}{dy}$ $\Rightarrow$

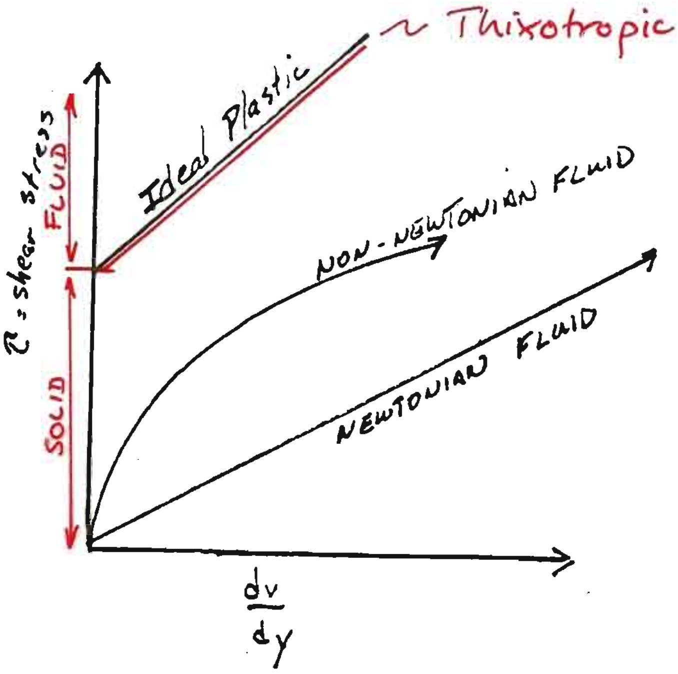
Units ?

$$\mu = \frac{\tau}{dv/dy} \doteq \frac{ML^{-1}T^{-2}}{LT^{-1}/L}$$

$$\mu \doteq ML^{-1}T^{-1}$$

Note: du may be represented by dv
Velocity may be represented as U or V.





Viscosity

The viscosity of a fluid is a measure of its resistance to shear or angular deformation. Friction forces in a fluid result from the cohesion and momentum interchange between fluid molecules.

As temperature increases the viscosity of:

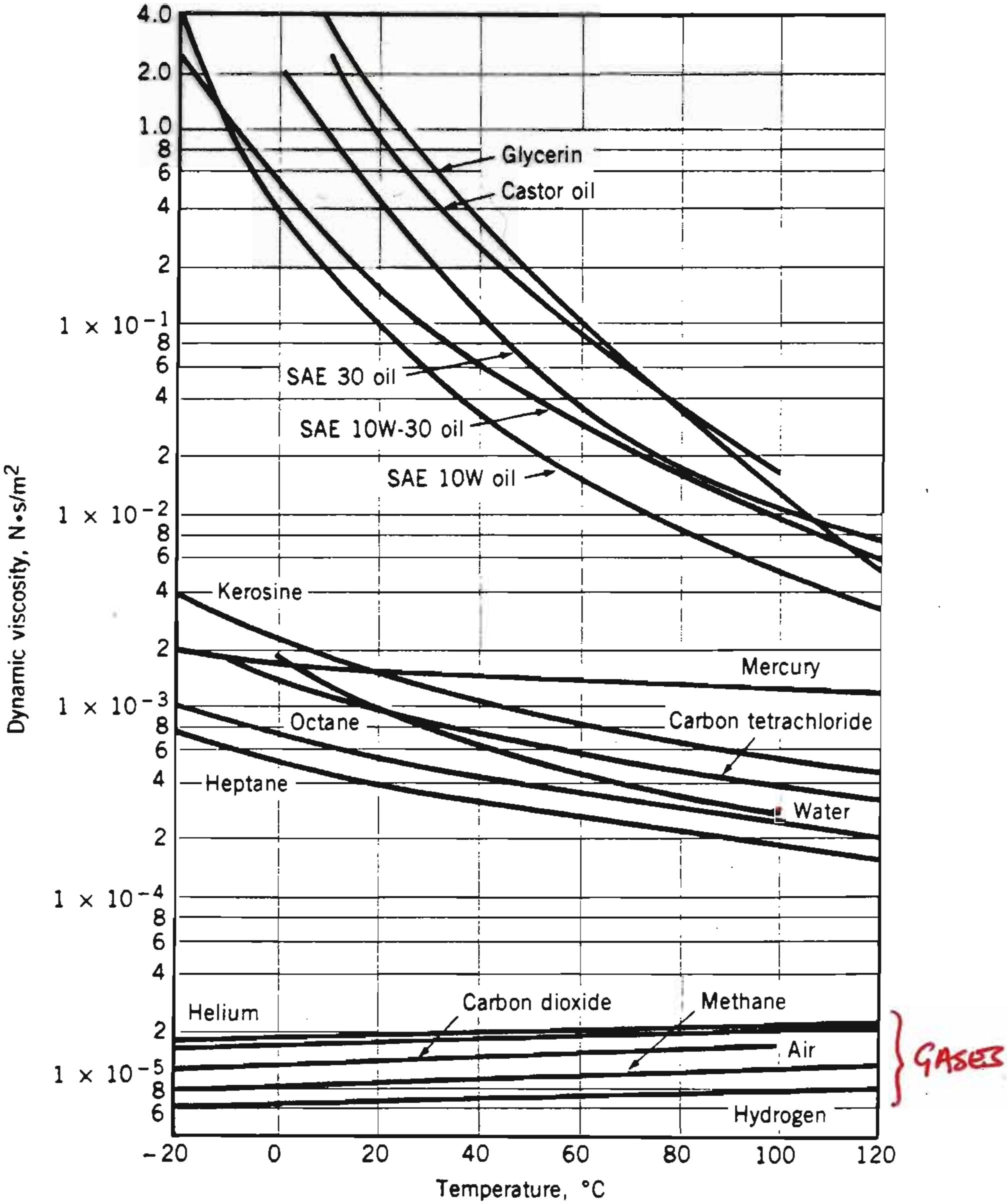
Gases - Increases.
Liquids - Decrease. } Figure B.1.

The viscosity (μ) is also called the coefficient of viscosity, the absolute viscosity, or the dynamic viscosity.

The kinematic viscosity (υ) is the viscosity divided by the density

$$\nu = \mu/\rho$$

□ Viscosity only exhibited with motion.



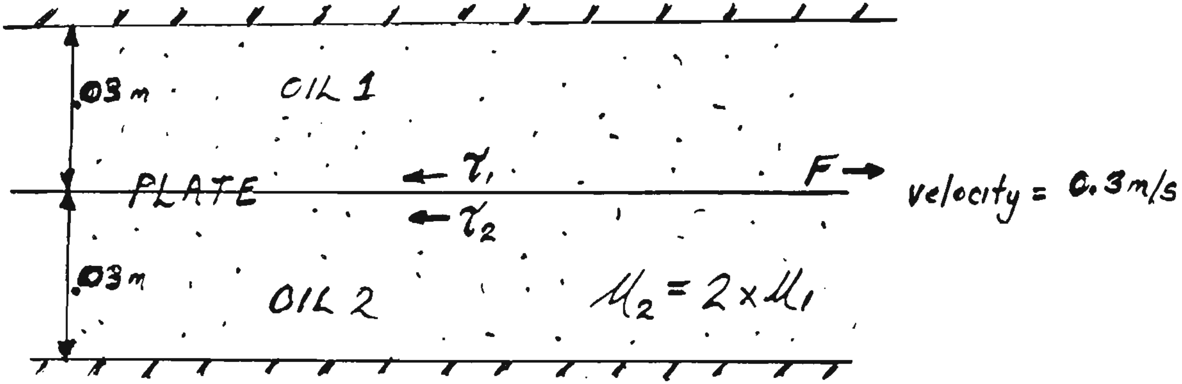
■ **FIGURE B.1** Dynamic (absolute) viscosity of common fluids as a function of temperature. To convert to BG units of lb·s/ft² multiply N·s/m² by 2.089×10^{-2}. Curves from R. W. Fox, and A. T. McDonald, *Introduction to Fluid Mechanics*, Third Edition, Wiley, New York, 1985. Used by permission.

Example Problem

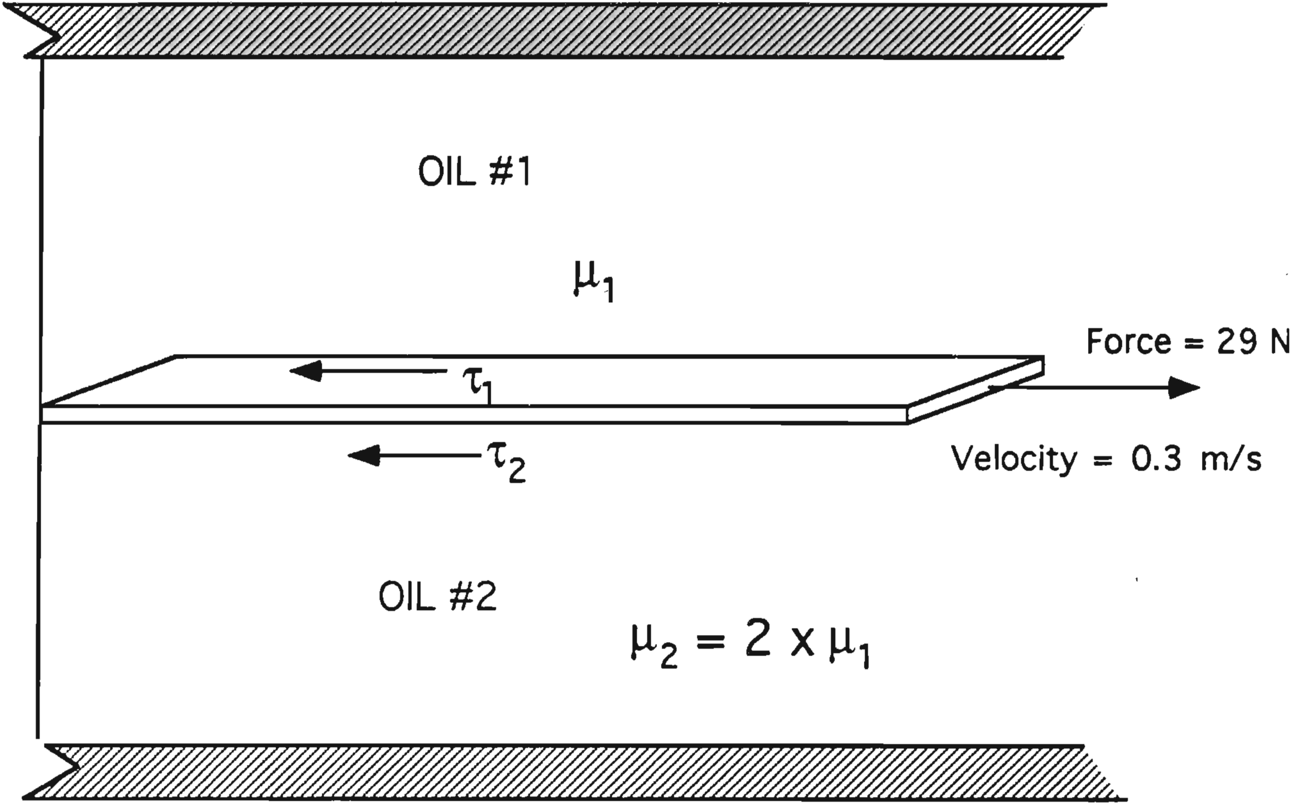
Given: A thin plate is centered in a gap of 0.06m between different oils at different viscosities. The viscosity of one oil is twice that of the other. The plate is being pulled through the oils at 0.3 m/sec. The resultant force due to viscous shear (on one square meter) is 29 N.

Assume: Neglect plate thickness

Diagram:



Required: Calculate the viscosities of the oils.



OIL #1
μ_1
Force = 29 N
τ_1
τ_2
Velocity = 0.3 m/s
OIL #2
$\mu_2 = 2 \times \mu_1$

Solution: $\tau = \mu \dfrac{dv}{dy}$; dy = 0.03 m dv = 0.3 m/sec

Sum forces "x" direction (per unit area)

$$F = \tau_1 A + \tau_2 A$$

$$F = A (\tau_1 + \tau_2)$$

$$F = A (\mu_1 \, dv/dy + 2\mu_1 \, dv/dy)$$

$$F = A \, dv/dy \, (3\mu_1)$$

$$A = 1 \text{ m}^2$$

$$dv/dy = \frac{0.3 \ m/\text{sec}}{0.03m} = 10 \text{ sec}^{-1}$$

$$29 \text{ N} = 1 \text{ m}^2 \; 10 \text{ sec}^{-1} \; (3 \; \mu_1)$$

$$\mu_1 = 0.967 \;\; \text{N} \bullet \text{s/m}^2$$

$$\mu_2 = 1.934 \;\; \text{N} \bullet \text{s/m}^2$$

Check units:

$$\text{N} = \text{m}^2 \bullet \text{sec}^{-1} \bullet \mu$$

$$\mu = \text{N} \bullet \text{s/m}^2 = \text{Pa} \bullet \text{s}$$

Note: $\text{Pa} = \text{N/m}^2$

Pressure = Force/Area

Compressibility of Liquids

The compressibility of liquid is its change in volume due to change in pressure and is inversely proportional to the bulk modulus (vol. modulus of elasticity).

$$E_v = -(v/dv)dp$$

v = ~~specific~~ volume

p = pressure

$$\boxed{dv/v_o \approx -dp/E_v} \Longrightarrow \underbrace{\frac{E_v}{1} = \frac{-dp}{dv/v_o}}$$

$$-\left(\frac{v_2 - v_1}{v_1}\right) \approx \frac{p_2 - p_1}{E_v}$$

For WATER:

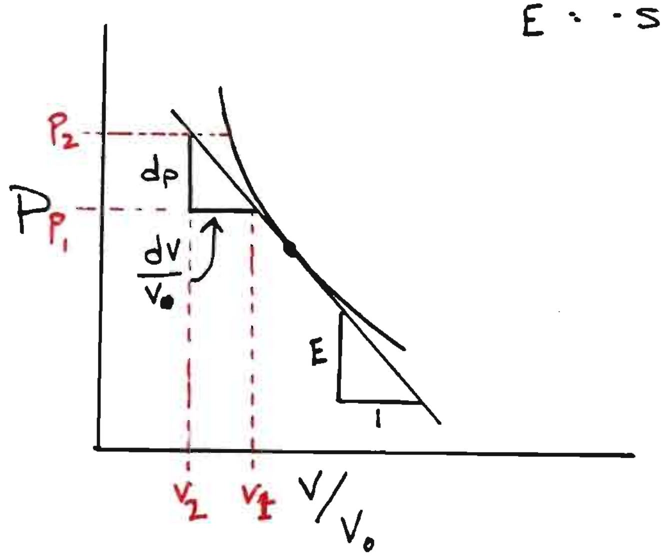
At 68°F: $E_v = 320{,}000$ psi

Increasing the pressure $(p_2 - p_1)$ by 1000 psi.

results in a volume change of 1000/320,000 or 0.3%

therefore, the assumption regarding water as incompressible.

E : -slope



$$E_v = +\frac{dp}{d\rho/\rho_o} = \frac{dp}{d\rho}\rho_o$$

Vapor Pressure, p_v

- any free surface has a continual interchange of molecules

- if more molecules leave liquid than enter ==> evaporation

- if more molecules enter liquid than leave ==> condensation

- the molecules impinging on the liquid surface create a pressure known as the partial pressure of the liquid vapor

- this partial pressure along with the partial pressure of the other gases in the atmosphere make up atmospheric pressure

- the molecules at the liquid surface create a vapor pressure, p_v

- when p_v = the partial pressure of the liquid vapor in the atmosphere ==> saturation of the atmosphere with that substance

- if we can reduce the partial pressure then its possible to boil the liquid at room temperature

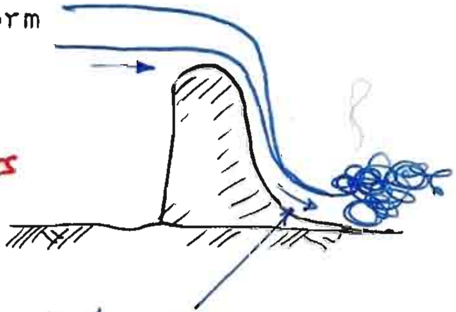
- when the pressure above the liquid equals the vapor pressure, the liquid will boil

@ p_v = 14.7 psia water boils @ 212 F
@ p_v = 0.364 psia water boils @ 70 F

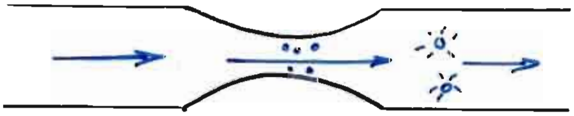
- this becomes a boundary condition for the minimum pressure

- CAVITATION = vaporization of a liquid to form bubbles which burst at a high pressure

☐ Dam spillways
☐ Submarine propellers



↑ velocity ∴ ↓ pressure ∴ cavitati



Velocity ↑
Pressure ↓ to P_v
Cavitation

Velocity ↓
Pressure ↑
Bubbles collapse.

Vapor Pressure of Liquids

Molecular activity in a liquid tends to free some surface molecules. The partial pressure caused by the formation of vapor by the escaping molecules is known as the vapor pressure and is a function of temperature ONLY.

At equilibrium the number molecules leaving the liquid equals the number of molecules reentering the fluid - the vapor pressure at this point is called the saturation pressure. A rapid rate of evaporation is termed "boiling." The boiling point of a liquid is a function of temperature and pressure.

Example Problem (pg. 18)

Given: Water at elevation of 10,000 ft.

Required: At approximately what temperature will the water boil?

Solution: (From Table C.1) @ 10,000 ft the psia = 10.11 psia
(From Table B.1) the saturation pressure (vapor pressure) of water is 10.11 psia at about 193.5°F.
Hence water will boil at 193.5°F at 10,000 ft; this explains why it takes longer to cook at higher elevations. ... but shorter to bring to the boil!!

APPENDIX C

Properties of the U.S. Standard Atmosphere

■ TABLE C.1

Properties of the U.S. Standard Atmosphere (BG Units)[a]

Altitude (ft)	Temperature (°F)	Acceleration of Gravity, g (ft/s^2)	Pressure, p [lb/in.^{2}(abs)]	Density, ρ (slugs/ft^3)	Dynamic Viscosity, μ (lb·s/ft^2)
−5,000	76.84	32.189	17.554	2.745 E − 3	3.836 E − 7
0	59.00	32.174	14.696	2.377 E − 3	3.737 E − 7
5,000	41.17	32.159	12.228	2.048 E − 3	3.637 E − 7
10,000	23.36	32.143	10.108	1.756 E − 3	3.534 E − 7
15,000	5.55	32.128	8.297	1.496 E − 3	3.430 E − 7
20,000	−12.26	32.112	6.759	1.267 E − 3	3.324 E − 7
25,000	−30.05	32.097	5.461	1.066 E − 3	3.217 E − 7
30,000	−47.83	32.082	4.373	8.907 E − 4	3.107 E − 7
35,000	−65.61	32.066	3.468	7.382 E − 4	2.995 E − 7
40,000	−69.70	32.051	2.730	5.873 E − 4	2.969 E − 7
45,000	−69.70	32.036	2.149	4.623 E − 4	2.969 E − 7
50,000	−69.70	32.020	1.692	3.639 E − 4	2.969 E − 7
60,000	−69.70	31.990	1.049	2.256 E − 4	2.969 E − 7
70,000	−67.42	31.959	0.651	1.392 E − 4	2.984 E − 7
80,000	−61.98	31.929	0.406	8.571 E − 5	3.018 E − 7
90,000	−56.54	31.897	0.255	5.610 E − 5	3.052 E − 7
100,000	−51.10	31.868	0.162	3.318 E − 5	3.087 E − 7
150,000	19.40	31.717	0.020	3.658 E − 6	3.511 E − 7
200,000	−19.78	31.566	0.003	5.328 E − 7	3.279 E − 7
250,000	−88.77	31.415	0.000	6.458 E − 8	2.846 E − 7

[a] Data abridged from *U.S. Standard Atmosphere*, 1976, U.S. Government Printing Office, Washington, D.C.

■ TABLE B.1
Physical Properties of Water (BG Units)[a]

Temperature (°F)	Density, ρ (slugs/ft³)	Specific Weight[b], γ (lb/ft³)	Dynamic Viscosity, μ (lb·s/ft²)	Kinematic Viscosity, ν (ft²/s)	Surface Tension[c], σ (lb/ft)	Vapor Pressure, p_v [lb/in.²(abs)]	Speed of Sound[d], c (ft/s)
32	1.940	62.42	3.732 E − 5	1.924 E − 5	5.18 E − 3	8.854 E − 2	4603
40	1.940	62.43	3.228 E − 5	1.664 E − 5	5.13 E − 3	1.217 E − 1	4672
50	1.940	62.41	2.730 E − 5	1.407 E − 5	5.09 E − 3	1.781 E − 1	4748
60	1.938	62.37	2.344 E − 5	1.210 E − 5	5.03 E − 3	2.563 E − 1	4814
70	1.936	62.30	2.037 E − 5	1.052 E − 5	4.97 E − 3	3.631 E − 1	4871
80	1.934	62.22	1.791 E − 5	9.262 E − 6	4.91 E − 3	5.069 E − 1	4819
90	1.931	62.11	1.500 E − 5	8.233 E − 6	4.86 E − 3	6.979 E − 1	4960
100	1.927	62.00	1.423 E − 5	7.383 E − 6	4.79 E − 3	9.493 E − 1	4995
120	1.918	61.71	1.164 E − 5	6.067 E − 6	4.67 E − 3	1.692 E + 0	5049
140	1.908	61.38	9.743 E − 6	5.106 E − 6	4.53 E − 3	2.888 E + 0	5091
160	1.896	61.00	8.315 E − 6	4.385 E − 6	4.40 E − 3	4.736 E + 0	5101
180	1.883	60.58	7.207 E − 6	3.827 E − 6	4.26 E − 3	7.507 E + 0	5195
200	1.869	60.12	6.342 E − 6	3.393 E − 6	4.12 E − 3	1.152 E + 1	5089
212	1.860	59.83	5.886 E − 6	3.165 E − 6	4.04 E − 3	1.469 E + 1	5062

10.108 psi

---

[a] Based on data from *Handbook of Chemistry and Physics*, 69th Ed., CRC Press, 1988. Where necessary, values obtained by interpolation.

[b] Density and specific weight are related through the equation $\gamma = \rho g$. For this table, $g = 32.174$ ft/s².

[c] In contact with air.

[d] From R. D. Blevins, *Applied Fluid Dynamics Handbook*, Van Nostrand Reinhold Co., Inc., New York, 1984.

■ TABLE B.2
Physical Properties of Water (SI Units)[a]

Temperature (°C)	Density, ρ (kg/m³)	Specific Weight[b], γ (kN/m³)	Dynamic Viscosity, μ (N·s/m²)	Kinematic Viscosity, ν (m²/s)	Surface Tension[c], σ (N/m)	Vapor Pressure, p_v [N/m²(abs)]	Speed of Sound[d], c (m/s)
0	999.9	9.806	1.787 E − 3	1.787 E − 6	7.56 E − 2	6.105 E + 2	1403
5	1000.0	9.807	1.519 E − 3	1.519 E − 6	7.49 E − 2	8.722 E + 2	1427
10	999.7	9.804	1.307 E − 3	1.307 E − 6	7.42 E − 2	1.228 E + 3	1447
20	998.2	9.789	1.002 E − 3	1.004 E − 6	7.28 E − 2	2.338 E + 3	1481
30	995.7	9.765	7.975 E − 4	8.009 E − 7	7.12 E − 2	4.243 E + 3	1507
40	992.2	9.731	6.529 E − 4	6.580 E − 7	6.96 E − 2	7.376 E + 3	1526
50	988.1	9.690	5.468 E − 4	5.534 E − 7	6.79 E − 2	1.233 E + 4	1541
60	983.2	9.642	4.665 E − 4	4.745 E − 7	6.62 E − 2	1.992 E + 4	1552
70	977.8	9.589	4.042 E − 4	4.134 E − 7	6.44 E − 2	3.116 E + 4	1555
80	971.8	9.530	3.547 E − 4	3.650 E − 7	6.26 E − 2	4.734 E + 4	1555
90	965.3	9.467	3.147 E − 4	3.260 E − 7	6.08 E − 2	7.010 E + 4	1550
100	958.4	9.399	2.818 E − 4	2.940 E − 7	5.89 E − 2	1.013 E + 5	1543

---

[a] Based on data from *Handbook of Chemistry and Physics*, 69th Ed., CRC Press, 1988.

[b] Density and specific weight are related through the equation $\gamma = \rho g$. For this table, $g = 9.807$ m/s².

[c] In contact with air.

[d] From R. D. Blevins, *Applied Fluid Dynamics Handbook*, Van Nostrand Reinhold Co., Inc., New York, 1984.

Surface Tension (SECTION 1·9)

Surface tension is the "skin" which seems to form on the free surface of a fluid due to intermolecular cohesive and adhesive forces.

Surface tension is the cause of capillarity, which occurs when a liquid comes into contact with a vertical surface.

If adhesive forces dominate (as in water) the fluid will rise as it tries to wet the interior surface.

If cohesive forces dominate (as in mercury) the fluid at the interior surface is below the fluid level.

The curved surface is termed the meniscus of the fluid.

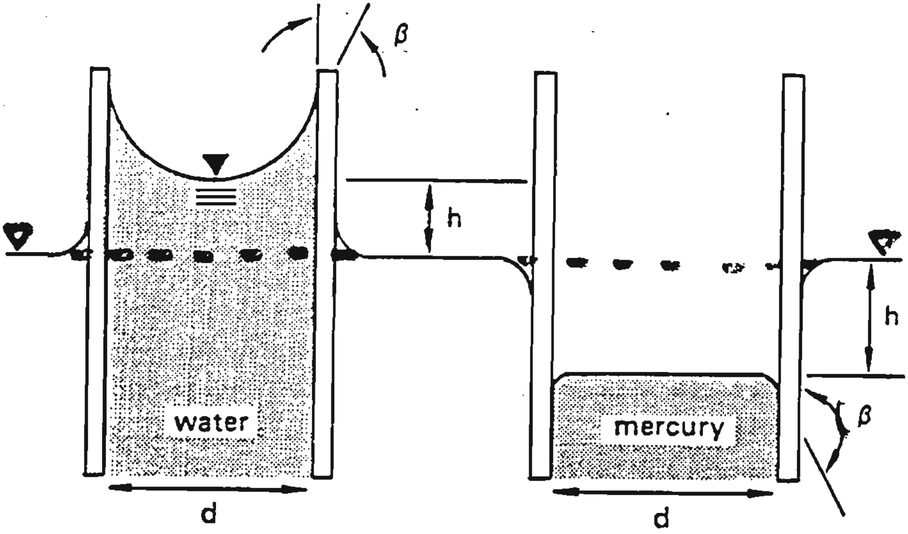


Figure 3.1 Capillarity in Thin-Wall Tubes

The capillary rise (or depression) as shown in the figure below is expressed as:

$$h = \frac{2\sigma \cos \theta}{\gamma r}$$

σ = surface tension in units of force per unit length
θ = wetting angle
γ = specific weight of the liquid
r = radius of the tube
h = capillary rise (from inflection point of meniscus)

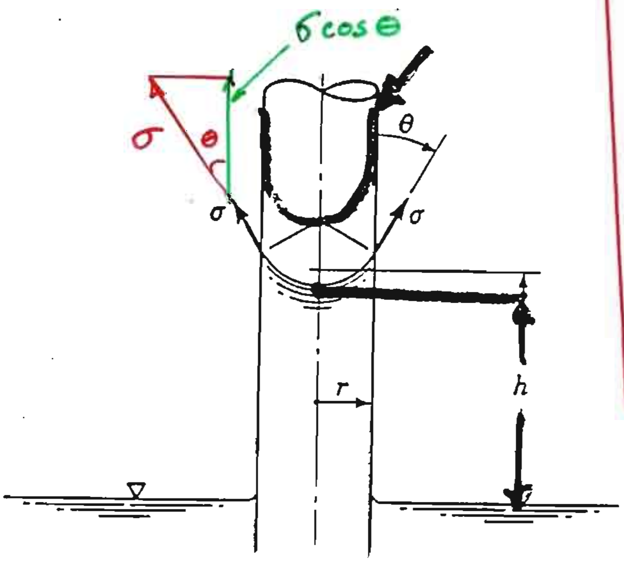


Figure 1.8 Capillary rise.

FORCE BALANCE

$$\gamma\ \underbrace{h\pi r^2}_{\text{Column volume}} = \underbrace{2\pi r}_{\text{Circumference}}\ \overbrace{\sigma \cos \theta}^{\text{Vertical force}}$$

REARRANGING

$$h = \frac{2\sigma \cos \theta}{\gamma r}$$

For a clean tube $\theta = 0°$ H_2O , 140° Hg

For tube diameters > ½" (12mm) capillary effects are negligible.

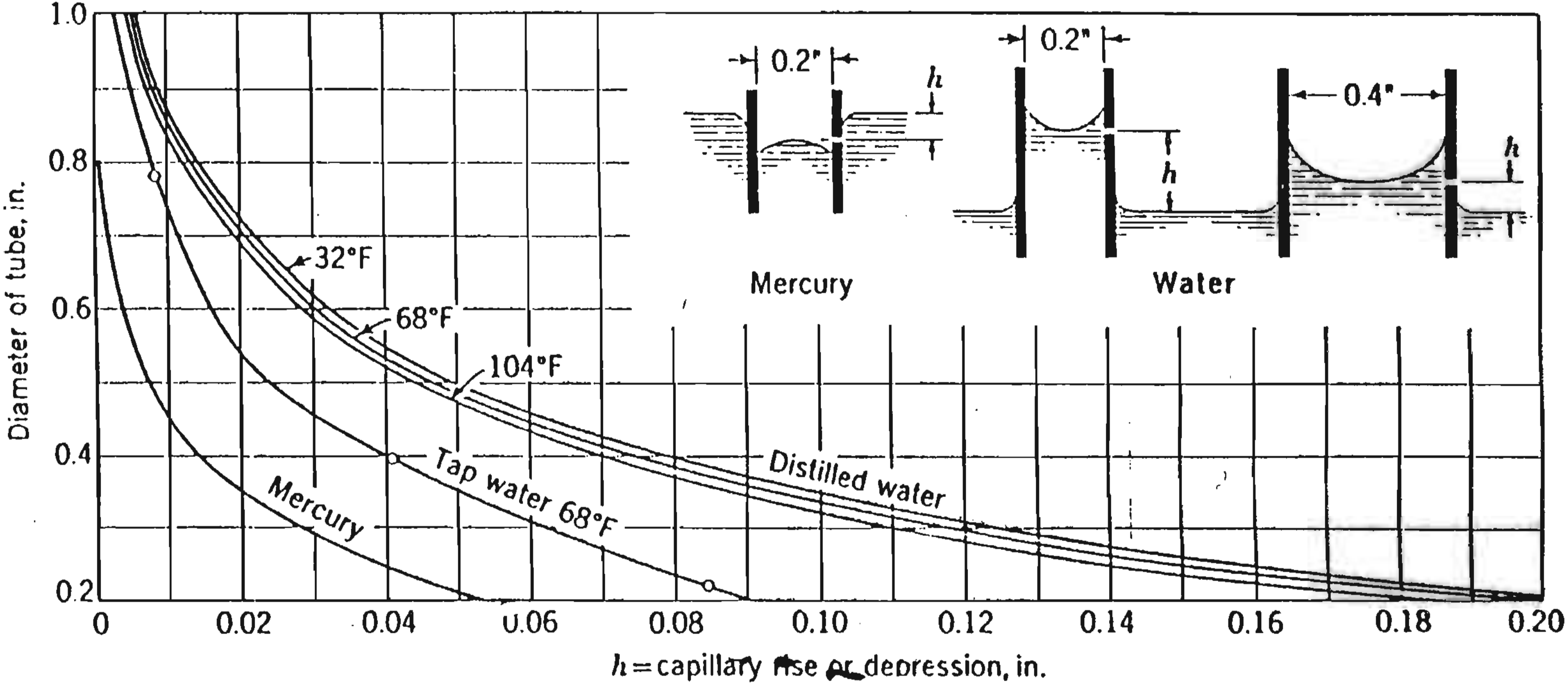


Figure 1.7 Capillarity in clean circular glass tubes.

EXAMPLE 1.8

Pressures are sometimes determined by measuring the height of a column of liquid in a vertical tube. What diameter of clean glass tubing is required so that the rise of water at 20 °C in a tube due to capillary action (as opposed to pressure in the tube) is less than 1.0 mm?

SOLUTION

From Eq. 1.22

$$h = \frac{2\sigma \cos\,\theta}{\gamma R}$$

so that

$$R = \frac{2\sigma \cos\,\theta}{\gamma h}$$

For water at 20 °C (from Table B.2), $\sigma = 0.0728$ N/m and $\gamma = 9.789$ kN/m^3. Since $\theta \approx 0°$ it follows that for $h = 1.0$ mm,

$$R = \frac{2(0.0728\ \mathrm{N/m})(1)}{(9.789\times 10^3\ \mathrm{N/m^3})(1.0\ \mathrm{mm})(10^{-3}\ \mathrm{m/mm})} = 0.0149\ \mathrm{m}$$

and the minimum required tube diameter, D, is

$$D = 2R = 0.0298\ \mathrm{m} = 29.8\ \mathrm{mm}$$

[2]

Fluid Statics

Fluid Statics [2,3]

$$p_x = p_y = p_z = p$$

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \mathbf{a} \text{ or } -\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} p - \gamma \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \rho \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

Simplifies when: $a_x = a_y = a_z = 0$ to $\frac{dp}{dx} = \frac{dp}{dy} = 0$ and $\frac{dp}{dz} = -\gamma; \frac{dz}{dx} = -\frac{a_x}{g + a_z}$

Rigid body rotation: $\frac{\partial p}{\partial r} = \rho r \omega^2; \frac{\partial p}{\partial \theta} = 0; \frac{\partial p}{\partial z} = -\gamma; \begin{cases} z = \frac{\omega r^2}{2g} + c \\ p = \frac{p \omega^2 r^2}{2} - \gamma z + c \end{cases}$

Incompressible fluid: $p = \gamma h + p_0$

Compressible fluid: $\frac{dp}{dz} = -\frac{gp}{RT}$ and integrate w.r.t (p, z).

Manometer rules: $(\uparrow -ve)(\downarrow +ve); \frac{dp}{dx} = \frac{dp}{dy} = 0; p_v$ if evacuated; $\gamma_{gas} \to 0$.

$$F_R = \gamma h_c A; \quad F_R = \sqrt{F_H^2 + F_V^2}$$

F_R acts through center of pressure $\begin{cases} y_R = \frac{I_{xc}}{y_c A} + y_c \\ x_R = \frac{I_{xyc}}{y_c A} + x_c \end{cases}$

$$F_B = \gamma \cancel{V}$$

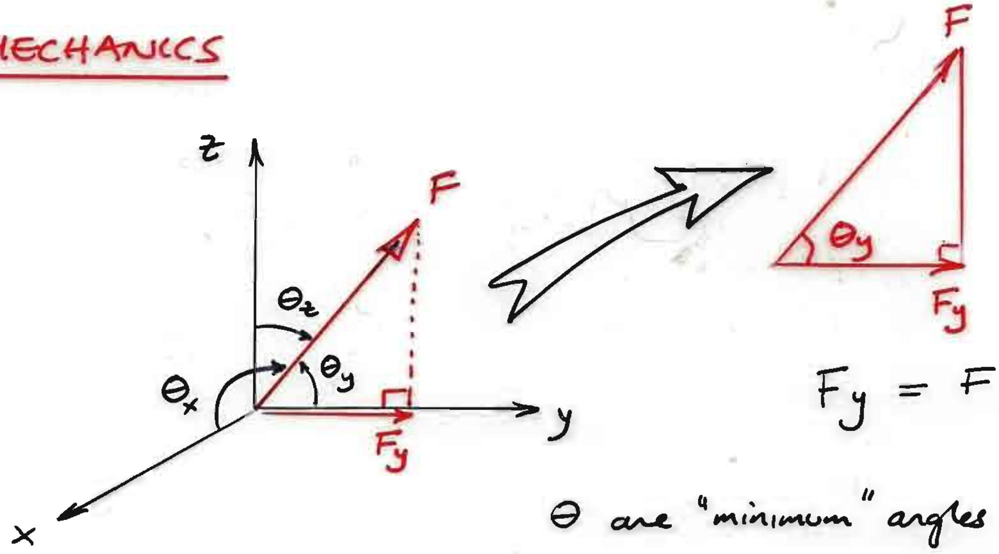
[2:1] Fluid Statics

Outline

Fluid pressure at a point $\frac{\partial p}{\partial z} = -\gamma$

- Incompressible (water)
- Compressible (atmosphere)

VECTOR MECHANICS



$$F_y = F\cos\theta_y$$

θ are "minimum" angles and $\cos\theta$ are "direction cosines".

Property of direction cosines: $(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2 = 1$

i.e. Same as "unit" vector $\hat{f} = \begin{Bmatrix} \cos\theta_x \\ \cos\theta_y \\ \cos\theta_z \end{Bmatrix}$

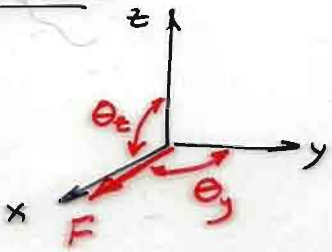
What are the components of $\vec{F}$ in the x, y, z directions?

$$F_x = \overset{\text{magnitude (scalar).}}{F}\cos\theta_x \quad ; \quad F_y = F\cos\theta_y \quad ; \quad F_z = F\cos\theta_z$$

Rewrite as an "orthogonal" force vector $\underline{F}$

$$\underline{F} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{Bmatrix} \cos\theta_x \\ \cos\theta_y \\ \cos\theta_z \end{Bmatrix} \underset{\text{scalar}}{F} + \begin{Bmatrix} \\ \\ \\ \end{Bmatrix} F_2 \; ---- \; \text{etc.}$$

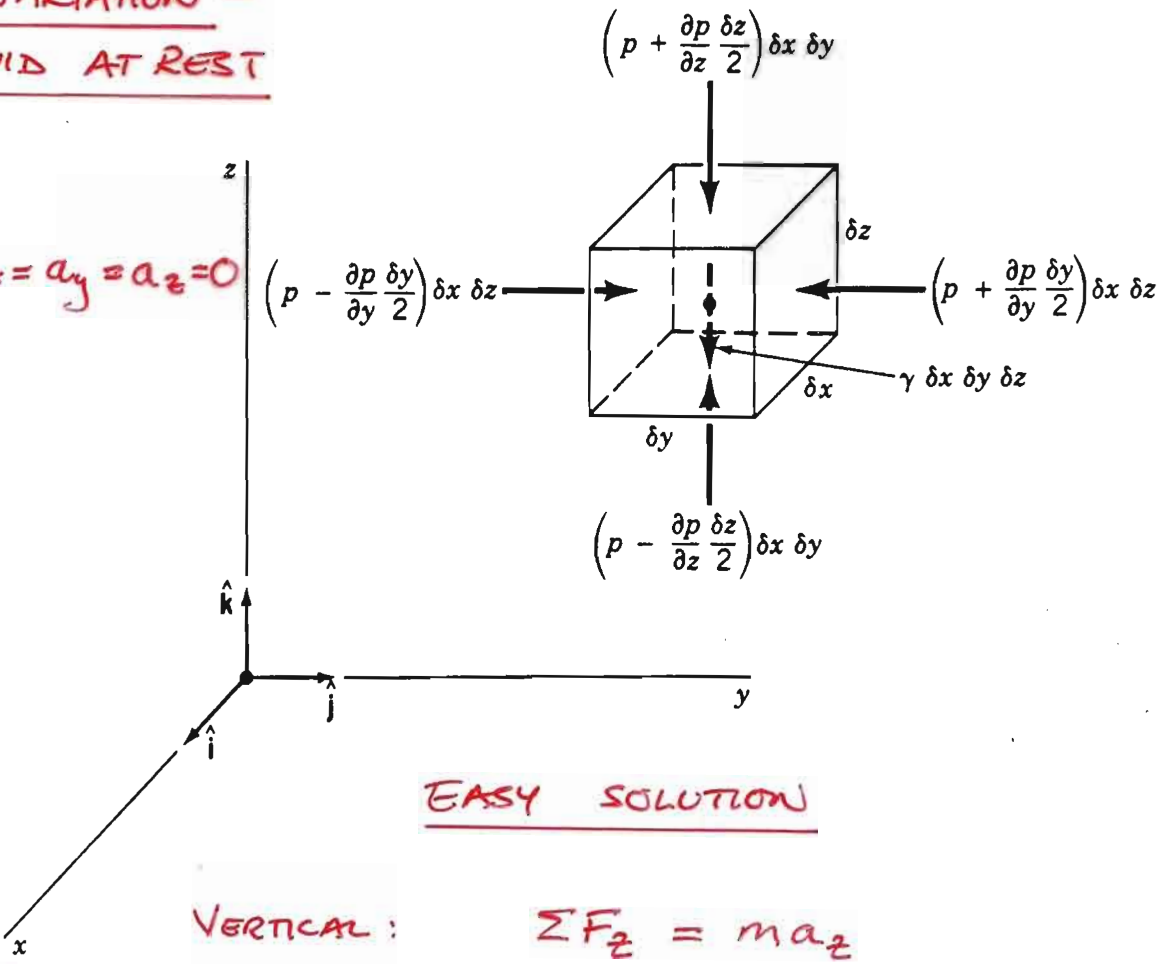
Special case



$$\underline{F} = F\begin{Bmatrix} \cos\theta_x \\ \cos\theta_y \\ \cos\theta_z \end{Bmatrix} = F\underbrace{\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}_{\hat{i}}$$

PRESSURE VARIATION – FLUID AT REST

At rest ∴ $a_x = a_y = a_z = 0$



EASY SOLUTION

VERTICAL: $\Sigma F_z = m a_z$

Force balance:

$$\Sigma F_z = \left(p - \frac{\partial p}{\partial z}\frac{\delta z}{2}\right)\overset{dxdy}{\wedge} - \left(p + \frac{\partial p}{\partial z}\frac{\delta z}{2}\right) dxdy - \gamma\, dxdydz - m\cancelto{0}{a_z} = 0$$

Summing terms:

$$-\frac{\partial p}{\partial z}\, dxdy\, dz - \gamma\, dxdy\, dz = 0$$

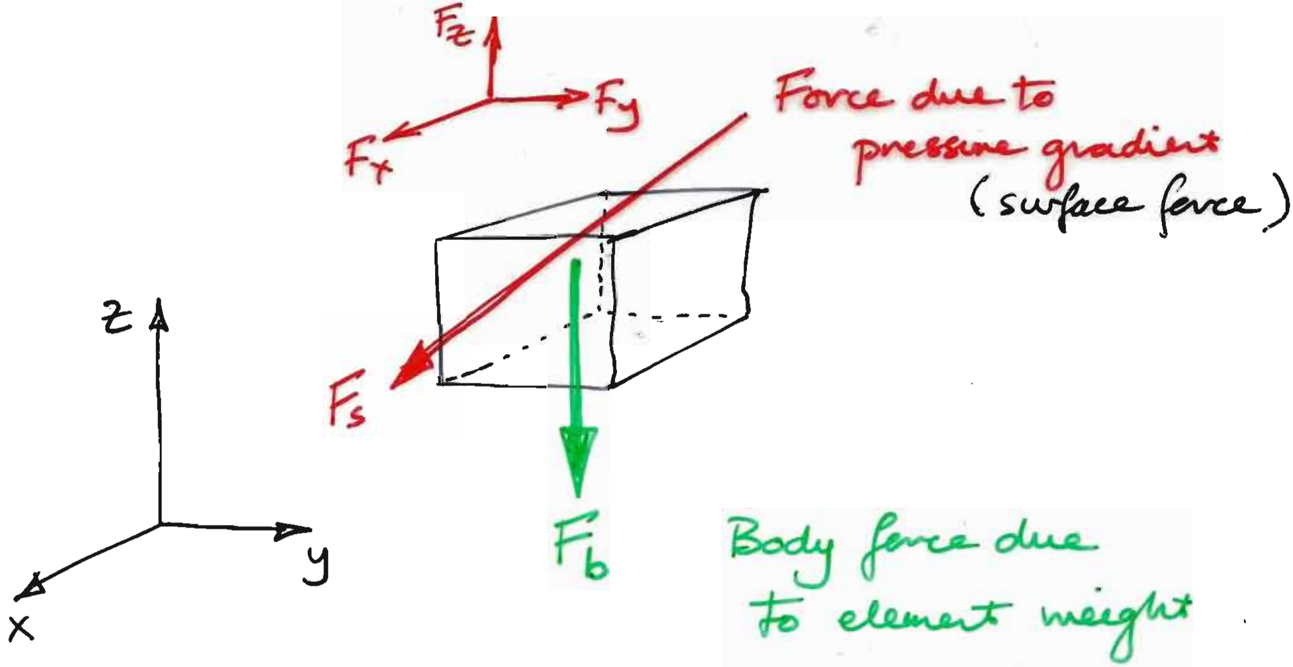
$$\boxed{\frac{\partial p}{\partial z} = -\gamma} \equiv \frac{dp}{dz} = -\gamma = -\rho g$$

Horizontal components:

$$\Sigma F_x = [(\quad) - (\quad)]\, dydz - m\cancelto{0}{a_x} = 0$$

$$\therefore \boxed{\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0}$$

EQUATION OF MOTION FOR FLUIDS



$$\Sigma \underline{F} = m\underline{a}$$

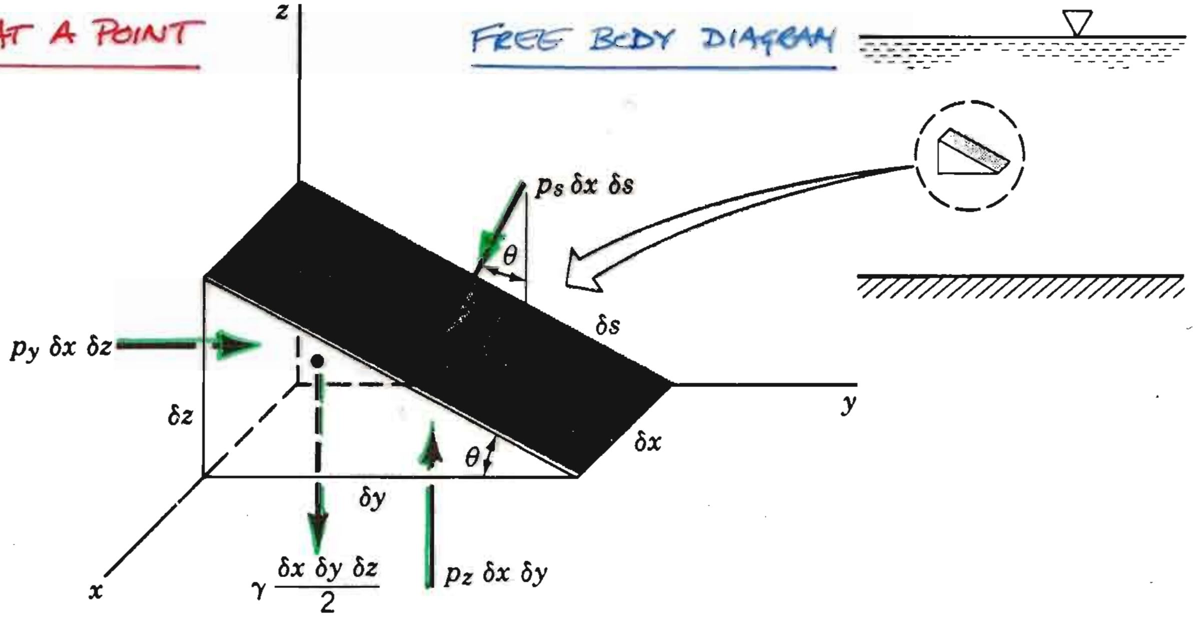
$$\underline{F}_s + \underline{F}_b = m\underline{a}$$

Note that "force balance" involves vectoral quantities !!

2. FLUID STATICS

2.1 PRESSURE AT A POINT

FREE BODY DIAGRAM



SUM FORCES IN -y- DIRECTION

$$\Sigma F_y = m a_y$$

$$\Sigma F_y = p_y\,dx\,dz - p_s\,dx\,\underbrace{ds\ \sin\theta} = \overbrace{\rho\,\tfrac{1}{2}\,dx\,dy\,dz}^{m}\ a_y$$

Noting that $\underbrace{dy = ds\ \cos\theta}$; $dz = ds\ \sin\theta$

Then $(p_y - p_s)\,\cancel{dx}\,\cancel{dz} = \frac{1}{2}\rho\,dy\,\cancel{dx}\,\cancel{dz}\,\overset{=0\ \text{static!!}}{\cancel{a_y}}$

$$\boxed{\therefore\ p_y = p_s}$$

In limit as $dx\,dy\,dz \rightarrow 0$

SUM FORCES IN -z- DIRECTION

$$\Sigma F_z = p_z\,dx\,dy - p_s\,dx\,\overbrace{ds\ \cos\theta} - \tfrac{1}{2}\gamma\,dx\,dy\,dz = \rho\,\tfrac{1}{2}\,dx\,dy\,dz\,a_z$$

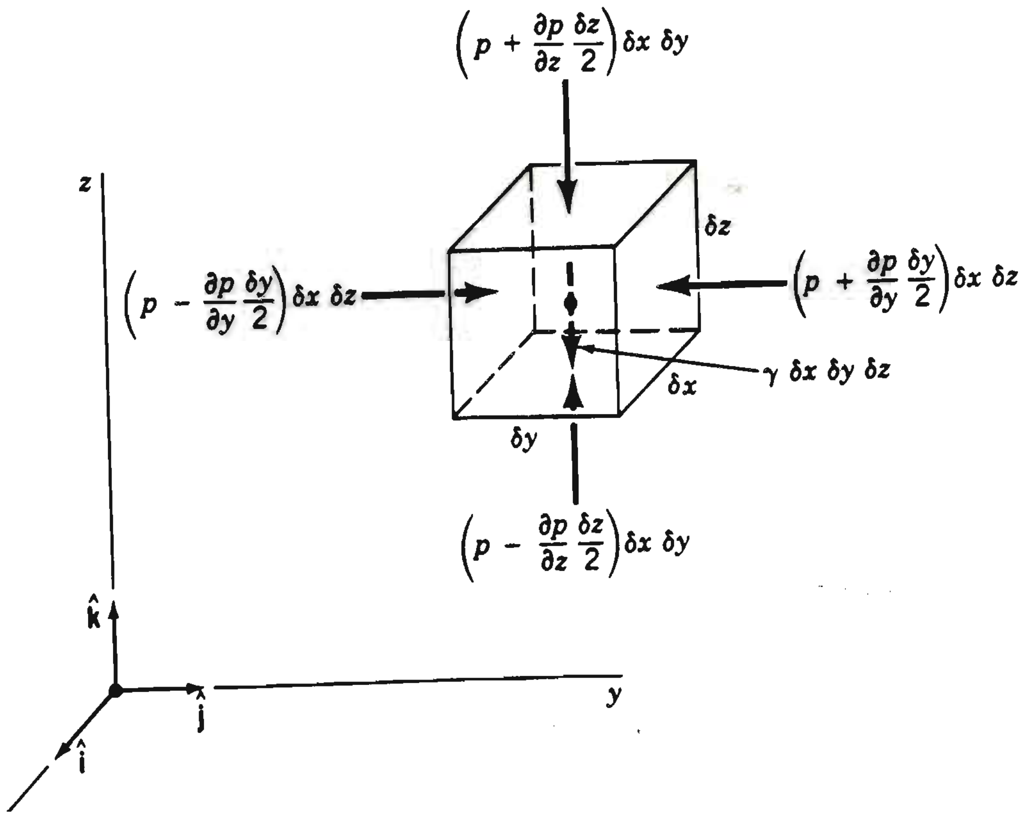
Then $(p_z - p_s)\,\cancel{dx}\,\cancel{dy} = \frac{1}{2}\left(\rho\,\overset{=0}{\cancel{a_z}} + \gamma\right)\cancel{dx}\,\cancel{dy}\ dz$

$$\boxed{\therefore\ p_z = p_s}$$

In limit as $dx\,dy\,dz \rightarrow 0$

CONCLUDE: UNIFORM PRESSURE @ POINT IN ALL DIRECTIONS IF $\tau = 0$. (Pascal's Law)

GENERAL PRESSURE FIELD DERIVATION



FORCE BALANCE IN -y- DIRECTION

$$\delta F_y = \left(p - \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z - \left(p + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z = \boxed{-\frac{\partial p}{\partial y}\,\delta x\,\delta y\,\delta z} \quad (1)$$

FORCE BALANCE IN -x- & -z- DIRECTIONS

$$\boxed{\delta \underline{F_x} = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z} \qquad \boxed{\delta F_z = -\frac{\partial p}{\partial z}\,\delta x\,\delta y\,\delta z} \quad (2)$$

Resulting force vector, $\delta \underline{F}_s$

$$\delta \underline{F}_s = \delta F_x\, \hat{i} + \delta F_y\, \hat{j} + \delta F_z\, \hat{k} \quad (3)$$

Substituting (1) and (2) into (3)

$$\delta \underline{F}_s = -\left[\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right]\delta x\, \delta y\, \delta z \quad (4)$$

Writing equn (4) in longhand:

$$\frac{1}{\delta x\, \delta y\, \delta z}\, \delta \underline{F}_s = -\frac{\partial p}{\partial x}\begin{Bmatrix}1\\0\\0\end{Bmatrix} - \frac{\partial p}{\partial y}\begin{Bmatrix}0\\1\\0\end{Bmatrix} - \frac{\partial p}{\partial z}\begin{Bmatrix}0\\0\\1\end{Bmatrix} \qquad (5)$$

This collapses to:

$$\frac{1}{\delta x\, \delta y\, \delta z}\, \delta \underline{F}_s = -\begin{Bmatrix}\frac{\partial p}{\partial x}\\ \frac{\partial p}{\partial y}\\ \frac{\partial p}{\partial z}\end{Bmatrix} = -\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix} p = -\underline{\nabla} p \qquad (6)$$

'del' operator

$\underline{\nabla}^T = [\partial/\partial x\,;\ \partial/\partial y\,;\ \partial/\partial z\,]$

Note that the vector $\delta \underline{F}_s$ contains force components in 3 orthogonal directions. Force per unit volume given as $\delta \underline{F}_s / (\delta x\, \delta y\, \delta z)$.

Now define body forces – due to weight of element $\delta x\, \delta y\, \delta z$.

$$-\delta W\, \hat{\underline{k}} = -\underbrace{\gamma\, \delta x\, \delta y\, \delta z}_{\delta W}\ \underset{\hat{\underline{k}}}{\begin{Bmatrix}0\\0\\1\end{Bmatrix}} = \text{vertical body force} \qquad (7)$$

-ve since -ve z dirn.

---

(6) defines surface (traction) force vector.

(7) defines body force vector.

Invoke Newton's 2nd Law. $\delta F = \delta m\, a$

In vector notation ($F=ma$)

$$\Sigma\, \delta \underline{F} = \delta m\, \underline{a} \quad \Rightarrow \quad \Sigma\,\delta \begin{Bmatrix}F_x\\F_y\\F_z\end{Bmatrix} = \delta m \begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix} \qquad (8)$$

Mass of cubic element, $\delta m = \rho\,\delta x\,\delta y\,\delta z$ (9)

Substituting (6) and (7) into (8)

$$\Sigma\,\delta\underline{F} = \delta\underline{F}_s - \delta W\,\hat{\underline{k}} = \delta m\,\underline{a} \qquad (10)$$

(6) (7)

$$-\underline{\nabla} p\, \cancel{\delta x\,\delta y\,\delta z} - \gamma\, \hat{\underline{k}}\,\cancel{\delta x\,\delta y\,\delta z} = \rho\,\underline{a}\,\cancel{\delta x\,\delta y\,\delta z} \qquad (2.2)$$

Longhand form

$$-\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{Bmatrix} p - \gamma \begin{Bmatrix}0\\0\\1\end{Bmatrix} = \rho \begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix} \qquad (2.2)$$

This is the general equation for motion, $-\underline{\nabla} p - \gamma\hat{\underline{k}} = \rho\underline{a}$

Fluid @ rest, $a_x = a_y = a_z = 0$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\gamma$$

QED.

PRESSURE AT A POINT

INCOMPRESSIBLE FLUID

$$\frac{dp}{dz} = -\gamma$$

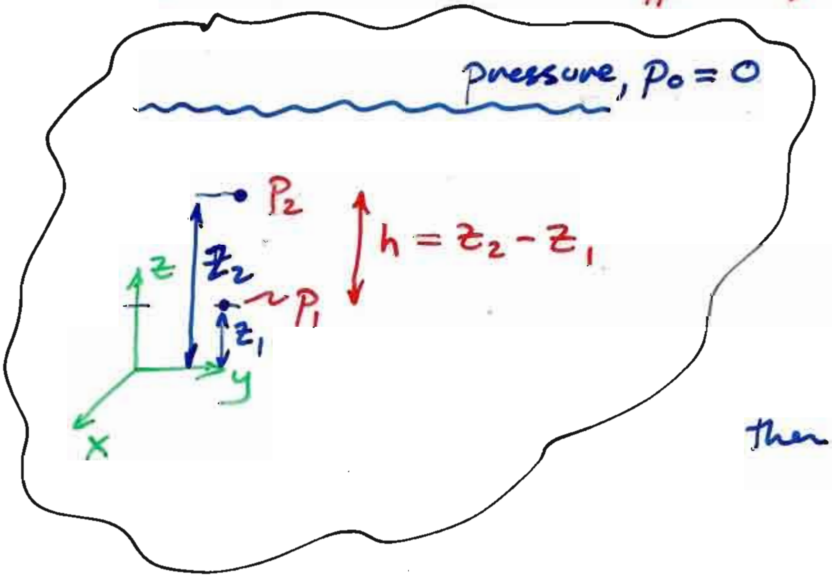
Integrate as:

$$\int_{p_1}^{p_2} dp = -\int_{z_1}^{z_2} \gamma \, dz$$

Insert limits as: $p_2 - p_1 = -\gamma(z_2 - z_1)$

or: $p_1 = \underbrace{\gamma(z_2 - z_1)}_{h} + p_2$

Static head or head difference, h.



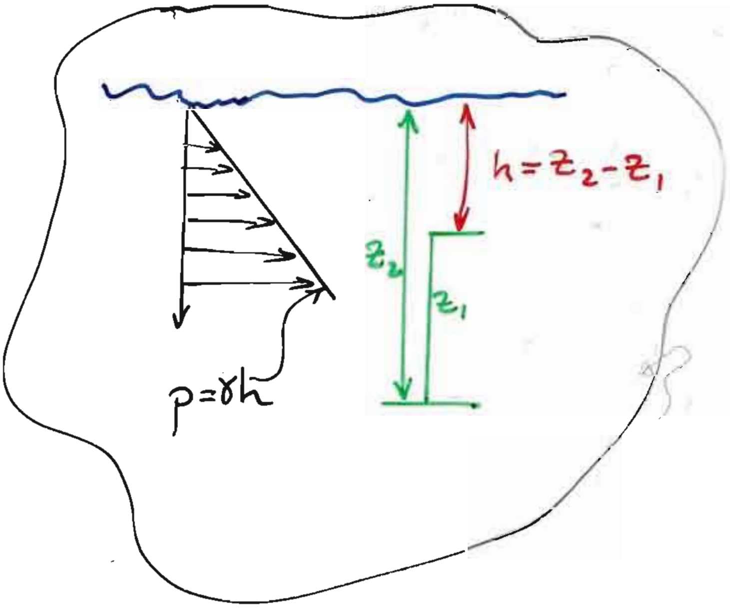
$$\boxed{p_1 = \gamma h + p_2}$$

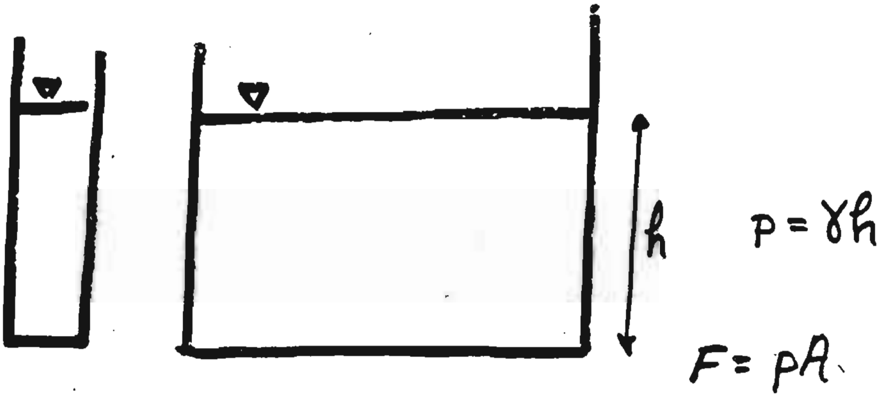
Set p_2 @ z_2 on surface $\cancelto{0}{p_2}$

then $p_1 = \gamma h + \cancelto{0}{p_2}$

Note change in p is linear with depth.

Pressure head; $h = \frac{p}{\gamma}$





The *hydrostatic paradox* is illustrated by figure 3.8. The pressure anywhere on the bottom of either container is the same. This pressure is dependent on only the maximum height of the fluid, not the volume.

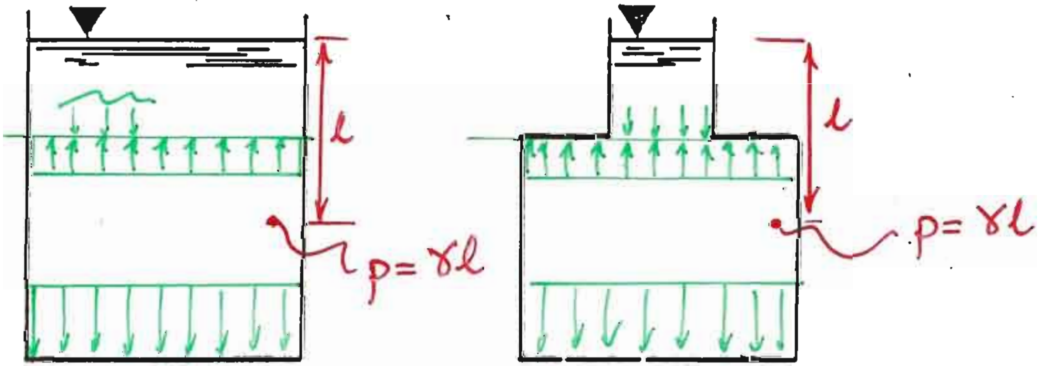
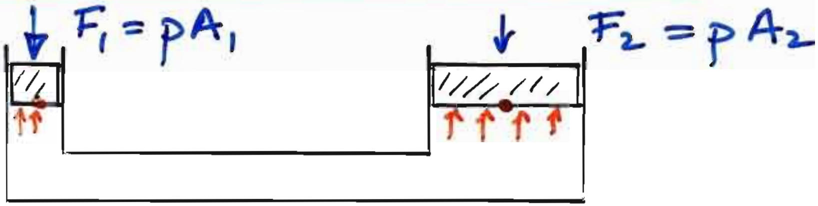


Figure 3.8 Hydrostatic Paradox

Pressure is NOT a function of the SHAPE of the tank. Only the depth below surface.

(For static fluids)!!

MECHANICAL (FLUID?) ADVANTAGE



Since p is uniform then

$$\frac{F_1}{A_1} = p = \frac{F_2}{A_2}$$

$$F_1 = \left(\frac{A_1}{A_2}\right) F_2 \quad \text{or} \quad F_2 = \left(\frac{A_2}{A_1}\right) F_1$$

$A_2/A_1 > 1 \quad \therefore \; F_2 > F_1$

[2:2] Fluid Statics

Recap

Fluid pressure at a point (static) $p_x = p_y = p_z = p$

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \mathbf{a} \text{ or } -\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} p - \gamma \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \rho \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

Simplifies when: $a_x = a_y = a_z = 0$ to $\dfrac{dp}{dx} = \dfrac{dp}{dy} = 0$ and $\dfrac{dp}{dz} = -\gamma$

Incompressible (water) Incompressible fluid: $p = \gamma h + p_0$

Compressible (atmosphere) Compressible fluid: $\dfrac{dp}{dz} = -\dfrac{gp}{RT}$ and integrate w.r.t (p, z).

Outline

Pressure measurement (manometry)

Manometer rules: $(\uparrow -ve)(\downarrow +ve); \dfrac{dp}{dx} = \dfrac{dp}{dy} = 0; p_v$ if evacuated; $\gamma_{gas} \to 0.$

PRESSURE AT A POINT (Cont'd)

COMPRESSIBLE FLUID – GASES

$$\rho \text{ or } \gamma = f(p, T)$$

Important in column.

General equation: $\boxed{\dfrac{dp}{dz} = -\gamma}$

dp is small, even for large dz since γ_{gas} is small!!

$\gamma_{water} = 62.4 \text{ lb/ft}^3$

$\gamma_{air} = 0.0763 \text{ lb/ft}^3$

For large height variations (dz large)

$$\underbrace{p = \rho R T \quad ; \quad \frac{dp}{dz} = -\gamma = -\rho g}$$

$$-\rho g = -\frac{pg}{RT}$$

$$\boxed{\frac{-pg}{RT} = \frac{dp}{dz}}$$

Separate variables: $\displaystyle\int_{z_1}^{z_2} \frac{-g}{RT}\,dz = \int_{P_1}^{P_2} \frac{dp}{P}$

$$-\frac{g}{R}\int_{z_1}^{z_2} \frac{dz}{T} = (\ln p_2 - \ln p_1) = \ln\left(\frac{P_2}{P_1}\right)$$

How does temperature vary, T.

If T is constant in the range $z_1 \rightarrow z_2$ (ISOTHERMAL)

$$T = T_0$$

Then:

$$-\frac{g}{RT_0}\int_{z_1}^{z_2} dz = \ln\left(\frac{P_2}{P_1}\right)$$

$$\exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] = \exp\left[\ln\left(\frac{P_2}{P_1}\right)\right]$$

$$\boxed{P_2 = P_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right]}$$

Q.E.D.

U.S. STANDARD ATMOSPHERE

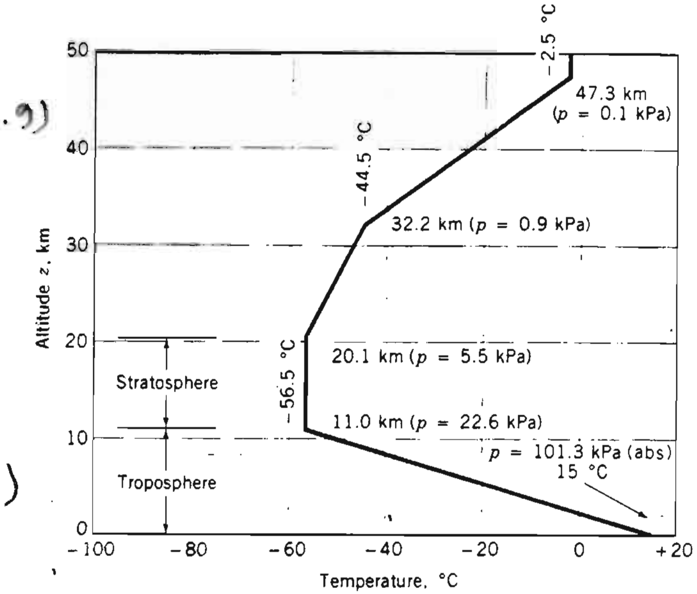
Temperature defined $T = f(z)$

Use $$\int_{P_1}^{P_2} \frac{dp}{p} = \ln\frac{P_2}{P_1} = -\frac{g}{R}\int_{z_1}^{z_2}\frac{1}{T}\,dz \quad (2.9)$$

$T =$ not constant.

If $T = T_a - \beta z$ (2.11)

Then $$\boxed{p = p_a\left(1 - \frac{\beta z}{T_a}\right)^{g/R\beta}} \quad (2.12.)$$



p_a = absolute pressure @ $z = 0$

■ TABLE 2.1

Properties of U.S. Standard Atmosphere at Sea Level[a]

Property	SI Units	BG Units
Temperature, T	288.15 K (15 °C)	518.67 °R (59.00 °F)
Pressure, p	101.33 kPa (abs)	2116.2 lb/ft^2 (abs) [14.696 lb/in.^{2} (abs)]
Density, ρ	1.225 kg/m^3	0.002377 slugs/ft^3
Specific weight, γ	12.014 N/m^3	0.07647 lb/ft^3
Viscosity, μ	1.789×10^{-5} N·s/m^2	3.737×10^{-7} lb·s/ft^2

[a] Acceleration of gravity at sea level = 9.807 m/s^2 = 32.174 ft/s^2.

SEE TABLES C.1 and C.2

Absolute and Gage Pressures (Section 2.4)

If pressure is measured relative to absolute zero (a perfect vacuum) it is called absolute pressure.

If pressure is measured relative to atmospheric pressure as a base it is called gage pressure.

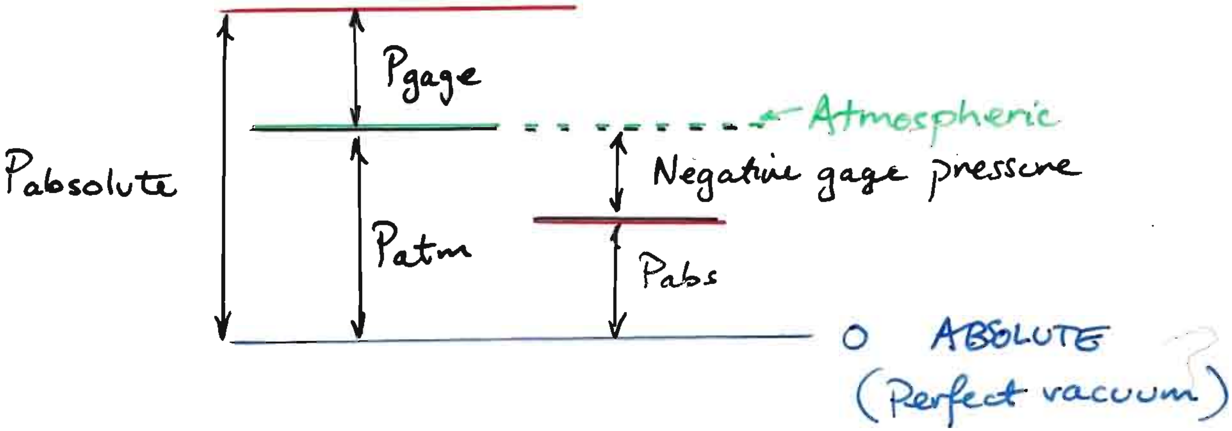
$$p\ (\text{abs}) = p\ (\text{atm}) + p\ (\text{gage})$$

p (gage) may be positive or negative (vacuum) atmospheric pressure is also called barometric pressure.

At sea level atmospheric pressure may be expressed as:

14.7 psia	or	101.3 kN/m^2 (abs) [1,013 mbars,* abs]
29.9 in of Hg	or	750 mm of Hg (0.76 m of Hg)
33.9 ft of H_20	or	10.3 m of H_20

*1 millibar = 100 N/m^2 = 0.1 kN/m^2

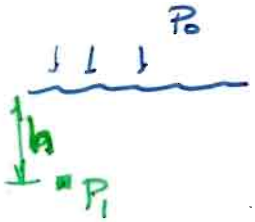


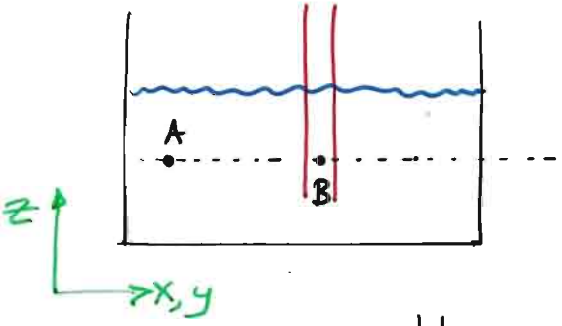
WHAT DO WE KNOW ?

$p_x = p_y = p_z$ @ a point

$$\frac{dp}{dx} = \frac{dp}{dy} = 0 \qquad \frac{dp}{dz} = -\gamma$$

$$p_1 = \gamma h + p_0$$

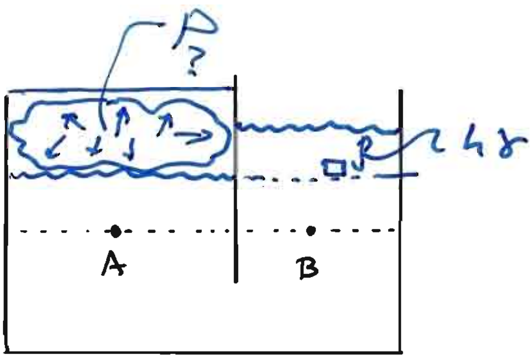




$$\frac{dp}{dx} = \frac{dp}{dy} = 0$$

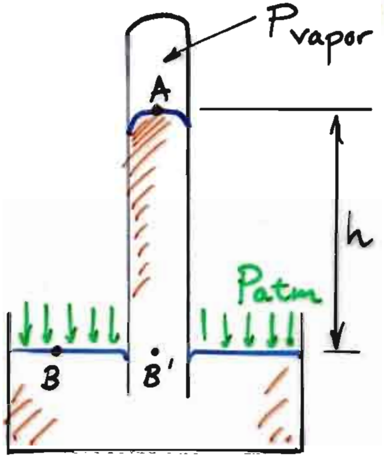
$P_A = P_B$!!

?



$P_A = P_B$?

PRESSURE MEASUREMENT — MERCURY BAROMETER



(Evacuated crest)

$$p = \gamma h + p_o$$

Pressure @ $B \equiv B'$

$$P_{atm} = \gamma h + P_{vapor}$$

Mercury used:

1. High density, ρ, ∴ small column, h
2. Vapor pressure small, $P_{vapor} = 2.3 \times 10^{-5}$ psia @ 68°F

$$\therefore \quad \boxed{P_{atm} \simeq \gamma h}$$

$$\frac{P_{atm}}{\gamma} \simeq 29.9 \text{ in Hg} \quad (\text{or } 34\text{ft water})$$

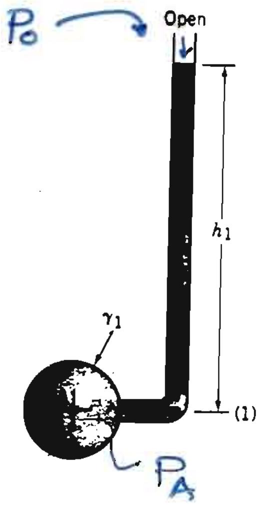
MANOMETRY – Static columns of liquids to measure pressures (gage).

PIEZOMETER

$$p = \gamma h + \cancel{p_o}$$

gage ≡ 0

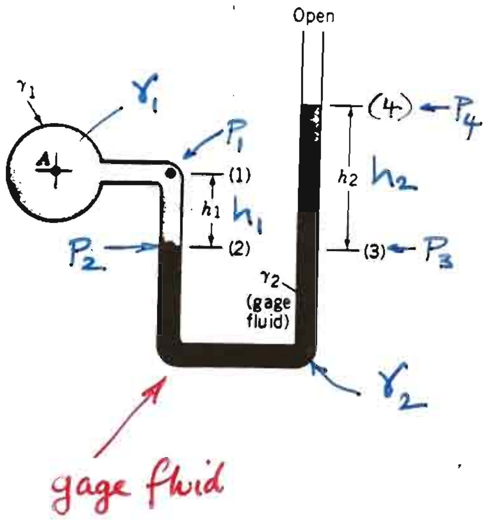
$$\boxed{P_A = \gamma h_1}$$



- ☐ Cannot measure tension/suction
- ☐ Limited to low pressures (groundwater)
- ☐ Container fluid cannot be gas

U-TUBE MANOMETER:

$$p = \gamma h + p_o$$



AT POINT (1)

$$P_A = P_1 = -\gamma_1 h_1 + P_2 \quad (I)$$

AT POINT (2)

$$p_2 = p_3 = +\gamma_2 h_2 + \cancel{p_4}^{\,0} \quad (II)$$

Combining equations (I) and (II)

$$\boxed{p_1 = -\gamma_1 h_1 + \gamma_2 h_2} \equiv P_A$$

- ☐ Gage fluid may be different from measured fluid/gas

If fluid (1) is gas then $\gamma_1 \ll \gamma_2$ and $\boxed{p_1 = \gamma_2 h_2}$

Choose gage fluid for γ_2 → Sensitivity/readability.

MANOMETER RULES:

1. Subtract when you go up.

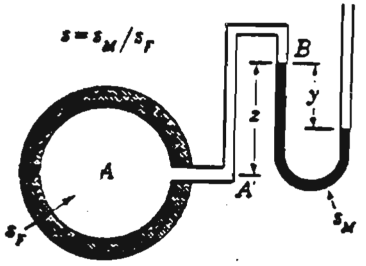
2. Add when you go down.

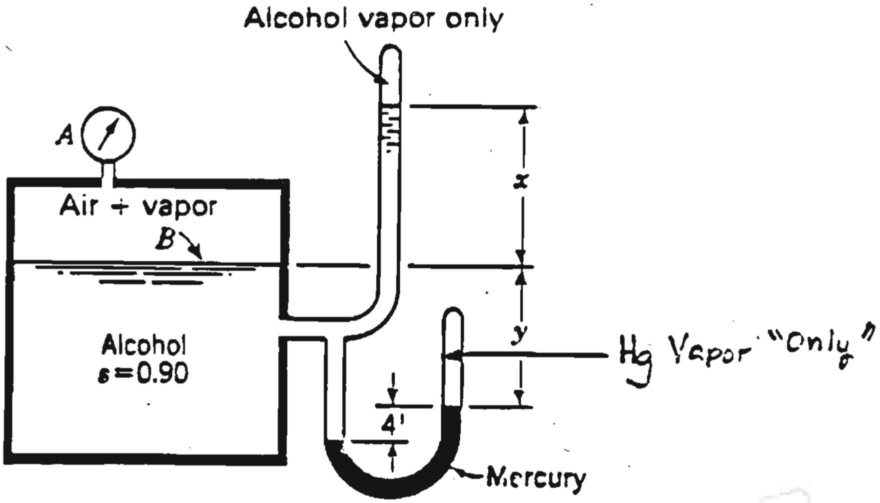
3. The first term is 0 for gage pressure; atmospheric pressure for absolute pressure.

4. In a closed vessel, the only pressure at the fluid surface is due to vapor pressure

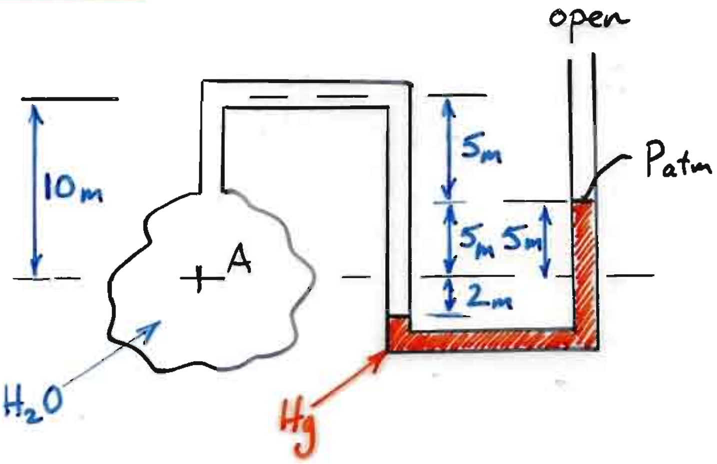
5. When measuring the pressure of a gas the "z" term may be neglected because of the relatively small specific gravities of gases.

Note: Many factors influence the accuracy of manometers, including capillarity and friction in the tube. However, they are neglected in most applications of engineering hydraulics. These factors are only significant when precise readings are required, such as in laboratories.





EXAMPLE



FROM TABLE 1.6

$\gamma_w = 9.8 \ KN/m^3$

$\gamma_{Hg} = 133 \ KN/m^3$

Rules : (−ve ↑) (+ve ↓)

$$P_A \underbrace{-10\gamma_w + 5\gamma_w + 5\gamma_w}_{} + 2\gamma_w - 7\gamma_{Hg} = \cancelto{0}{P_{atm}}$$

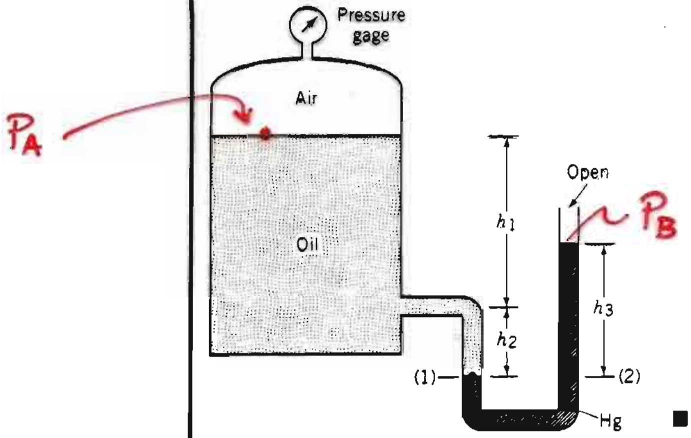
$$P_A + 2\gamma_w - 7\gamma_{Hg} = 0$$

$$P_A = -2\gamma_w + 7\gamma_{Hg} = -2(9.8) + 7(133) \ KPa$$

$$P_A = 911.4 \ KPa \equiv KN/m^2$$

EXAMPLE 2.4

A closed tank contains compressed air and oil ($SG_{\text{oil}} = 0.90$) as is shown in Fig. E2.4. A U-tube manometer using mercury ($SG_{\text{Hg}} = 13.6$) is connected to the tank as shown. For column heights $h_1 = 36$ in., $h_2 = 6$ in., and $h_3 = 9$ in., determine the pressure reading (in psi) of the gage.



■ FIGURE E2.4

Note :

$$\gamma_{oil} = \overset{0.9}{SG_{OIL}}\,\gamma_{H_2O}$$
$$\gamma_{Hg} = \underset{13.6}{SG_{Hg}}\,\gamma_{H_2O}$$

Start @ Air/Oil ; P_A :   ($\downarrow$ +ve) ; ($\uparrow$ −ve)

$$P_A + (h_1 + h_2)\gamma_{OIL} - h_3\gamma_{Hg} = \cancel{P_B} \quad \text{atmospheric} = 0 \qquad (1)$$

Rearrange:

$$P_A = [h_3 SG_{Hg} - (h_1 + h_2) SG_{OIL}]\gamma_{H_2O} \qquad (2)$$

Datum, P_A in psi ∴ Keep units for length in "in"
convert pcf to psi

$$62.4 \frac{lb}{ft^3} = \frac{62.4\ lb}{12\times12\times12\ in^3} \qquad 1ft^3 = 12\times12\times12\ in^3$$

Substituting into (2)

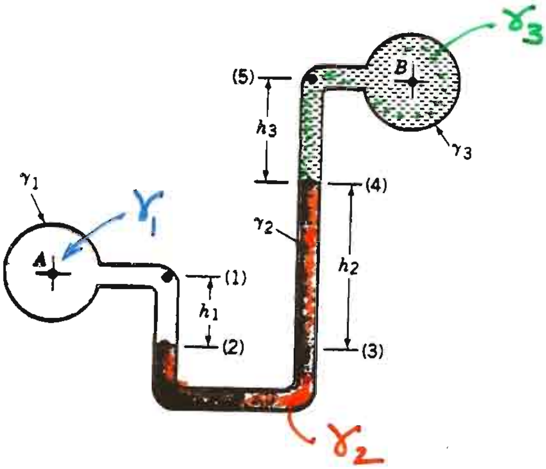
$$P_A = [(9)\ 13.6 - (36+6)\ 0.9]\ \frac{62.4}{12\times12\times12} = 3.06\ psi$$

UNITS: in   dimensionless $\frac{lb}{in^3} = 3.06 \frac{lb}{in^2}$

"DIFFERENTIAL" MANOMETER

Measures "difference" in pressures between 'A' and 'B'.

(−ve ↑)   (+ve ↓)



$$p_A + \gamma_1 h_1 - \gamma_2 h_2 - \gamma_3 h_3 = p_B$$

$$\underbrace{p_A - p_B}_{\text{"differential" pressure}} = -\gamma_1 h_1 + \gamma_2 h_2 + \gamma_3 h_3$$

CAPILLARITY

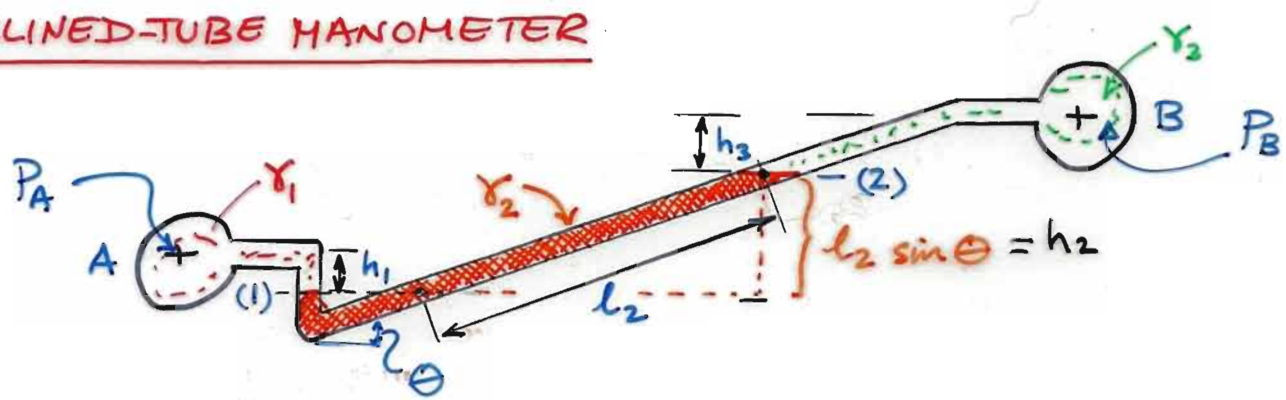
Reduce influence by:

- ☐ Two tubes ∴ effects cancel
- ☐ Large bore tubes.

DESIRED GAGE FLUID PROPERTIES

- ☐ Definite meniscus (Hg & dyed water)
- ☐ Immiscible

INCLINED-TUBE MANOMETER



☐ To increase sensitivity of instrument.
Δl, more sensitive than Δh.

$$P_A + \gamma_1 h_1 - \gamma_2 \underbrace{l_2 \sin\theta}_{h_2} - \gamma_3 h_3 = P_B$$

$$\boxed{P_A - P_B = \gamma_2 l_2 \sin\theta + \gamma_3 h_3 - \gamma_1 h_1}$$

If γ_1 and γ_3 gases then $\gamma_1 \ll \gamma_2$

$\gamma_3 \ll \gamma_2$

$$\boxed{P_A - P_B = \gamma_2 l_2 \sin\theta}$$

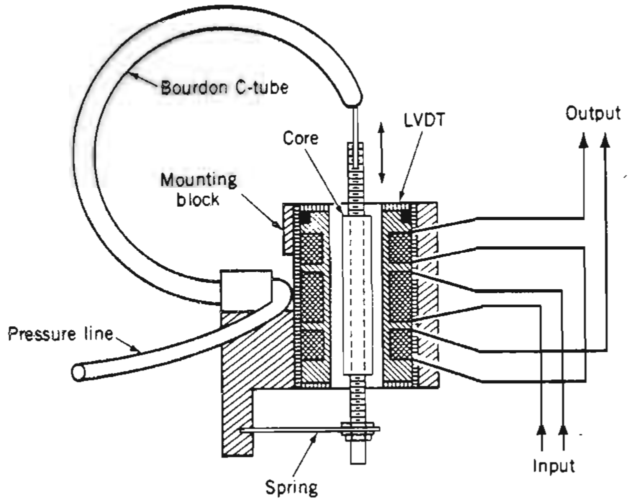
Flatten the tube $(\theta \to 0)$ $\therefore$ ↑ sensitivity

PRESSURE MEASUREMENT - Mechanical & Electronic devices.

Basic principle: Change in pressure on structure → deformation
→ pressure calibration.

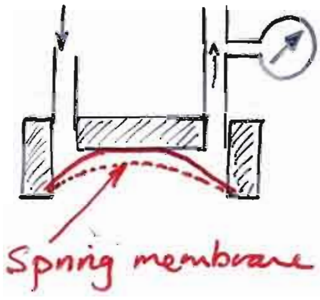
Bourdon Gage (Mechanical)

- Transient Δp difficult to measure due to volume of 'C' tube.



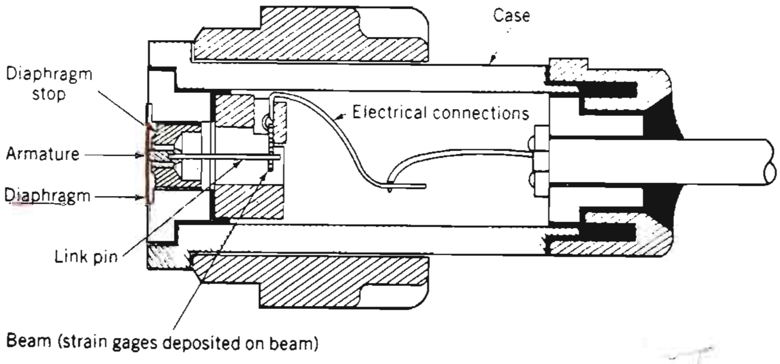
Pneumatic piezometer (Mechanical)

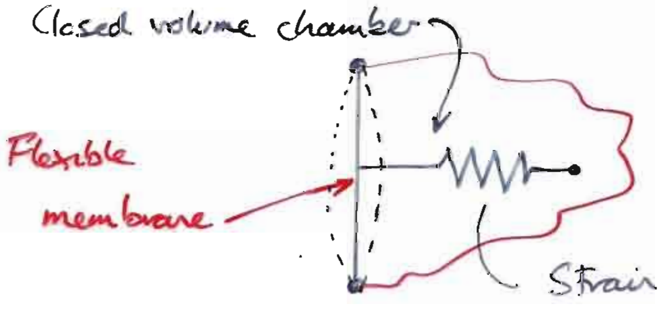
- Measure pressures remotely.



Pressure transducer

- Aneroid barometer (Elec.)





Strain gage (a) resistance changes with strain
(b) frequency changes with strain

Tire pressure gage (Mechanical)

[2:3] Fluid Statics

Recap

Fluid pressure at a point (static) $p_x = p_y = p_z = p$

When: $a_x = a_y = a_z = 0$ then $\frac{dp}{dx} = \frac{dp}{dy} = 0$ and $\frac{dp}{dz} = -\gamma$

Incompressible (water) Incompressible fluid: $p = \gamma h + p_0$

Compressible (atmosphere) Compressible fluid: $\frac{dp}{dz} = -\frac{gp}{RT}$ and integrate w.r.t (p,z).

Pressure measurement (manometry)

Manometer rules: $(\uparrow -ve)(\downarrow +ve); \frac{dp}{dx} = \frac{dp}{dy} = 0; p_v$ if evacuated; $\gamma_{gas} \to 0$.

Examples

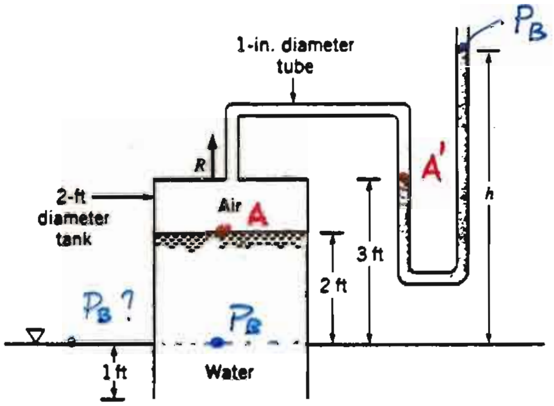
- Suction
- Micromanometry
- Mercury manometry
- Uniform pressure – step
- "Slightly" compressible fluids

2.30

2.30 An inverted open tank is held in place by a force R as shown in Fig. P2.30. If the specific gravity of the manometer fluid is 2.5, determine the value of h.

Gas (air) $\therefore \gamma_{AIR} \rightarrow 0$

$\therefore P_A = P_{A'}$



■ FIGURE P2.30

Area, A_T

$$\gamma_{gf}\,(h - 3\,ft) + \gamma_{H_2O}\,(2\,ft) = 0$$

Thus,

$$h = 3\,ft - \frac{\gamma_{H_2O}\,(2\,ft)}{(SG)\,\gamma_{H_2O}}$$

$$= 3\,ft - \frac{2\,ft}{2.5}$$

$$= \underline{\underline{2.20\,ft}}$$

At A': $\quad P_{AIR} + (3\,ft - h)\,\gamma_{gf} = \cancel{P_B}^{0}$

$P_{AIR} = -(3\,ft - h)\,\gamma_{gf} \qquad (1)$

At A: $\quad P_{AIR} + 2\,ft\,\gamma_w = \cancel{P_B}^{0}$

$P_{AIR} = -2\,ft\,\gamma_w \qquad (2)$

Equating $(1) \equiv (2) \Rightarrow \quad h = 3\frac{\cancel{\gamma_{gf}}}{\cancel{\gamma_{gf}}} - 2\frac{\gamma_w}{\gamma_{gf}} = 3 - 2\left(\frac{1}{2.5}\right) = 2.2\,ft$

What is force, R? $\qquad R = (2\,ft)(A_T)\,\gamma_w = 2\frac{\pi d^2}{4}\gamma_w$

2.36

2.36 Small differences in gas pressures are commonly measured with a *micromanometer* of the type illustrated in Fig. P2.36. This device consists of two large reservoirs each having a cross-sectional area, A_r, which are filled with a liquid having a specific weight, γ_1, and connected by a U-tube of cross-sectional area, A_t, containing a liquid of specific weight, γ_2. When a differential gas pressure, $p_1 - p_2$, is applied a differential reading, h, develops. It is desired to have this reading sufficiently large (so that it can be easily read) for small pressure differentials. Determine the relationship between h and $p_1 - p_2$ when the area ratio A_t/A_r is small, and show that the differential reading, h, can be magnified by making the difference in specific weights, $\gamma_2 - \gamma_1$, small. Assume that initially (with $p_1 = p_2$) the fluid levels in the two reservoirs are equal.

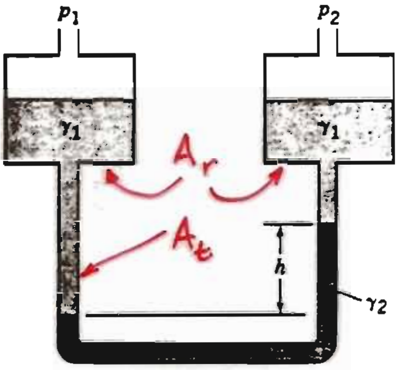
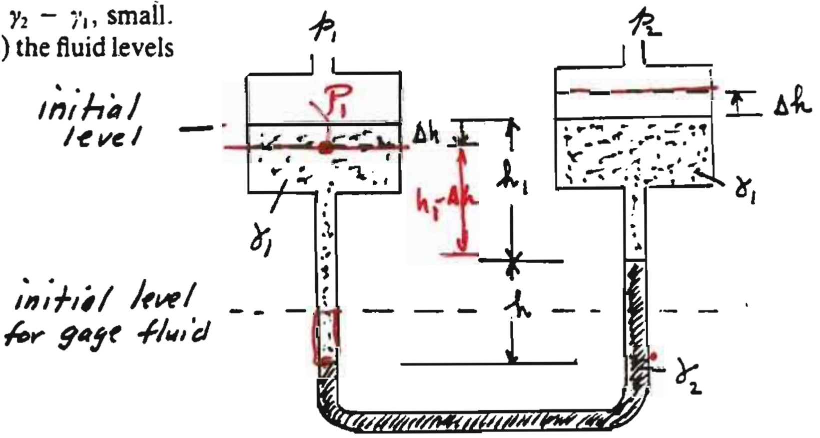


FIGURE P2.36



When a differential pressure, $p_1 - p_2$, is applied we assume that level in left reservoir drops by a distance, Δh, and right level rises by Δh. Thus, the manometer equation becomes

$$p_1 + \gamma_1 (h_1 + h - \Delta h) - \gamma_2 h - \gamma_1 (h_1 + \Delta h) = p_2$$

or

$$p_1 - p_2 = \gamma_2 h - \gamma_1 h + \gamma_1 (2 \Delta h) \qquad (1)$$

Since the liquids in the manometer are incompressible,

$$\Delta h \, A_r = \frac{h}{2} A_t \qquad \text{or} \qquad \frac{2 \Delta h}{h} = \frac{A_t}{A_r}$$

and if $\frac{A_t}{A_r}$ is small Then $2 \Delta h << h$ and last term in Eq.(1) can be neglected. Thus,

$$p_1 - p_2 = (\gamma_2 - \gamma_1) h$$

or

$$\underline{\underline{h = \frac{p_1 - p_2}{\gamma_2 - \gamma_1}}}$$

and large values of h can be obtained for small pressure differentials if $\gamma_2 - \gamma_1$ is small.

Note, however; If $\gamma_2 = \gamma_1$, then:

$$h = \frac{P_1 - P_2}{\cancel{(\gamma_2 - \gamma_1)}^{\,0}} \qquad h \rightarrow \infty$$

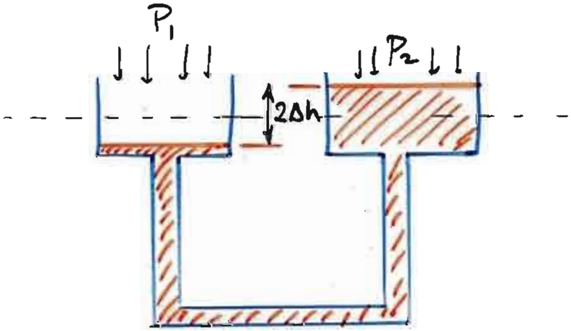
Suggests that system with one fluid is infinitely sensitive !!

Doesn't make sense. In the limit, the full form of the equation must be used.

$$p_1 - p_2 = (\gamma_2 - \gamma_1)h + \gamma_1 2\Delta h$$

if $\gamma_1 = \gamma_2$ then: $p_1 - p_2 = \gamma_1 2\Delta h$ NOT !!

$$\therefore \quad \frac{p_1 - p_2}{2\Delta h} = \gamma_1 \quad (\text{or } \gamma_2)$$



Which makes sense !!

2.38

2.38 A mercury manometer is used to measure the pressure difference in the two pipelines of Fig. P2.38. Fuel oil (specific weight = 53.0 lb/ft³) is flowing in *A* and SAE 30 lube oil (specific weight = 57.0 lb/ft³) is flowing in *B*. An air pocket has become entrapped in the lube oil as indicated. Determine the pressure in pipe *B* if the pressure in *A* is 15.3 psi.

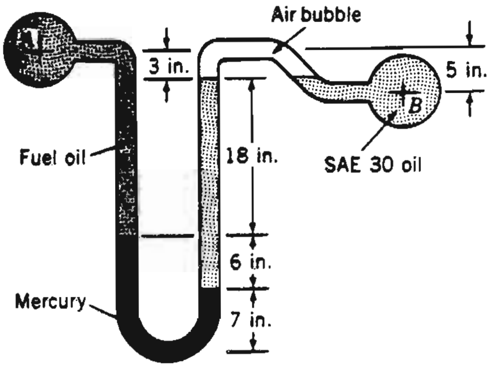


FIGURE P2.38

$$p_A + \gamma_{\substack{fuel\\oil}}\left(\frac{3+18}{12}\text{ ft}\right) + \gamma_{Hg}\left(\frac{6}{12}\text{ ft}\right) - \gamma_{SAE30}\left(\frac{6+18}{12}\text{ ft}\right) + \gamma_{SAE30}\left(\frac{2}{12}\text{ ft}\right) = p_B$$

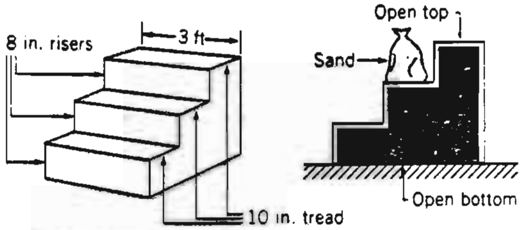
Thus,

$$p_B = \left(15.3\,\frac{lb}{in.^2}\right)\left(144\,\frac{in^2}{ft^2}\right) + \left(53.0\,\frac{lb}{ft^3}\right)\left(\frac{21}{12}\text{ ft}\right) + \left(847\,\frac{lb}{ft^3}\right)\left(\frac{6}{12}\text{ ft}\right) - \left(57.0\,\frac{lb}{ft^3}\right)\left(\frac{22}{12}\text{ ft}\right)$$

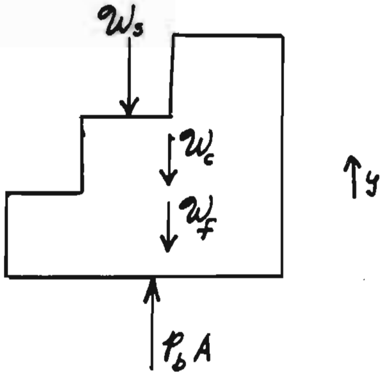
$$= 2615\,\frac{lb}{ft^2} = \left(2615\,\frac{lb}{ft^2}\right)\left(\frac{1\text{ ft}^2}{144\text{ in.}^2}\right) = \underline{\underline{18.2\text{ psi}}}$$

2.51

2.51 Concrete is poured into the forms as shown in Fig. P2.51 to produce a set of steps. Determine the weight of the sandbag needed to keep the bottomless forms from lifting off the ground. The weight of the forms is 85 lb, and the specific weight of the concrete is 150 lb/ft³.



■ FIGURE P2.51



From the free-body-diagram

$$\left(\downarrow +\right) \Sigma F_y = 0$$

$$\mathcal{W}_s + \mathcal{W}_c + \mathcal{W}_f - p_b A = 0 \qquad (1)$$

Where:

$\mathcal{W}_s$ = weight of sandbag

$\mathcal{W}_c$ = weight of concrete

$\mathcal{W}_f$ = weight of forms

p_b = pressure along bottom surface due to concrete

A = area of bottom surface

From the data given:

$$\mathcal{W}_c = \left(150 \frac{lb}{ft^3}\right)(\text{Vol. concrete})$$

$$= \left(150 \frac{lb}{ft^3}\right)(3 ft) \frac{\left[(10 in.)(24 in.) + (10 in.)(16 in.) + (10 in.)(8 in.)\right]}{144 \frac{in.^2}{ft^2}}$$

$$= 1500 \; lb$$

$$\mathcal{W}_f = 85 \; lb$$

$$pA_b = \left(150 \frac{lb}{ft^3}\right)\left(\frac{24}{12} ft\right) = 300 \frac{lb}{ft^2}$$

$$A = \left(\frac{30}{12} ft\right)(3 ft) = 7.5 ft^2$$

Thus, from Eq. (1)

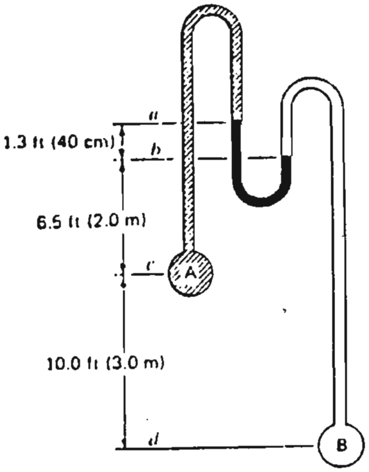
$$\mathcal{W}_s = \left(300 \frac{lb}{ft^2}\right)(7.5 ft^2) - 1500 \; lb - 85 \; lb$$

$$= \underline{\underline{665 \; lb}}$$

Example 2.2 (pg. 37)

Given: Liquid A weighs 53.5 lb/ft^3. Liquid B weighs 78.8 lb/ft^3. The manometer liquid is Mercury. The pressure at B is 30 psi. Express all pressure heads in terms of the liquid in bulb B.

Diagram:



Illustrative Example 2.2

pg. 37

solution on next page

Required: The pressure at A.

Solution: Pressure @ B = (30 lb/in^2 x 144 in^2/ft^2) $\div$ 78.8 lb/ft^3 = 54.8 ft.

Now work through manometer:

$54.8 - 16.5 - SZ$ (a to b) $+ S^1Z$ (a to c) $= P_A/\gamma_B$
$S = S(m)/S(B) = 13.56/(78.8/62.4) = 10.74$
$S^1 = S(A)/S(B) = \gamma(A)/\gamma(B) = 53.5/78.8 = 0.679$

$54.8 - 16.5 - 13.96 + 5.30 = 29.64$ ft

$P_A = 29.64$ ft $\gamma_B = 2335.6$ pcf $= 16.2$ psi

2.8

2.8 For the great depths that may be encountered in the ocean the compressibility of seawater may become an important consideration. **(a)** Assume that the bulk modulus for seawater is constant and derive a relationship between pressure and depth which takes into account the change in fluid density with depth. **(b)** Make use of part (a) to determine the pressure at a depth of 6 km assuming seawater has a bulk modulus of 2.3×10^9 Pa, and a density of 1030 kg/m^3 at the surface. Compare this result with that obtained by assuming a constant density of 1030 kg/m^3.

(a)

$$\frac{dp}{dz} = -\gamma = -\rho g \qquad (Eq.\ 2.4)$$

Thus,

$$\frac{dp}{\rho} = -g\,dz \qquad (1)$$

If ρ is a function of p, we must determine $\rho = f(p)$ before integrating Eq.(1). Since,

$$E_v = \frac{dp}{d\rho/\rho} \qquad (Eq.\ 1.13)$$

then

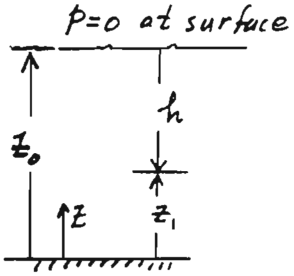
$$\int_0^p dp = E_v \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$$

so that

$$p = E_v \ln \frac{\rho}{\rho_0}$$

Thus,

$$\rho = \rho_0 e^{\frac{p}{E_v}} \quad \text{where } \rho = \rho_0 \text{ at } p = 0$$



From Eq.(1)

$$\int_{p_1}^{0} \frac{dp}{\rho_0 e^{\frac{p}{E_v}}} = -g \int_{z_1}^{z_0} dz$$

or

$$\int_{p_1}^{0} e^{-\frac{p}{E_v}}\,dp = -\rho_0 g \int_{z_1}^{z_0} dz$$

so that

$$\underline{\underline{p = -E_v \ln\left(1 - \frac{\rho_0 g h}{E_v}\right)}}$$

where, $h = z_0 - z_1$, the depth below surface

(con't)

2.8 (con't)

(b) From part (a),

$$p = -E_v \ln\left(1 - \frac{\rho_o g h}{E_v}\right)$$

so that at $h = 6\,km$

$$p = -\left(2.3\times10^9 \frac{N}{m^2}\right)\ln\left[1 - \frac{\left(1.03\times10^3 \frac{kg}{m^3}\right)\left(9.81 \frac{m}{s^2}\right)\left(6\times10^3 m\right)}{2.3\times10^9 \frac{N}{m^2}}\right]$$

$$= 6.14\times10^7 \frac{N}{m^2} = \underline{\underline{61.4\ MPa}}$$

(c) For constant density

$$p = \gamma h = \rho g h = \left(1.03\times10^3 \frac{kg}{m^3}\right)\left(9.81 \frac{m}{s^2}\right)\left(6\times10^3 m\right)$$

$$= \underline{\underline{60.6\ MPa}}$$

2.9

2.9 Blood pressure is usually given as the ratio of the maximum pressure (systolic pressure) to the minimum pressure (diastolic pressure). For example, a typical value for this ratio for a human would be 120/70, where the pressures are in mm Hg. What would these pressures be in pascals and in psi?

$$p = \gamma h$$

For 120 mm Hg: $p = \left(133\times10^3 \frac{N}{m^3}\right)(0.120 m) = \underline{\underline{16.0\ kPa}}$

or

$$p = \left(16.0\times10^3 \frac{N}{m^2}\right)\left(1.450\times10^{-4} \frac{\frac{lb}{in.^2}}{\frac{N}{m^2}}\right) = \underline{\underline{2.32\ psi}}$$

For 70 mm Hg: $p = \left(133\times10^3 \frac{N}{m^3}\right)(0.070 m) = \underline{\underline{9.31\ kPa}}$

or

$$p = \left(9.31\times10^3 \frac{N}{m^2}\right)\left(1.450\times10^{-4} \frac{\frac{lb}{in.^2}}{\frac{N}{m^2}}\right) = \underline{\underline{1.35\ psi}}$$

[3]

Fluid Pressure & Buoyancy

Fluid Statics [2,3]

$$p_x = p_y = p_z = p$$

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \mathbf{a} \text{ or } -\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} p - \gamma \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \rho \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

Simplifies when: $a_x = a_y = a_z = 0$ to $\dfrac{dp}{dx} = \dfrac{dp}{dy} = 0$ and $\dfrac{dp}{dz} = -\gamma; \dfrac{dz}{dx} = -\dfrac{a_x}{g + a_z}$

Rigid body rotation: $\dfrac{\partial p}{\partial r} = \rho r \omega^2; \dfrac{\partial p}{\partial \theta} = 0; \dfrac{\partial p}{\partial z} = -\gamma; \begin{cases} z = \frac{\omega r^2}{2g} + c \\ p = \frac{p\omega^2 r^2}{2} - \gamma z + c \end{cases}$

Incompressible fluid: $p = \gamma h + p_0$

Compressible fluid: $\dfrac{dp}{dz} = -\dfrac{gp}{RT}$ and integrate w.r.t (p, z).

Manometer rules: $(\uparrow -ve)(\downarrow +ve); \dfrac{dp}{dx} = \dfrac{dp}{dy} = 0; p_v$ if evacuated; $\gamma_{gas} \to 0$.

$$F_R = \gamma h_c A; \quad F_R = \sqrt{F_H^2 + F_V^2}$$

F_R acts through center of pressure $\begin{cases} y_R = \dfrac{I_{xc}}{y_c A} + y_c \\ x_R = \dfrac{I_{xyc}}{y_c A} + x_c \end{cases}$

$$F_B = \gamma \cancel{V}$$

[3:1] Fluid Statics & Pressures

Recap

Pressure measurement (manometry)

$$\text{Manometer rules: } (\uparrow -ve)(\downarrow +ve); \frac{dp}{dx} = \frac{dp}{dy} = 0; p_v \text{ if evacuated; } \gamma_{gas} \to 0.$$

Outline

Pressures on structures – why should we care?

Disasters: Malpasset; Vaiont disasters; Dambusters

Routine: Turbines (gas, liquid) – but not static

How big are the forces? $F_R = \gamma h_c A$

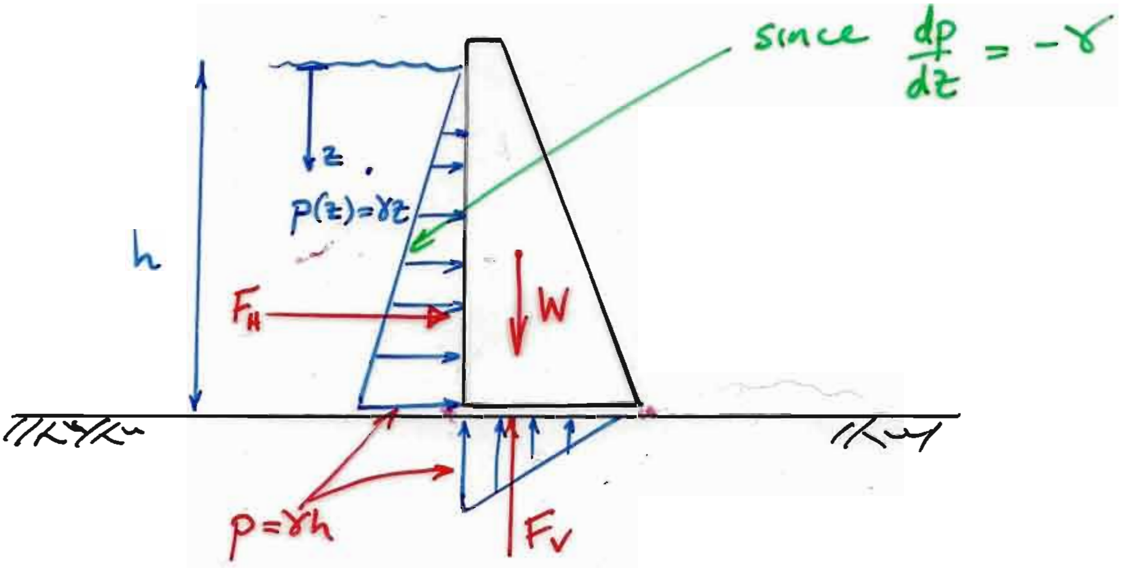
How do we resolve them? $F_R = \sqrt{F_H^2 + F_V^2}$

Where do they act?

$$F_R \text{ acts through center of pressure } \begin{cases} y_R = \dfrac{I_{xc}}{y_c A} + y_c \\ x_R = \dfrac{I_{xyc}}{y_c A} + x_c \end{cases}$$

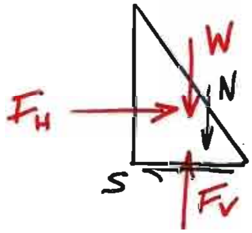
HYDROSTATIC FORCES ACTING ON SURFACES.

Why?



Questions:

1. What is magnitude of force?



$$S = N \tan\phi$$

Resolve vertically: $N = W - F_V$

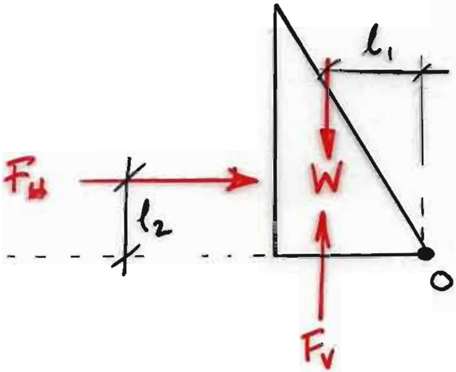
Strength of base $= S = (W - F_V)\tan\phi$

if $F_H > S$ Translational failure.

$F_H \leq S$ Stable equilibrium.

∴ Need to know force magnitudes!!

2. Where does the force $\underset{\approx}{\underline{\text{act}}}$?



$$\overset{\curvearrowright}{\sum} M_o = 0$$

$$F_H \ell_2 - (W - F_V)\ell_1 = 0$$

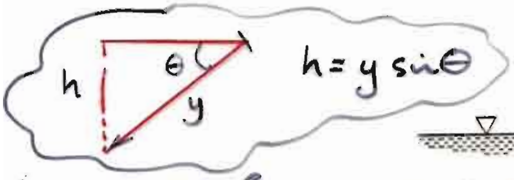
$$F_H = (W - F_V)\frac{\ell_1}{\ell_2}$$

If F_H is larger than this magnitude limit then the dam "fails" by overturning.

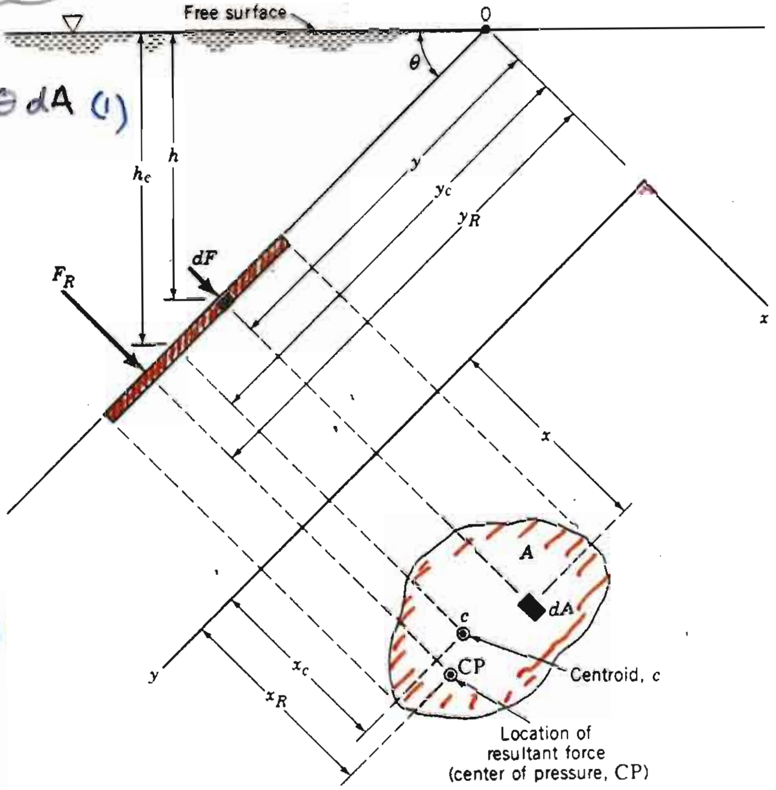
∴ Need to know where the forces $\underset{\approx}{\underline{\text{act}}}$!!

HYDROSTATIC FORCE ON PLANE SURFACE

$$F_R = \int p\, dA = \int \gamma h\, dA$$



$$h = y \sin\theta$$



$$F_R = \int_A \gamma h\, dA = \int_A \gamma y \sin\theta\, dA \quad (1)$$

Assuming γ; θ constant:

$$F_R = \gamma \sin\theta \underbrace{\int_A y\, dA}_{\substack{\text{first moment of area}\\ \text{w.r.t. } x\text{-axis}}} \quad (2)$$

i.e. $\int_A y\, dA = y_c A \quad (3)$

location of centroid from where x-axis crosses i.e. $y = 0$.
i.e. The free surface.

Substituting (3) into (2)

$$F_R = \gamma A \overbrace{y_c \sin\theta}^{h_c}$$

or $\boxed{F_R = \gamma h_c A}$

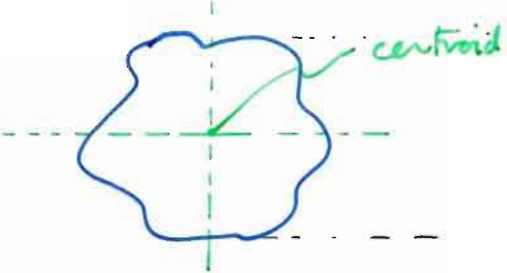
depth to centroid (below surface, $p=0$)

NOTE: This gives resultant force F_R

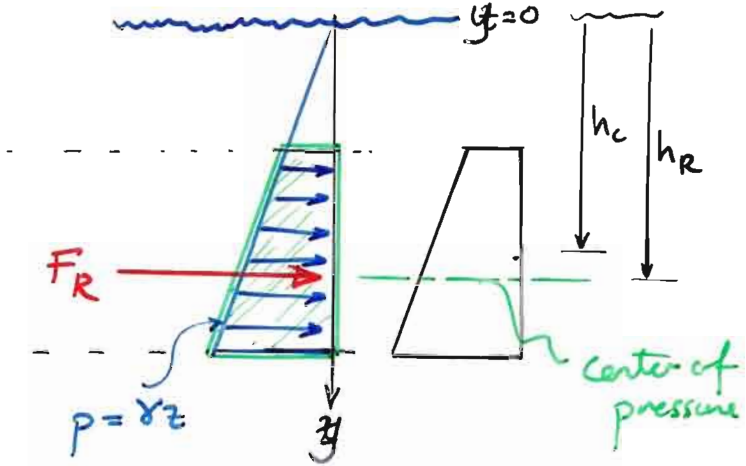
but F_R acts through "Center of Pressure", not

the centroid!!

CENTER OF PRESSURE



Centroid is the "balance" point of the plate.!



"Center of pressure" is the "balance" point for the pressure distribution

Determine C of Pressure by summing moments around 'x' axis.

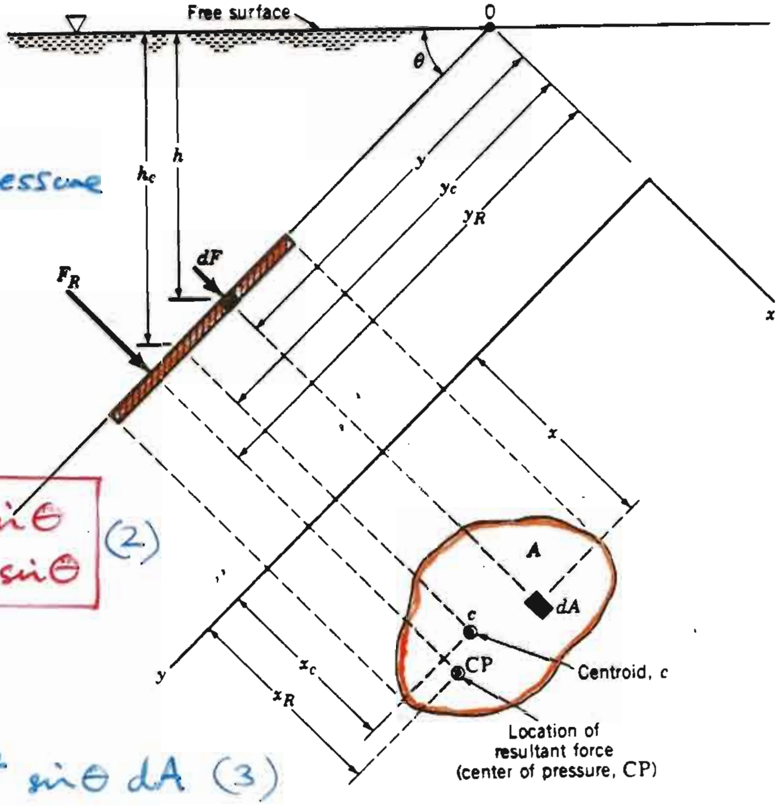
CENTER OF PRESSURE

Take moments about $y = 0$.

$$\sum M_{F_R} = \sum M_{\text{due to pressure}}$$

ie.

$$F_R \, y_R = \int_A y \, dF \qquad (1)$$



From previous: $F_R = \gamma A y_c \sin\theta$ (2)

OR $dF = \gamma \, dA \, y \sin\theta$

Define the term: $\int_A y \, dF$

$$\int_A y \, dF = \int_A \gamma y^2 \sin\theta \, dA \quad (3)$$

Substitute (2) & (3) into (1) gives:

$$F_R \, y_R = \int_A y \, dF$$

$$y_R = \frac{1}{F_R} \int_A y \, dF = \frac{1}{\cancel{\gamma} A y_c \cancel{\sin\theta}} \int_A \cancel{\gamma} y^2 \cancel{\sin\theta} \, dA$$

$$\boxed{y_R = \frac{I_x}{A y_c}}$$

$I_x = \int_A y^2 \, dA$ = second M of A about surface. (moment of inertia).

Parallel axes theorem: $I_x = I_{xc} + A y_c^2$

$$\therefore \boxed{y_R = \frac{I_{xc} + A y_c^2}{A y_c}}$$

I_{xc} available for specific shapes. Fig 2.18.

[3:2] Fluid Statics & Pressures

Recap

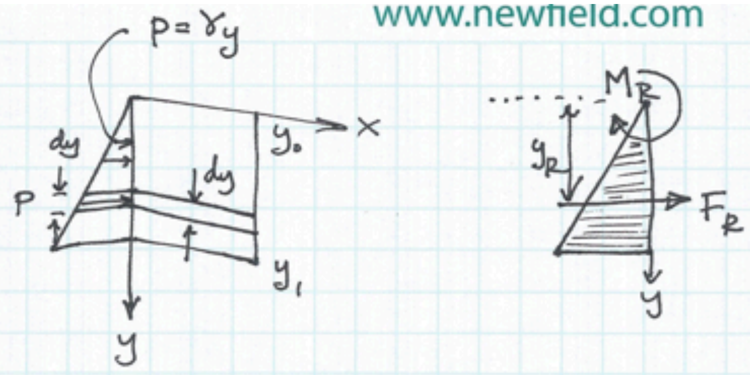
How big are the forces? $F_R = \gamma h_c A$

How do we resolve them? $F_R = \sqrt{F_H^2 + F_V^2}$

Where do they act? F_R acts through center of pressure $\left\{ \begin{array}{l} y_R = \dfrac{I_{xc}}{y_c A} + y_c \\ x_R = \dfrac{I_{xyc}}{y_c A} + x_c \end{array} \right.$

Outline

Buoyancy and stability $F_B = \gamma \cancel{V}$



RESULTANT:

FORCE: $F_R = \int p\, dA = \int \underbrace{\gamma y}_{P}\, \underbrace{w\, dy}_{dA} = \gamma w \int y\, dy$

MOMENT: $M_R = \int p y\, dA = \int \gamma y\, y\, w\, dy = \gamma w \int y^2 dy$

$F_R y_R = M_R$

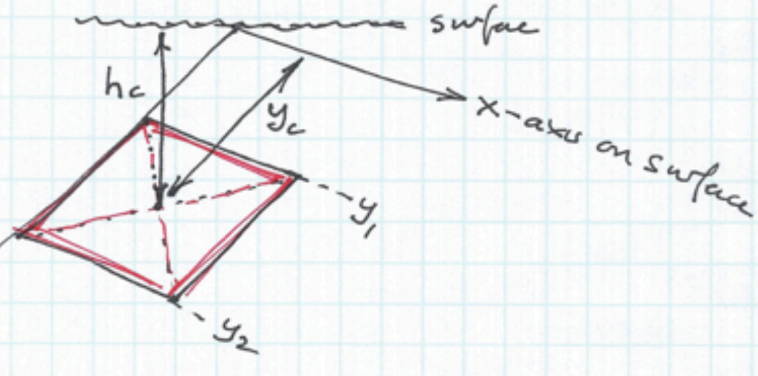
$y_R = \frac{1}{F_R} M_R$

$y_R = \frac{\int y^2 dy}{\int y\, dy} \cdot \frac{w}{w}$

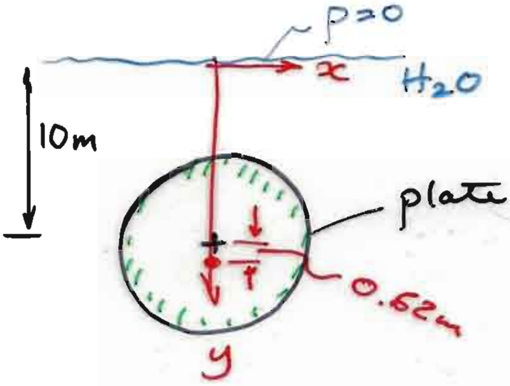
second moment of area about sofaa

$$y_R = \frac{I_x}{A y_c} = \frac{I_{xc} + A y_c^2}{A y_c}$$

← second moment of area about centroid.



EXAMPLE



R = 5m

Locate centroid, C of P and determine F_R.

1) Centroid is center of area of plate ⊕

2) C of P

$$y_R = \frac{I_{xc}}{y_c A} + y_c = \frac{\pi R^4}{4}\,\frac{1}{\underset{10m}{y_c}\,\underset{\pi R^2}{A}} + y_c$$

$$y_R = \frac{\pi R^4}{4}\,\frac{1}{y_c \pi R^2} + y_c = \frac{1}{4}\frac{R^2}{y_c} + y_c$$

$$\boxed{y_R = \frac{1}{4}\frac{5^2}{10} + 10 = \frac{25}{40} + 10 = 10.62\,m}$$

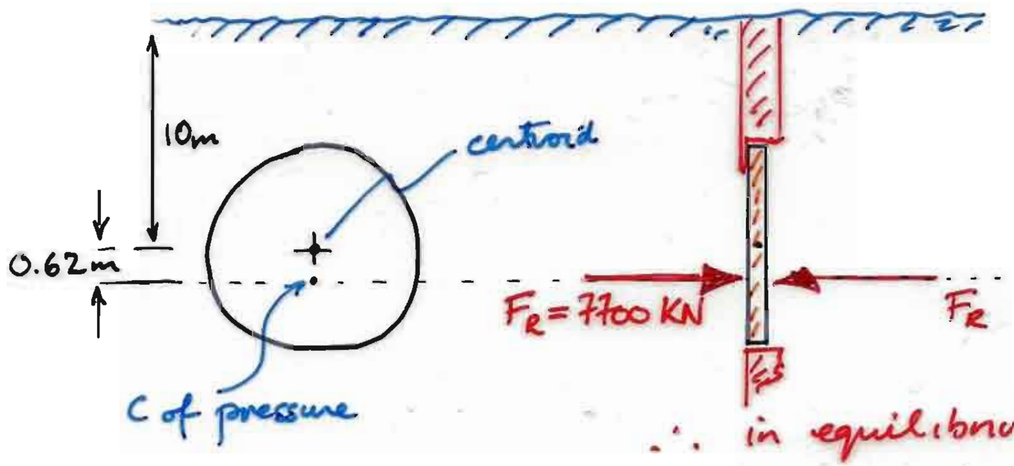
$$\boxed{x_R = \frac{I_{xyc}}{y_c A} + x_c = \underline{\underline{x_c}}}$$

3) F_R

$$F_R = \gamma h_c A = 9.81\,(KN/m^3)\ 10_{(m)}\ \pi R^2\,(m^2)$$

$$F_R = 7700.85\ KN$$

FINAL FORM

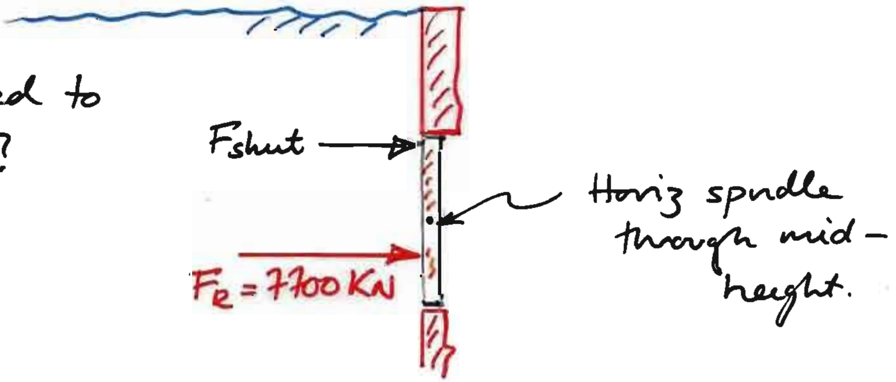


∴ in equilibrium if water on both sides

i.e. $\Sigma M = 0 \quad \Sigma F = 0$

If R.H.S. of tank is emptied:

What F_{shut} is needed to keep gate closed?



$\circlearrowright$

Taking moments: $F_{shut}\ 5m - F_R\ 0.62m = 0$

$$F_{shut} = F_R \frac{0.62}{5} = 954.8 \text{ KN}$$

CENTER OF PRESSURE – X-DIRECTION (X-COORDINATE)

As before : $F_R x_R = \int_A \gamma \sin\theta \underset{\xi}{x} y \, dA$

Pressure varies from surface (y), but moment taken about $\underset{\xi}{x}$

$$x_R = \frac{\int_A xy \, dA}{y_c A} = \frac{I_{xy}}{y_c A}$$

Parallel axes theorem

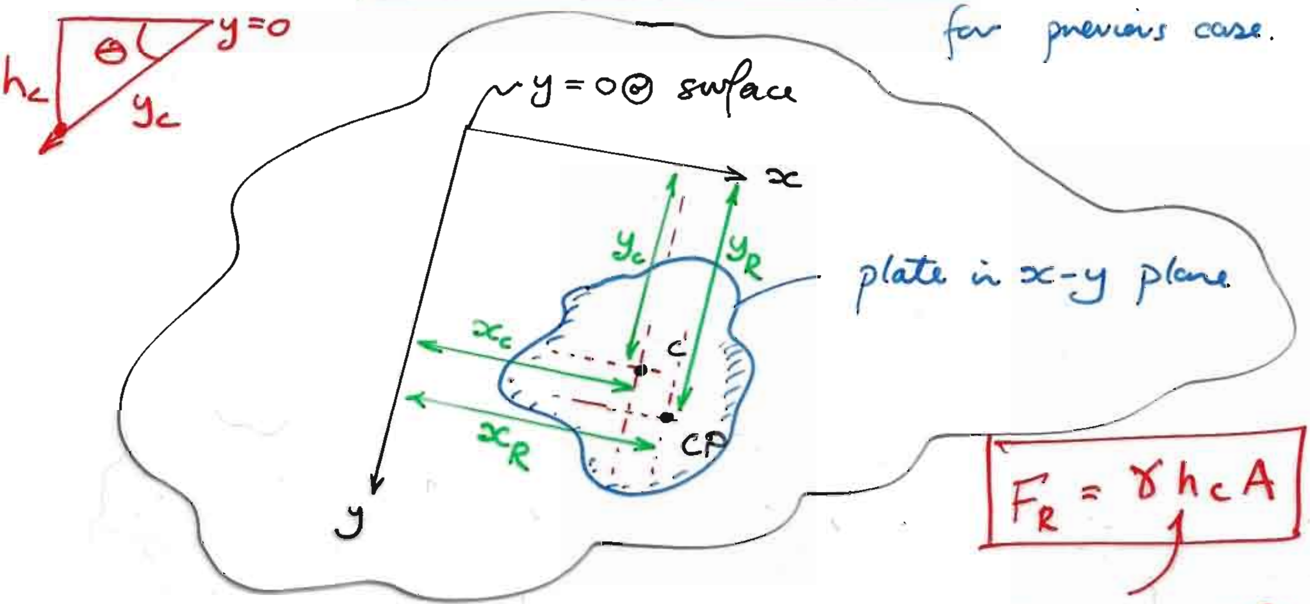
$$\boxed{x_R = \frac{I_{xyc}}{y_c A} + x_c}$$

I_{xyc} = product of moment of inertia with respect to x & y axes.

Compares to:

$$\boxed{y_R = \frac{I_{xc}}{y_c A} + y_c}$$

for previous case.

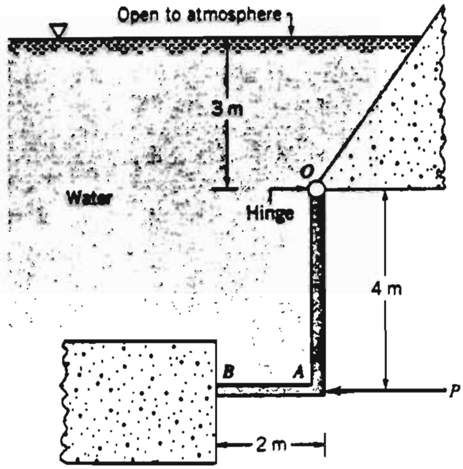


$$\boxed{F_R = \gamma h_c A}$$

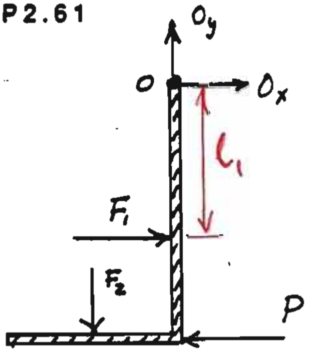
$h_c = y_c \sin\theta$

2.61

2.61 The rigid gate, *OAB*, of Fig. P2.61 is hinged at *O* and rests against a rigid support at *B*. What minimum horizontal force, *P*, is required to hold the gate closed if its width is 3 m? Neglect the weight of the gate and friction in the hinge. The back of the gate is exposed to the atmosphere.



■ FIGURE P2.61



$$F_1 = \gamma h_{c_1} A_1 \quad \text{where} \quad h_{c_1} = 5m$$

Thus,

$$F_1 = (9800 \tfrac{N}{m^3})(5m)(4m \times 3m)$$

$$= 5.88 \times 10^5 N$$

$$F_2 = \gamma h_{c_2} A_2 \quad \text{where} \quad h_{c_2} = 7m$$

so that

$$F_2 = (9800 \tfrac{N}{m^3})(7m)(2m \times 3m)$$

$$= 4.12 \times 10^5 N$$

To locate F_1,

$$y_{R_1} = \frac{I_{xc}}{y_{c_1} A_1} + y_{c_1} = \frac{\frac{1}{12}(3m)(4m)^3}{(5m)(4m \times 3m)} + 5m = 5.267m$$

The force F_2 acts at the center of the AB section. Thus,

$$\Sigma M_O = 0 \quad \circlearrowleft$$

and

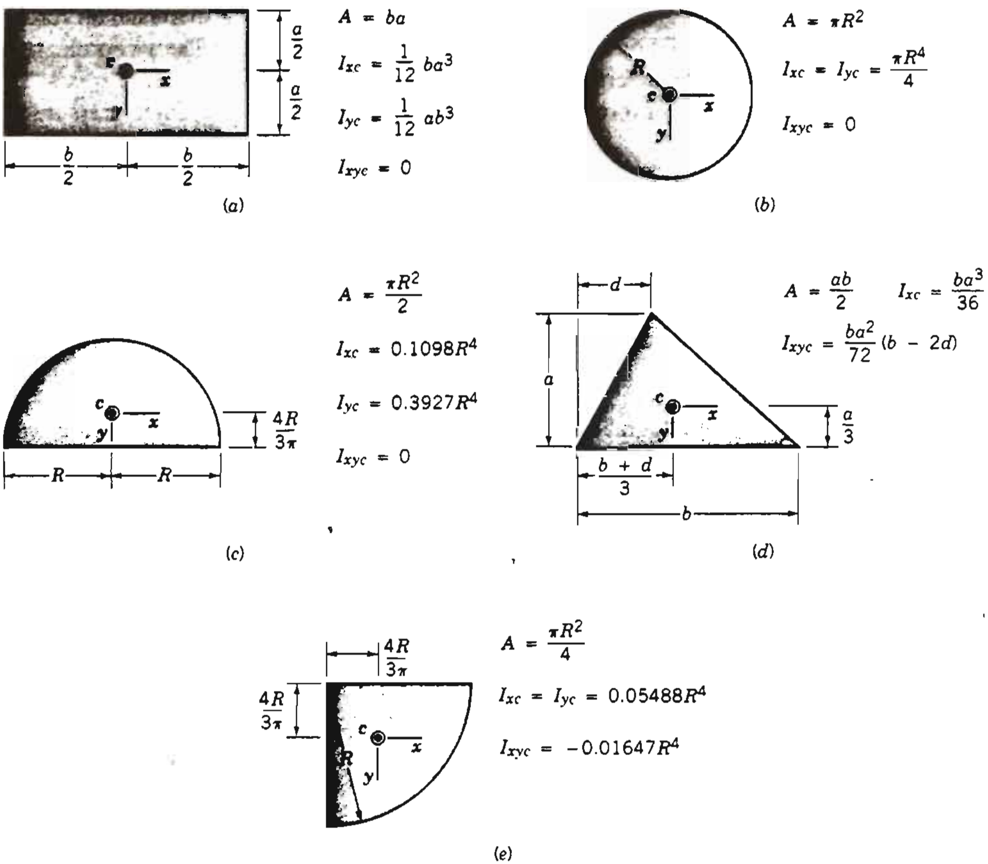
$$F_1 (\underbrace{5.267m - 3m}_{l_1}) + F_2 (1m) = P(4m)$$

so that

$$P = \frac{(5.88 \times 10^5 N)(2.267m) + (4.12 \times 10^5 N)(1m)}{4m}$$

$$= \underline{\underline{436 \, kN}}$$

AREAS & MOMENTS OF INERTIA



(a)

$$A = ba \qquad I_{xc} = \frac{1}{12} ba^3 \qquad I_{yc} = \frac{1}{12} ab^3 \qquad I_{xyc} = 0$$

(b)

$$A = \pi R^2 \qquad I_{xc} = I_{yc} = \frac{\pi R^4}{4} \qquad I_{xyc} = 0$$

(c)

$$A = \frac{\pi R^2}{2} \qquad I_{xc} = 0.1098R^4 \qquad I_{yc} = 0.3927R^4 \qquad I_{xyc} = 0$$

(d)

$$A = \frac{ab}{2} \qquad I_{xc} = \frac{ba^3}{36} \qquad I_{xyc} = \frac{ba^2}{72}(b - 2d)$$

(e)

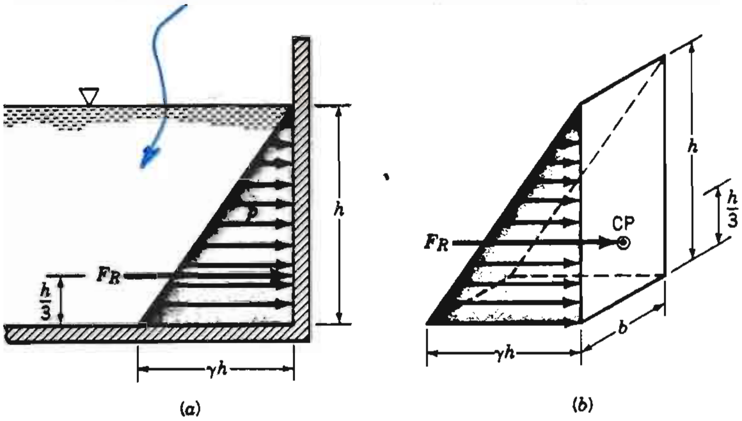
$$A = \frac{\pi R^2}{4} \qquad I_{xc} = I_{yc} = 0.05488R^4 \qquad I_{xyc} = -0.01647R^4$$

■ **FIGURE 2.18** Geometric properties of some common shapes.

For any shape symmetric w.r.t. $x = 0$ then $I_{xyc} = 0$

PRESSURE PRISM

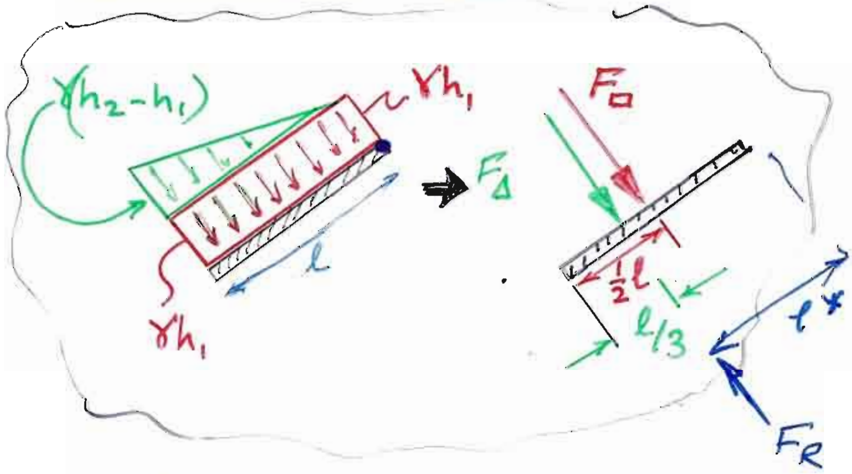
Fluid pressures same if sand filled ?



$$F_R = \frac{1}{2}(\gamma h)\, h b = \frac{1}{2}\gamma h^2 \times b$$

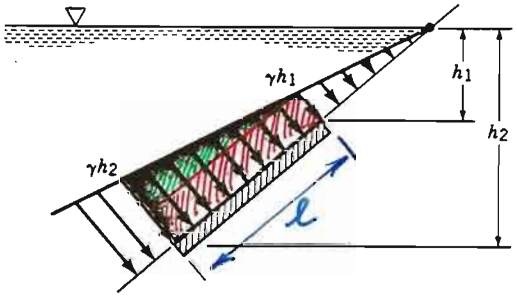
F_R acts @ Center of Pressure.

FOR SUBMERGED BODY



Decompose components

1. To determine $F_\triangle$ and $F_\square$
2. Locate center of pressure



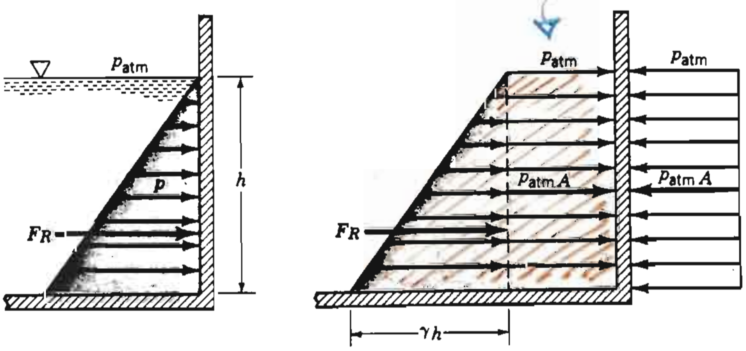
$$F_\square \frac{\ell}{2} + F_\triangle \frac{2}{3}\ell = \underbrace{F_R}_{F_\square + F_\triangle} \ell^*$$

Influence of atmospheric pressure:

$$p = p_{surface} + \gamma h$$

$$P_{abs} = P_{atm} + \gamma h$$

Absolute pressure



[3:3] Fluid Statics & Pressures

Recap

How big are the forces? $F_R = \gamma h_c A$

How do we resolve them? $F_R = \sqrt{F_H^2 + F_V^2}$

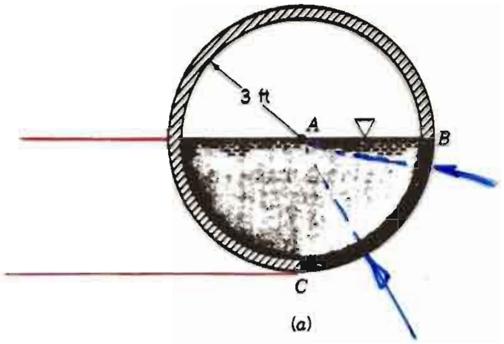
Where do they act? F_R acts through center of pressure $\left\{ \begin{array}{l} y_R = \dfrac{I_{xc}}{y_c A} + y_c \\ x_R = \dfrac{I_{xyc}}{y_c A} + x_c \end{array} \right.$

Outline

Buoyancy and stability $F_B = \gamma \cancel{V}$

EXAMPLE 2.9

The 6-ft-diameter drainage conduit of Fig. E2.9*a* is half full of water at rest. Determine the magnitude and line of action of the resultant force that the water exerts on a 1-ft length of the curved section *BC* of the conduit wall.

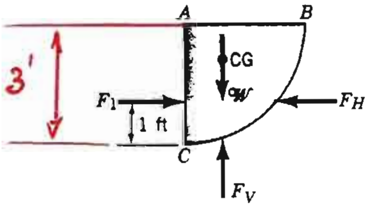


(a)

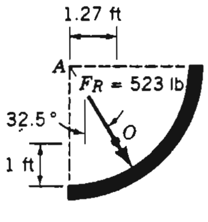
FROM GEOMETRY

$$F_1 = F_H$$

$$F_V = W$$



(b)



(c)

$$F_1 = \gamma h_c A \Rightarrow$$

$$F_1 = (62.4)\frac{lb}{ft^3}\left(\frac{3}{2}\right) ft \; (3) ft^2$$

Unit length along pipe

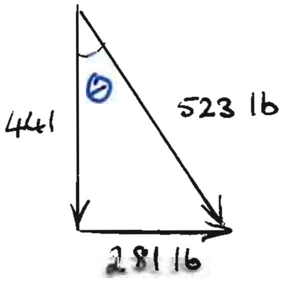
$$\boxed{F_1 = 281 \text{ lb}} = F_H$$

$$W = \gamma \frac{\pi r^2}{4} = (62.4)\frac{lb}{ft^3}\left(\frac{\pi}{4} 3^2\right) ft^2 \, (1) ft = 441 \text{ lb} = F_V$$

Resultant $F_R = \sqrt{F_H^2 + F_V^2} = 523 \text{ lb}$

Direction of resultant: Pressure ⊥ to conduit wall ∴ all pressure "vectors" pass through 'O'.

Consequently Resultant passes through O.

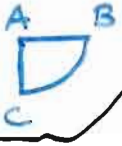


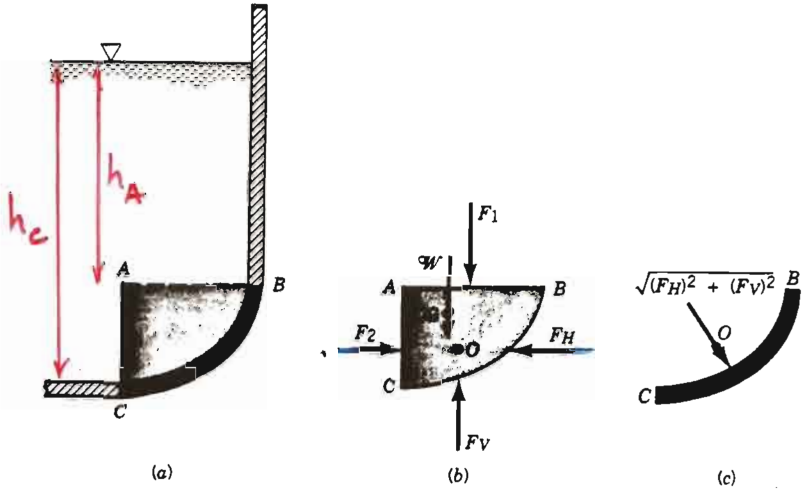
$523 \cos\theta = 441$ $\cos\theta = 441/523$

$\theta = 32.5°$

HYDROSTATIC FORCE ON CURVED SURFACE

Isolate free body



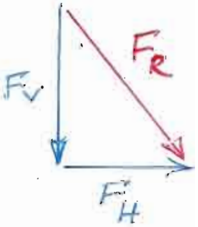


■ **FIGURE 2.23** **Hydrostatic force on a curved surface.**

Horizontal $\Sigma F_H = 0$ $F_H = F_2$

Vertical $\Sigma F_V = 0$ $F_V = F_1 + W$

Resultant, F_R $F_R = \sqrt{F_H^2 + F_V^2}$

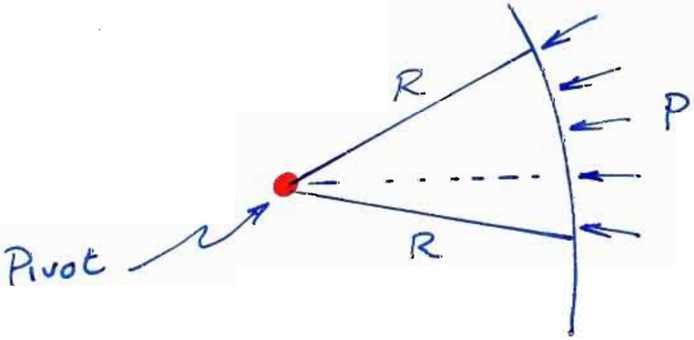


(F_R is a magnitude).

Point of action, O

Determined by summing F_H, F_V and F_R (all known) about an appropriate axis.

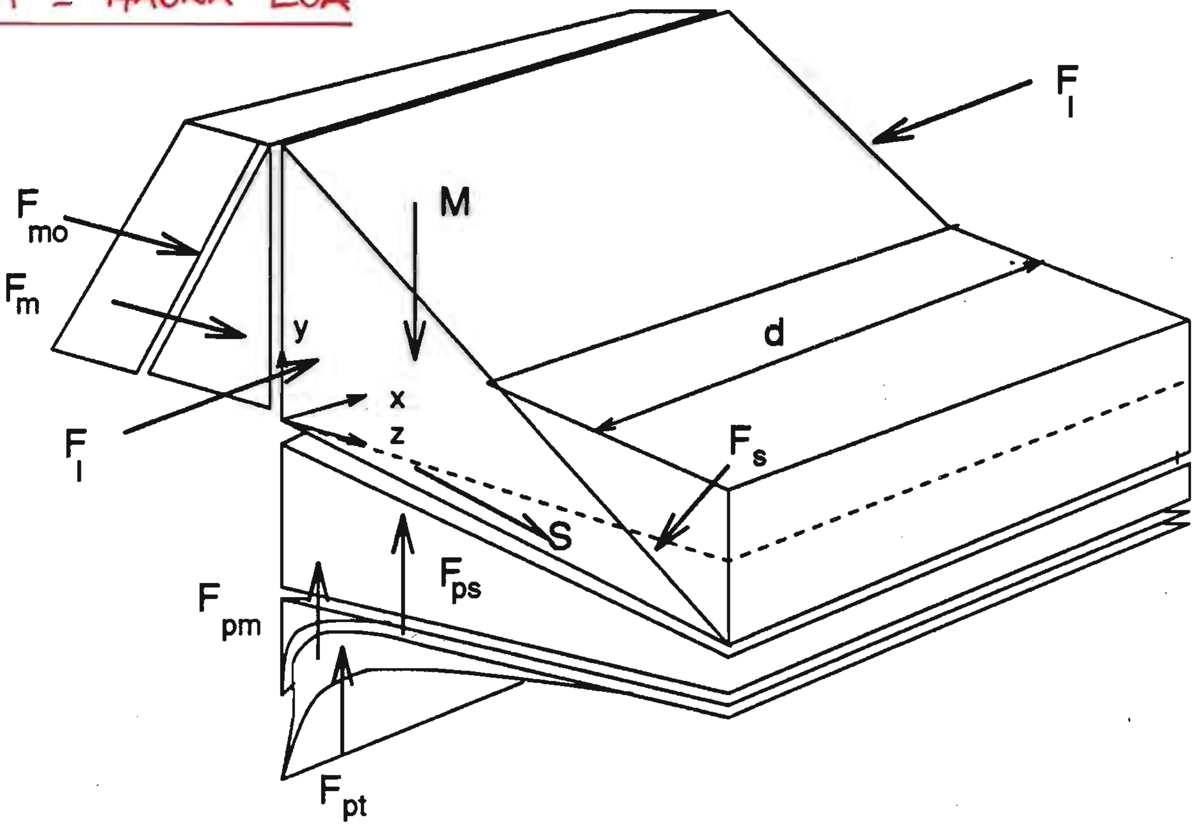
TAINTER GATE

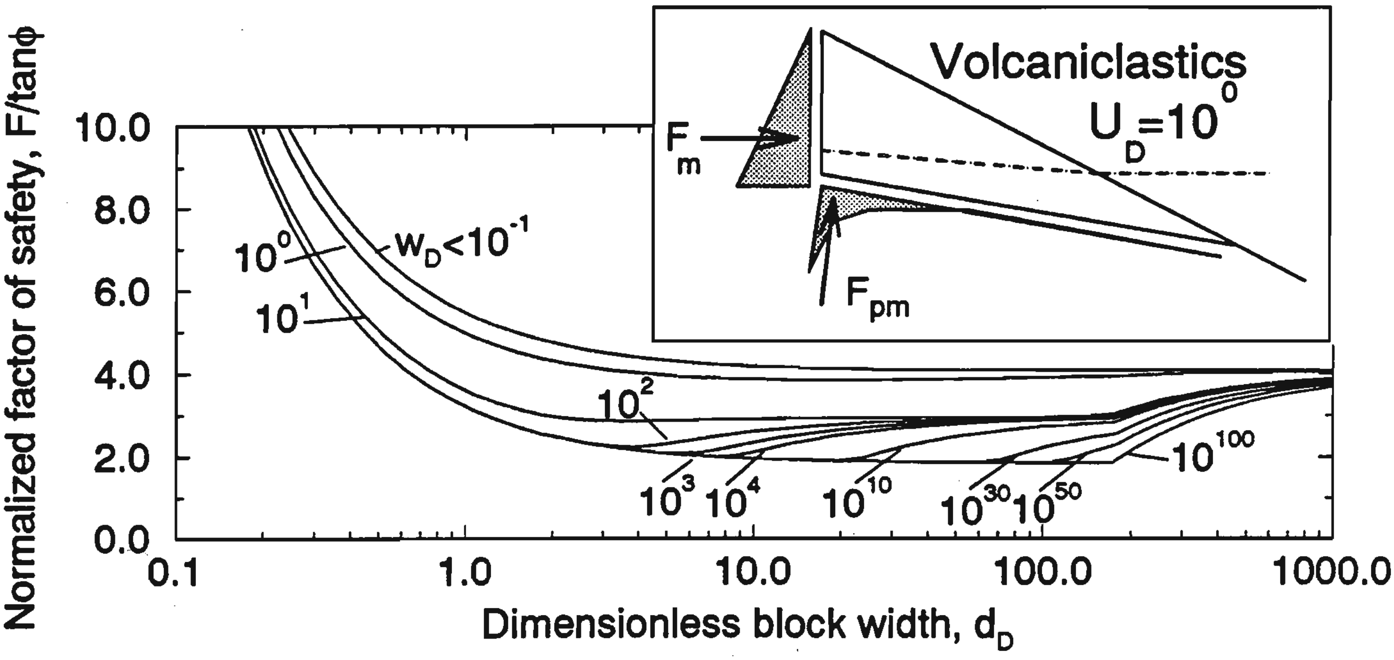


All resultant forces due to pressure distribution act through pivot if circular

∴ no net moment !!

HAWAII - MAUNA LOA





2.63*

2.63* A 200-lb homogeneous gate of 10-ft. width and 5-ft length is hinged at point *A* and held in place by a 12-ft-long brace as shown in Fig. P2.63. As the bottom of the brace is moved to the right, the water level remains at the top of the gate. The line of action of the force that the brace exerts on the gate is along the brace. **(a)** Plot the magnitude of the force exerted on the gate by the brace as a function of the angle of the gate, θ, for $0 \le \theta \le 90°$. **(b)** Repeat the calculations for the case in which the weight of the gate is negligible. Comment on the results as $\theta \to 0$.

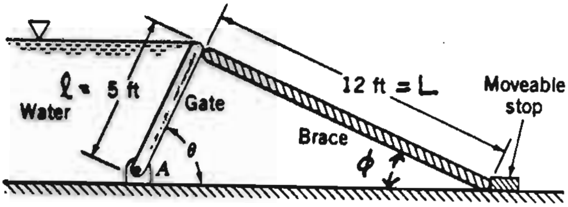
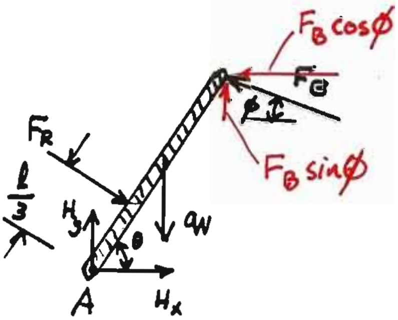


FIGURE P2.63



(a) For the free-body-diagram of the gate (see figure),

$$\overset{\curvearrowright}{\sum F_A} = 0$$

so that

$$F_R\left(\frac{\ell}{3}\right) + \mathcal{W}\left(\frac{\ell}{2}\cos\theta\right) = (F_B\cos\phi)(\ell\sin\theta) + (F_B\sin\phi)(\ell\cos\theta) \qquad (1)$$

Also,

$$\ell\sin\theta = L\sin\phi$$ (assuming hinge and end of brace at same elevation)

or

$$\sin\phi = \frac{\ell}{L}\sin\theta$$

and

$$F_R = \gamma h_c A = \gamma\left(\frac{\ell\sin\theta}{2}\right)(\ell w)$$

where w is the gate width. Thus, Eq. (1) can be written as

$$\gamma\left(\frac{\ell^3}{6}\right)(\sin\theta)w + \frac{\mathcal{W}\ell}{2}\cos\theta = F_B\,\ell\left(\cos\phi\sin\theta + \sin\phi\cos\theta\right)$$

so that

$$F_B = \frac{\left(\frac{\gamma\ell^2 w}{6}\right)\sin\theta + \frac{\mathcal{W}}{2}\cos\theta}{\cos\phi\sin\theta + \sin\phi\cos\theta} = \frac{\left(\frac{\gamma\ell^2 w}{6}\right)\tan\theta + \frac{\mathcal{W}}{2}}{\cos\phi\tan\theta + \sin\phi} \qquad (2)$$

For $\gamma = 62.4\ \text{lb/ft}^3$, $\ell = 5\ \text{ft}$, $w = 10\ \text{ft}$, and $\mathcal{W} = 200\ \text{lb}$,

$$F_B = \frac{\frac{(62.4\frac{\text{lb}}{\text{ft}^3})(5\text{ft})^2(10\text{ft})}{6}\tan\theta + \frac{200\ \text{lb}}{2}}{\cos\phi\tan\theta + \sin\phi} = \frac{2600\tan\theta + 100}{\cos\phi\tan\theta + \sin\phi} \qquad (3)$$

(con't)

2.71

2.71 The inclined face *AD* of the tank of Fig. P2.71 is a plane surface containing a gate *ABC*, which is hinged along line *BC*. The shape of the gate is shown in the plan view. If the tank contains water, determine the magnitude of the force that the water exerts on the gate.

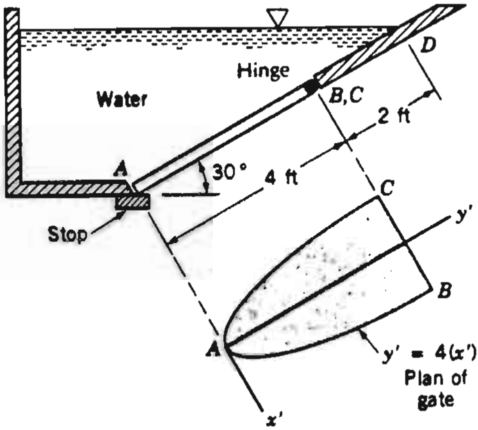
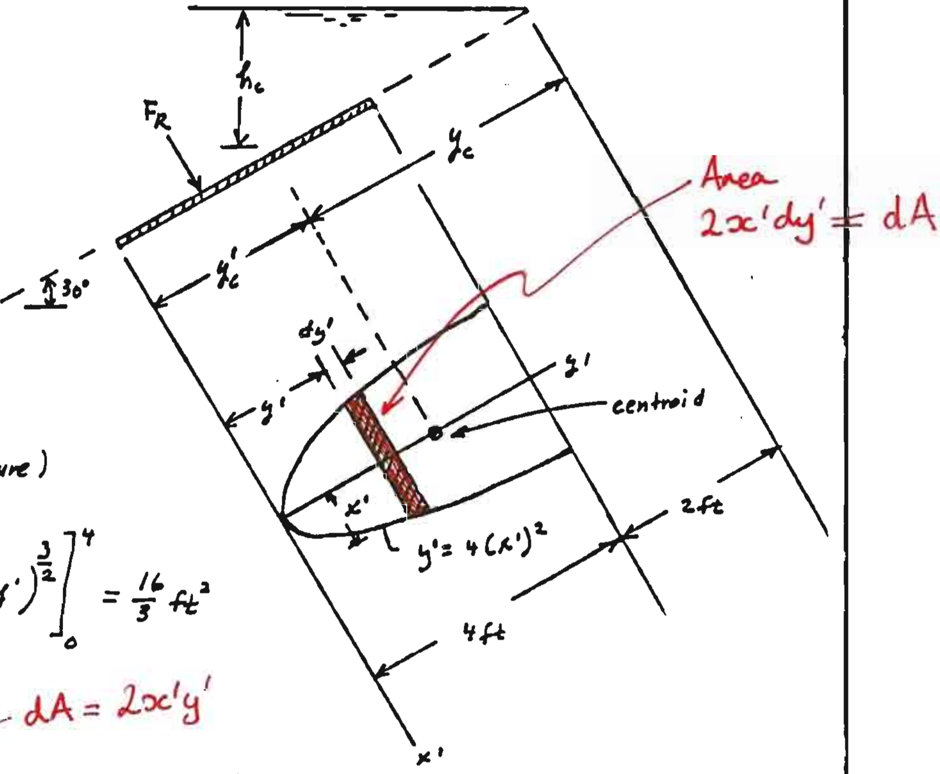


FIGURE P2.71

$$\boxed{\begin{aligned} y' &= 4(x')^2 \\ x' &= \tfrac{1}{2}\sqrt{y'} \end{aligned}}$$



$F_R = \gamma\, h_c\, A$

where

$A = \int dA \rightarrow A = \int_0^4 2x'\,dy'$ (see figure)

$$= \int_0^4 2\left(\tfrac{1}{2}\right)\sqrt{y'}\,dy' = \frac{2}{3}(y')^{\frac{3}{2}}\Big]_0^4 = \frac{16}{3}\text{ ft}^2$$

To locate centroid: $dA = 2x'y'$

$$y_c' A = \int_0^4 y'\,dA = \int_0^4 2y'x'\,dy' = \int_0^4 (y')^{3/2}\,dy' = \frac{2}{5}(y')^{\frac{5}{2}}\Big]_0^4 = \frac{64}{5}\text{ ft}^3$$

Thus, $$y_c' = \frac{\frac{64}{5}\text{ ft}^3}{\frac{16}{3}\text{ ft}^2} = 2.4\text{ ft}$$

and $$y_c = 6\text{ft} - 2.4\text{ ft} = 3.6\text{ ft}$$

Since $h_c = y_c \sin 30°$,

$$F_R = \left(62.4\,\tfrac{\text{lb}}{\text{ft}^3}\right)(3.6\text{ ft})(\sin 30°)\left(\tfrac{16}{3}\text{ft}^2\right) = \underline{599\text{ lb}}$$

2.84

2.84 The 9-ft-long cylinder of Fig. P2.84 floats in oil and rests against a wall. Determine the horizontal force the cylinder exerts on the wall at the point of contact, *A*.

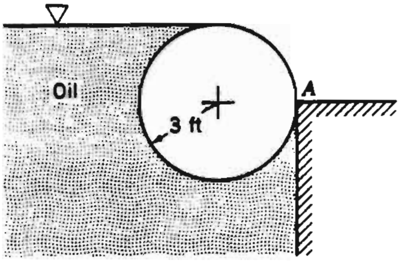
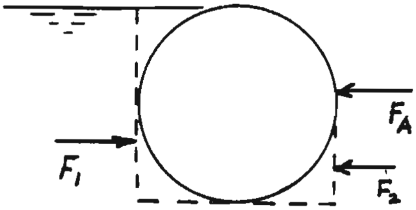


FIGURE P2.84



The horizontal forces acting on the free-body-diagram are shown on the figure. For equilibrium,

$$F_A = F_1 - F_2$$

where F_A is the horizontal force the wall exerts on the cylinder.

Since,

$$F_1 = \gamma h_{c_1} A_1$$
$$= \left(57.0 \frac{lb}{ft^3}\right)\left(\frac{6 ft}{2}\right)(6 ft \times 9 ft)$$
$$= 9230 \ lb$$

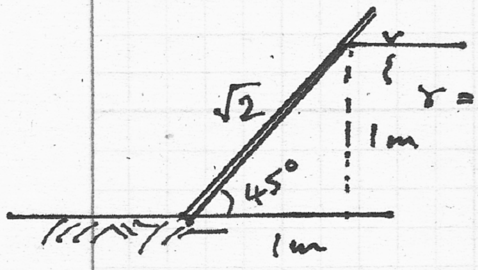
and

$$F_2 = \gamma h_{c2} A_2$$
$$= \left(57.0 \frac{lb}{ft^3}\right)\left(3 ft + \frac{3}{2} ft\right)(3 ft \times 9 ft)$$
$$= 6930 \ lb$$

then

$$F_A = 9230 \ lb - 6930 \ lb = \underline{\underline{2300 \ lb \longrightarrow \text{on the wall}}}$$

Determine the horizontal and vertical forces acting on this gate

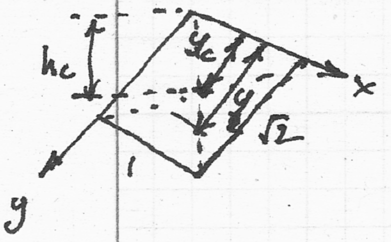


$\gamma = 10\,kN/m^3$

W = 1m into page

Three approaches to solve the same problem — they are equivalent.

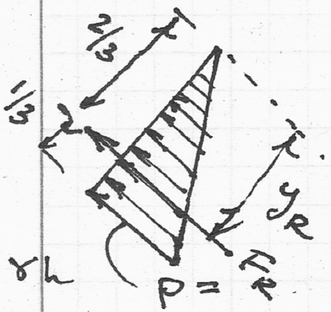
Centroid and Center of Pressure Approach



$$F_R = \gamma A h_c = 10\,KN/m^3\,(1\times\sqrt{2})m^2\,\frac{1}{2}m = \underline{\underline{5\times\sqrt{2}\,kN}}$$

$$y_R = y_c + \frac{\overset{\frac{1}{12}ba^3}{I_{xc}}}{y_cA} = \frac{1}{2}\sqrt{2}m + \frac{\frac{1}{12}\,1\,2^{3/2}}{\frac{1}{2}\sqrt{2}.(\sqrt{2})1} = \underline{\underline{\frac{1}{2}\sqrt{2}+\frac{1}{6}\sqrt{2}\,m}}$$

Pressure Prism Approach



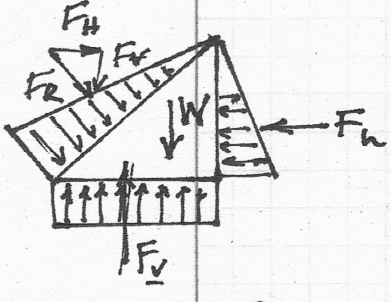
$p = \gamma h$

$p = 10kN/m^3 \times 1m = 10\,kN/m^2$

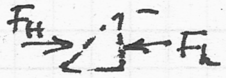
$$F_R = \frac{1}{2}\overset{\text{average pressure}}{p} \times A = \frac{10}{2}kPa \times \underbrace{\sqrt{2}\times1}_{A} = \underline{\underline{5\sqrt{2}\,kN}}$$

$$y_R = \frac{2}{3}\sqrt{2}m \equiv \underline{\underline{(\frac{1}{2}+\frac{1}{6})\sqrt{2}\,m}}$$

Free Body Diagram Approach



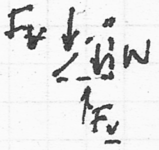
Resolving horizontally:



$$F_H = F_h = \gamma \overset{\text{area of vert plane}}{A}\,h_c$$

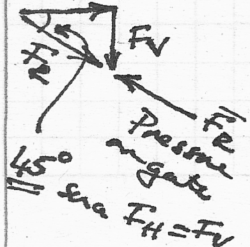
$$F_H = F_h = 10\,kN/m^3\,(1\times1)m^2\,\frac{1}{2}m = 5kN$$

Resolving vertically: $F_{\underline{V}} = F_V + W$ or $F_V = F_{\underline{V}} - W$



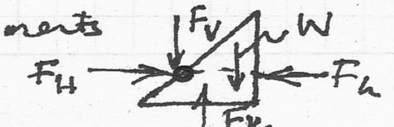
$$F_V = F_{\underline{V}} - W = \gamma A h_c - \gamma \forall$$

$$= 10kN/m^3\,(\underbrace{1\times1}_{A}\underbrace{\times1}_{h_c} - \underbrace{\frac{1}{2}(1\times1\times1)}_{\forall}) = 5\,kN$$



Resultant: $F_R = \sqrt{F_H^2+F_V^2} = \sqrt{5^2+5^2} = \sqrt{2\times25} = \underline{\underline{5\sqrt{2}\,kN}}$

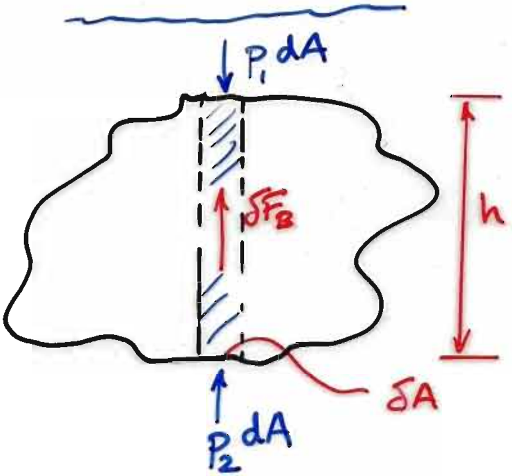
Assume F_H and F_V act @ 2/3 depth and check moments



$$W\frac{1}{3} + F_V\frac{2}{3} = F_{\underline{V}}\frac{1}{2}$$

$$5\frac{1}{3} + 5\frac{2}{3} = 10\frac{1}{2} \qquad \underline{QED.}$$

BUOYANCY, FLOTATION, STABILITY



$$\delta F_B = (p_1 - p_2)\,\delta A$$
$$= \gamma h\,\delta A$$
$$= \gamma\, d\forall$$

Integrating over prism:

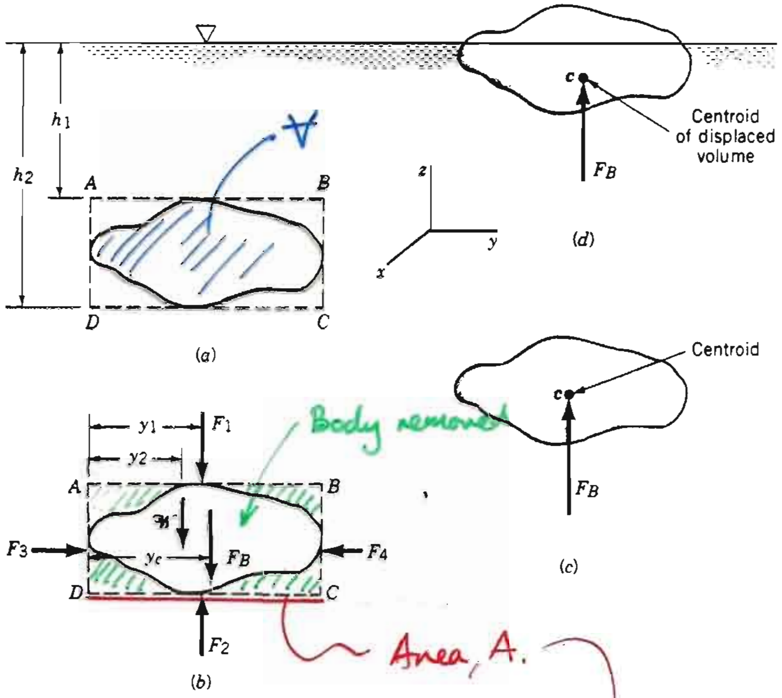
$$F_B = \int_{\forall} \delta F_B = \gamma \int_{\forall} d\forall = \gamma\forall$$

$$\boxed{F_B = \gamma\forall}$$

- ☐ No lateral forces (all cancel)
- ☐ Buoyant force acts through <u>centroid</u> of <u>displaced</u> volume. Center of buoyancy.

BUOYANCY, FLOTATION, STABILITY

ARCHIMEDES' PRINCIPLE



☐ Body of volume, $\forall$.

☐ Surround by

$AD \doteq BC$

$\therefore \quad F_3 = F_4$

Lateral equilibrium

☐ Resolve vertically:

$$F_B = \underbrace{F_2 - F_1} - W \qquad (1)$$

$$\overbrace{F_2 - F_1} = \underbrace{\gamma(h_2 - h_1)A}_{\text{Wt. of } \square} \qquad (2)$$

Substitute (2) in (1) $$F_B \doteq \gamma(h_2-h_1)A - \gamma[\underbrace{(h_2-h_1)A}_{\text{volume } \square} - \forall]$$

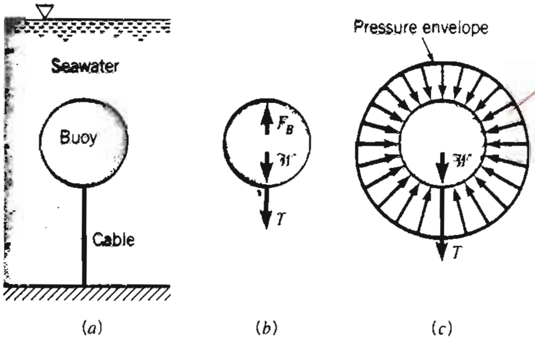
Rearranging : $\boxed{F_B = \gamma\forall}$ = Force on body.

☐ Buoyant force equal and opposite to F_{BODY}.

☐ Buoyant force acts through 'centroid' of displaced volume. Referred to as "CENTER OF BUOYANCY."

EXAMPLE 2.10

A spherical buoy has a diameter of 1.5 m, weighs 8.50 kN, and is anchored to the sea floor with a cable as is shown in Fig. E2.10*a*. Although the buoy normally floats on the surface, at certain times the water depth increases so that the buoy is completely immersed as illustrated. For this condition what is the tension of the cable?



Resultant of "collar" is F_B.

■ **FIGURE E2.10**

Resolving vertically

$$T + W = F_B$$

$$\uparrow$$

$$F_B = \gamma \forall$$

$$T = F_B - W = \gamma\forall - W$$

$$= 10.1\ \text{kN/m}^3 \quad \frac{4}{3}\pi(0.75)^3\ \text{m}^3 - 8.5\ \text{kN}$$

$$T = 17.85\ \text{kN} - 8.5\ \text{kN}$$

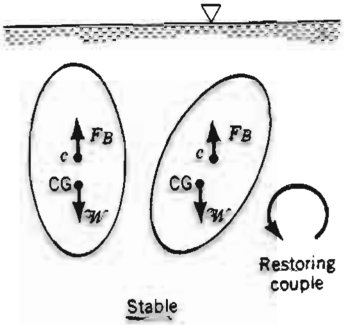
$$\boxed{T = 9.35\ \text{kN.}}$$

STABILITY

SUBMERGED BODIES

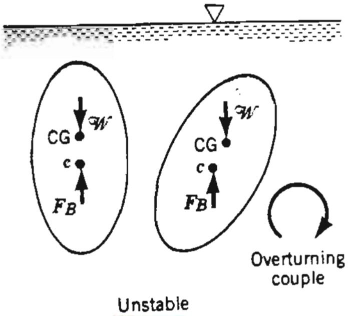
- ☐ Stable – Small displ. is restored to equilibrium.
- ☐ Unstable Equilibrium
  Small displ → new equilibrium posn.

C of G above C of B → Unstable equilibrium.



Stable

■ FIGURE 2.25 Stability of a completely immersed body—center of gravity below centroid.



Unstable

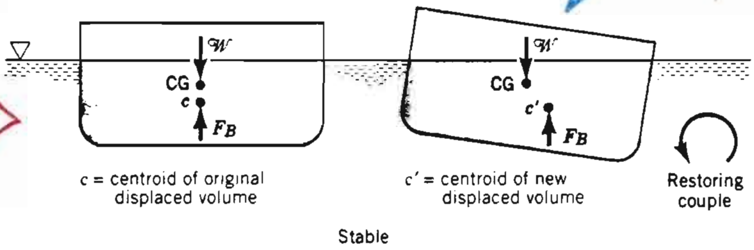
■ FIGURE 2.26 Stability of a completely immersed body—center of gravity above centroid.

FLOATING BODIES

More complex:
C of B moves as body rotates.

Buoyant force shifts to provide restoring couple

May be stable even if C of G above C of B. ⇨



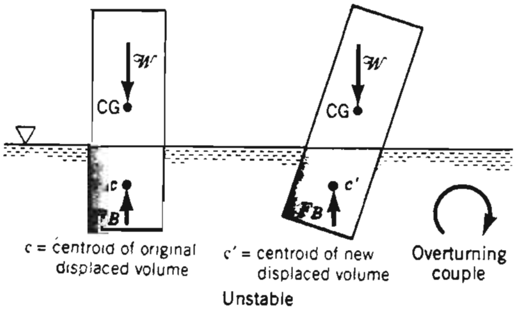
c = centroid of original displaced volume

c' = centroid of new displaced volume

Stable

■ FIGURE 2.27 Stability of a floating body—stable configuration.

or May be unstable if C of G above C of B ⇘



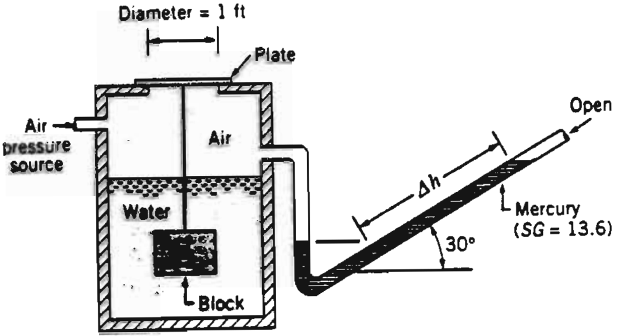
c = centroid of original displaced volume

c' = centroid of new displaced volume

Unstable

2.95

2.95 A plate of negligible weight closes a 1-ft diameter hole in a tank containing air and water as shown in Fig. P2.95. A block of concrete (specific weight = 150 lb/ft³), having a volume of 1.5 ft³, is suspended from the plate and is completely immersed in the water. As the air pressure is increased the differential reading, Δh, on the inclined-tube mercury manometer increases. Determine Δh just before the plate starts to lift off the hole. The weight of the air has a negligible effect on the manometer reading.



■ FIGURE P2.95

For equilibrium,

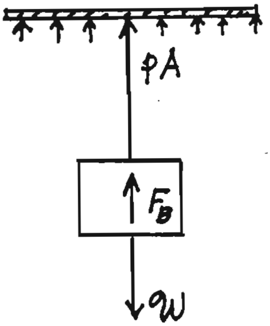
$$\Sigma F_{vertical} = 0$$

so that

$$\mathcal{W} = pA + F_B$$

where:

$\mathcal{W}$ ~ weight of concrete
p ~ air pressure
A ~ area of plate
F_B ~ b



Thus,

$$\underbrace{(150\,\tfrac{lb}{ft^3})(1.5\,ft^3)}_{W} = \overbrace{p\left(\tfrac{\pi}{4}\right)(1\,ft)^2}^{PA} + \overbrace{(62.4\,\tfrac{lb}{ft^3})(1.5\,ft^3)}^{F_B}$$

so that

$$p = 167\,\frac{lb}{ft^2}$$

The manometer equation is

$$p = \gamma_{Hg}\,\Delta h\,\sin 30°$$

so that

$$\Delta h = \frac{p}{\gamma_{Hg}\sin 30°}$$

$$= \frac{167\,\frac{lb}{ft^2}}{(847\,\frac{lb}{ft^3})\sin 30°} = \underline{\underline{0.394\,ft}}$$

[4:1] Linear and Rotational Accelerations

Outline

Pressures in accelerating (static) fluids

Linear

$$-\nabla p-\gamma\hat{\mathbf{k}}=\rho\mathbf{a} \text{ or } -\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z}\end{Bmatrix}p-\gamma\begin{Bmatrix}0\\0\\1\end{Bmatrix}=\rho\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix}$$

Simplifies when: $a_x=a_y=a_z=0$ to $\dfrac{dp}{dx}=\dfrac{dp}{dy}=0$ and $\dfrac{dp}{dz}=-\gamma;\ \dfrac{dz}{dx}=-\dfrac{a_x}{g+a_z}$

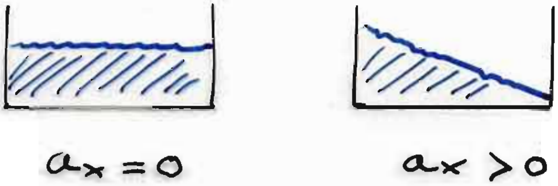
Rotational

Rigid body rotation: $\dfrac{\partial p}{\partial r}=\rho r\omega^2;\dfrac{\partial p}{\partial \theta}=0;\dfrac{\partial p}{\partial z}=-\gamma;\begin{cases} z=\frac{\omega r^2}{2g}+c \\ p=\frac{\rho\omega^2 r^2}{2}-\gamma z+c\end{cases}$

PRESSURE VARIATION WITH RIGID-BODY MOTION

"Rigid-body" → no relative movement within the fluid.
(always a short transient rearrangement)

i.e.



Recall that:

$$-\underline{\nabla} p - \gamma \underline{\hat{k}} = \rho \underline{a}$$

Longhand:

$$-\begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} p - \gamma \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \rho \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

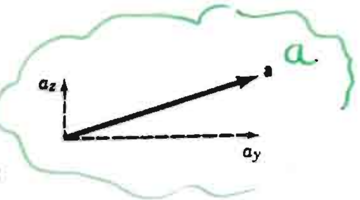
$$-\frac{\partial p}{\partial x} = \rho a_x$$

$$-\frac{\partial p}{\partial y} = \rho a_y$$

$$-\frac{\partial p}{\partial z} = \gamma + \rho a_z$$

LINEAR MOTION

Acceleration profile.

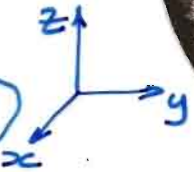


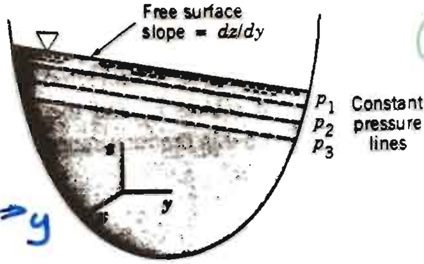
Accelerate:

$a_y \neq 0; \; a_z \neq 0$

$a_x = 0$

$\therefore \; \partial z/\partial x \text{ surface} = 0$





From basic relations:

$$\left.\begin{aligned} \partial p/\partial x &= -\cancelto{0}{\rho a_x} = 0 \\ \partial p/\partial y &= -\rho a_y \\ \partial p/\partial z &= -\rho(g+a_z) \end{aligned}\right\} \quad (1)$$

Evaluate change in pressure, dp, in x, y & z directions

$$dp = \frac{dp}{dx}dx + \frac{dp}{dy}dy + \frac{dp}{dz}dz \qquad (2)$$

Substitute (1) into (2):

$$dp = 0\,dx - \rho a_y\,dy - \rho(g+a_z)\,dz \qquad (3)$$

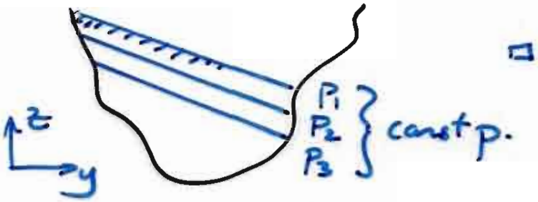
Along a line of <u>constant</u> pressure, $dp = 0$. Setting $dp = 0$ in (3) gives slope of line of constant pressure dz/dy, including free surface.

$$\rho a_y\,dy = -\rho(g+a_z)\,dz$$

$$\boxed{\frac{dz}{dy} = -\frac{a_y}{(g+a_z)}} \qquad (2.28)$$

Equation (2.28):

- $\frac{dz}{dy}$ is constant for constant accelerations a_y & a_z.
- Surfaces of constant pressure of inclination dz/dy = const.

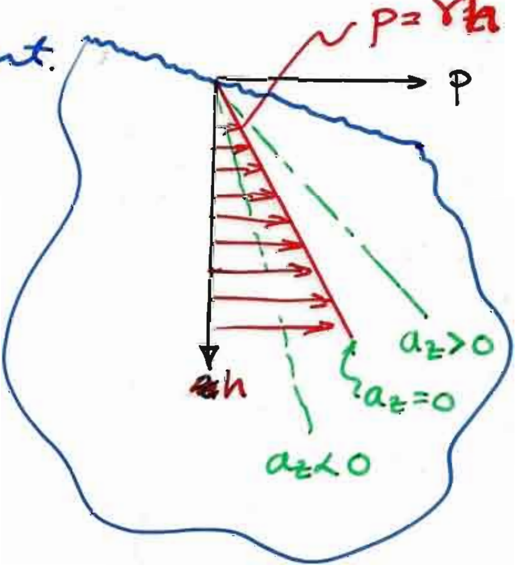


Free surface dz/dy

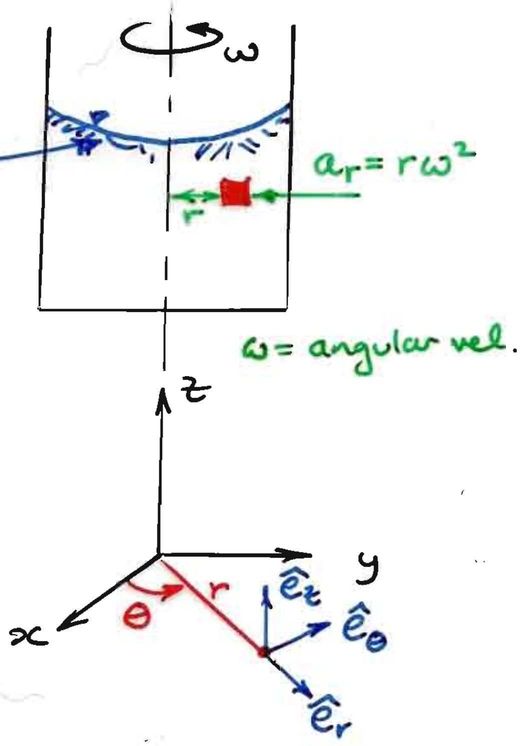
Pressure gradient down from the free surface is given by (1) as: $\frac{\partial p}{\partial z} = -\rho(g + a_z)$

If $a_z = 0$ then $\partial p/\partial z = -\gamma$ $\therefore$ "hydrostatic"

If $a_z \neq 0$ then extra component.



RIGID BODY ROTATION



Basic relations:

$$\left.\begin{aligned} \partial p/\partial r &= \rho r \omega^2 \\ \partial p/\partial \theta &= 0 \\ \partial p/\partial z &= -\gamma \end{aligned}\right\} (2.30)$$

Evaluate change in pressure, dp.

$$dp = \frac{dp}{dr}dr + \overset{0}{\cancel{\frac{dp}{d\theta}}}d\theta + \frac{dp}{dz}dz \qquad (1)$$

Setting dp = 0 for equi-pressures, isobars then

$$\overset{0}{\cancel{dp}} = \rho r \omega^2 \, dr - \gamma \, dz \qquad (2)$$

Rearrange for

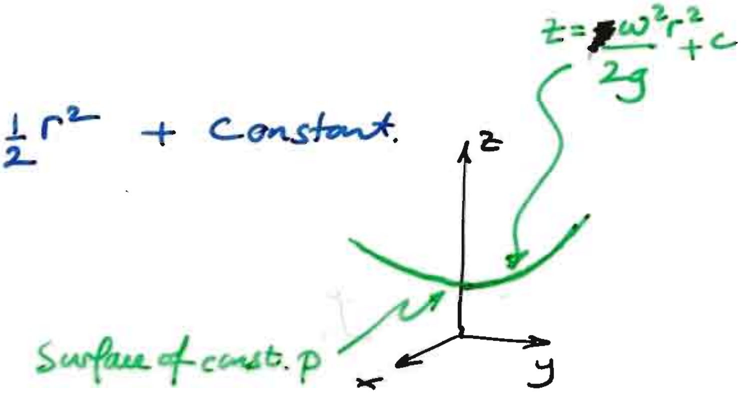
$$\frac{dz}{dr} = \frac{\rho r \omega^2}{\underset{\rho g}{\gamma}} = \frac{r\omega^2}{g} \qquad (3)$$

Integrating (3).

$$\int dz = \frac{\omega^2}{g}\int r \, dr$$

$$z = \frac{\omega^2}{g}\frac{1}{2}r^2 + \text{Constant}.$$

gives surface of equal pressure (parabolic)



With lines of constant pressure defined by (parabolic). $z = \frac{\omega^2 r^2}{2g} + c$

Integrate equation (2).

$$dp = \rho r \omega^2 dr - \gamma dz$$

$$\int dp = \rho \omega^2 \int r\, dr - \gamma \int dz$$

$$\boxed{p = \frac{\rho \omega^2 r^2}{2} - \gamma z + \underline{Const}}$$

To solve, define p @ specified r_0, z_0 and define <u>Const</u>

Then resubstitute to solve p @ any r and z.

- For r = constant, p varies linearly, since p = 0 @ surface and sets const. magnitude.

2.106

2.106 The U-tube of Fig. P2.106 is partially filled with water and rotates around the axis a–a. Determine the angular velocity that will cause the water to start to vaporize at the bottom of the tube (point A).

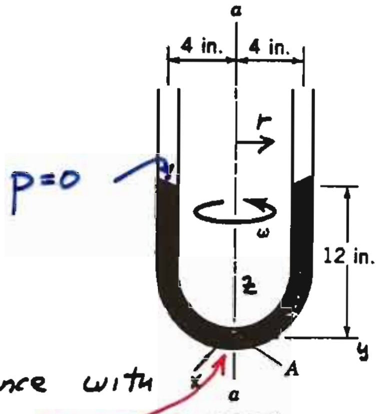


FIGURE P2.106

Pressure in a rotating fluid varies in accordance with the equation,

$$p = \frac{\rho\omega^2 r^2}{2} - \gamma z + \text{Constant} \qquad \text{(Eq. 2.33)}$$

With the coordinate system shown,

$p=0$ at $r = 4$ in. and $z = 12$ in., so that

$$\text{Constant} = -\frac{\rho\omega^2\left(\frac{4}{12}\text{ft}\right)^2}{2} + \gamma\left(\tfrac{12}{12}\text{ft}\right) = -\frac{\rho\omega^2}{18} + \gamma$$

Thus,

$$p = \frac{\rho\omega^2}{2}\left(r^2 - \frac{1}{9}\right) - \gamma(z-1)$$

At point A, $r=0$ and $z=0$, and

$$p_A = -\frac{\rho\omega^2}{18} + \gamma \qquad (1)$$

If p_A = vapor pressure = 0.256 psia, or

$$p_A = (0.256\text{ psi} - 14.7\text{psi})\left(144\frac{\text{in.}^2}{\text{ft}^2}\right) = -2080\ \frac{\text{lb}}{\text{ft}^2}\ (\text{gage})$$

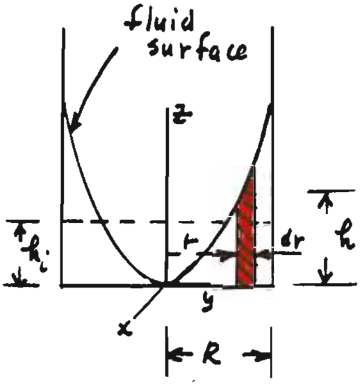
then from Eq. (1)

$$\omega = \sqrt{\frac{18(\gamma - p_A)}{\rho}}$$

$$= \sqrt{\frac{18\left[62.4\frac{\text{lb}}{\text{ft}^3} - \left(-2080\frac{\text{lb}}{\text{ft}^2}\right)\right]}{1.94\ \frac{\text{slugs}}{\text{ft}^3}}} = \underline{\underline{141\ \frac{\text{rad}}{\text{s}}}}$$

2.105

2.105 An open 1-m-diameter tank contains water at a depth of 0.5 m when at rest. As the tank is rotated about its vertical axis the center of the fluid surface is depressed. At what angular velocity will the bottom of the tank first be exposed? No water is spilled from the tank.



h_i ~ initial depth

Equation for surfaces of constant pressure (Eq. 2.32):

$$z = \frac{\omega^2 r^2}{2g} + \text{constant}$$

For free surface with $h=0$ at $r=0$,

$$h = \frac{\omega^2 r^2}{2g}$$

The volume of fluid in rotating tank is given by

$$\forall_f = \int_0^R 2\pi r h \, dr = \frac{2\pi\, \omega^2}{2g}\int_0^R r^3 dr = \frac{\pi \omega^2 R^4}{4g}$$

Since the initial volume, $\forall_i = \pi R^2 h_i$, must equal the final volume,

$$\forall_f = \forall_i$$

so that

$$\frac{\pi \omega^2 R^4}{4g} = \pi R^2 h_i$$

or

$$\omega = \sqrt{\frac{4 g h_i}{R^2}} = \sqrt{\frac{4\left(9.81 \frac{m}{s^2}\right)(0.5m)}{(0.5m)^2}} = \underline{\underline{8.86 \frac{rad}{s}}}$$

[4-5]

Fluid Dynamics

Elementary Fluid Mechanics [4-5]

$$\frac{dp}{ds}+\frac{1}{2}\rho\frac{d(V^2)}{ds}+\gamma\frac{dz}{ds}=0 \text{ (along streamline)}$$

$$\int\frac{dp}{\rho}+\frac{1}{2}V^2+gz=\text{constant (along streamline)}$$

$$\gamma\frac{dz}{dn}+\frac{\partial p}{\partial n}+\frac{\rho V^2}{R}=0 \text{ (normal to streamline)}$$

$$p+\rho\int\frac{V^2}{R}dn+\gamma z=\text{constant (normal to streamline)}$$

$$V=\sqrt{2gh} \text{ Free jets.}$$

$$A_1V_1=A_2V_2 \text{ Conservation of mass.}$$

$$Q=z_2b\sqrt{\frac{2g(z_1-z_2)}{1-(z_2/z_1)^2}} \text{ Sluice.} \qquad Q=C_1b\sqrt{2g}h^{\frac{3}{2}} \text{ Sharp crested weir.}$$

[4:2] Fluid Dynamics

Recap

Pressures in accelerating (static) fluids

Linear

Simplifies when: $a_x = a_y = a_z = 0$ to $\frac{dp}{dx} = \frac{dp}{dy} = 0$ and $\frac{dp}{dz} = -\gamma; \frac{dz}{dx} = -\frac{a_x}{g+a_z}$

Rotational

Rigid body rotation: $\frac{\partial p}{\partial r} = \rho r \omega^2; \frac{\partial p}{\partial \theta} = 0; \frac{\partial p}{\partial z} = -\gamma; \begin{cases} z = \frac{\omega r^2}{2g} + c \\ p = \frac{\rho \omega^2 r^2}{2} - \gamma z + c \end{cases}$

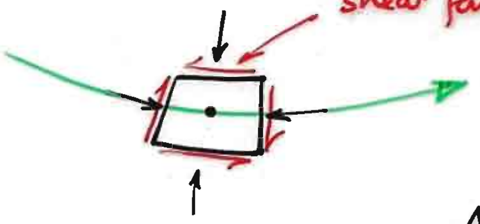
Outline

Moving fluid accelerations

- Along streamline
- Normal to streamline

FLUID DYNAMICS — BERNOULLI EQUATION

Behaviour of <u>INVISCID</u> fluids. i.e. $\mu = \nu \Rightarrow 0$



shear forces controlled by viscosity $(\mu \rightarrow 0)$

Flow path

Apply force balance: $\underline{F} = m\underline{a}$

Due to assumptions, apply only where:

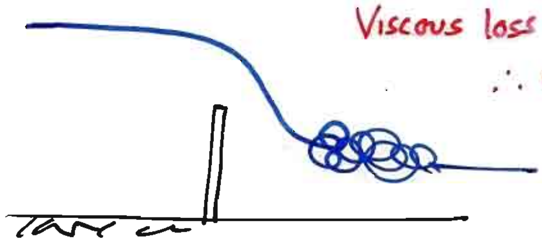
1. Incompressible / Isothermal.
2. Irrotational
3. Steady
4. Inviscid

Or <u>nearly</u> met !!

Eg: Viscosity $\rightarrow 0$?

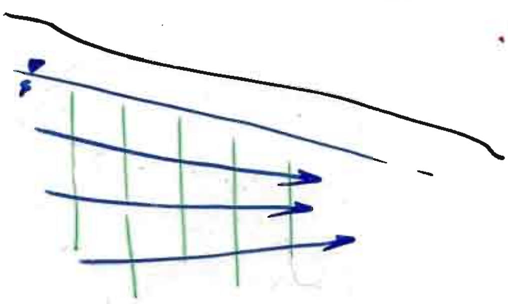
Both are water.

a)



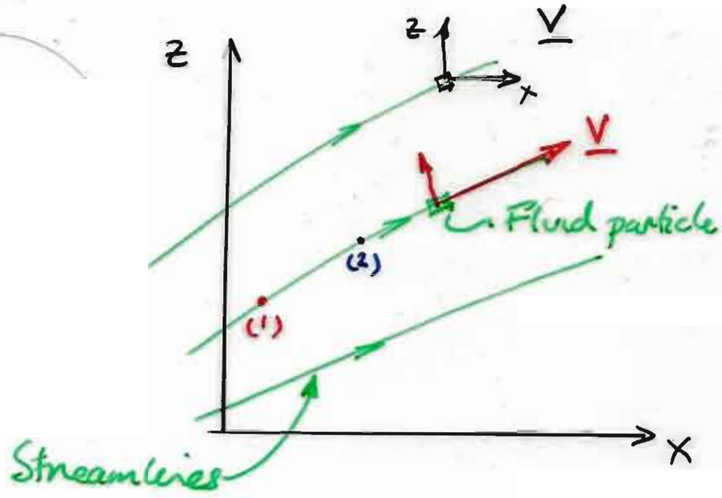
Viscous loss low ∴ OK.

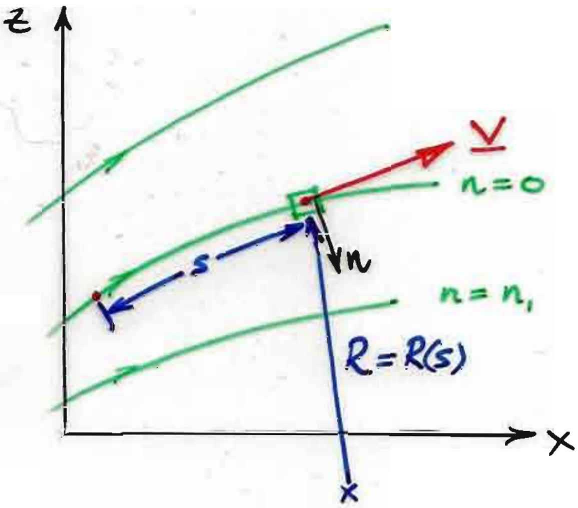
b)



Viscous loss high ∴ OK not!

FRAME OF REFERENCE





☐ Streamlines – path of a particle.

☐ Choose co-ordinate system. – choose convenient system // and ⊥ to streamlines.

∴ velocity vector $\underline{V}$ is tangent to streamline.

For streamline coordinate system:

Particle location: $s = s(t)$

Radius of curvature (local): $R = R(s)$

Velocity (along streamline): $V = \frac{ds}{dt}$ (not a vector)

Accelerations – in $(x-z)$ $\begin{Bmatrix} a_x \\ a_z \end{Bmatrix} = \begin{Bmatrix} dV_x/dt \\ dV_z/dt \end{Bmatrix} \Rightarrow \underline{a} = \frac{d}{dt}\underline{V}$

$$a_s = \frac{dV}{dt} = \left(\frac{\partial V}{\partial s}\right)\frac{ds}{dt} = \frac{\partial V}{\partial s} V$$

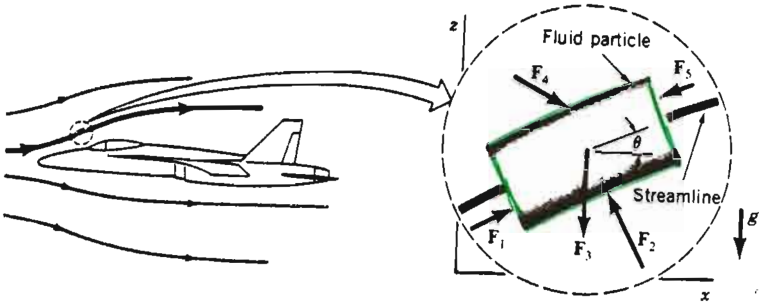
$$V = \frac{ds}{dt}$$

in $(n-s)$ $\begin{Bmatrix} a_s \\ a_n \end{Bmatrix} = \begin{Bmatrix} V\frac{\partial V}{\partial s} \\ \frac{V^2}{R} \end{Bmatrix}$

$\underline{F} = m\underline{a}$ ALONG STREAMLINE   i.e. a_s

Apply force balance to a component of the flow system.

Isolate as a free body.



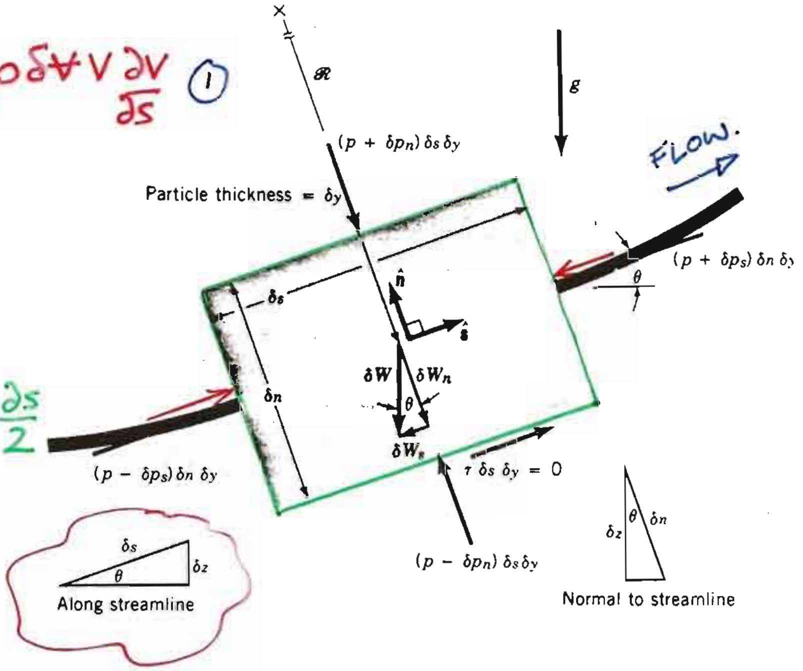
① Newton's law of motion

$$\Sigma \delta F_s = \delta m\, a_s = \delta m\, V \frac{\partial V}{\partial s} = \rho \delta \forall V \frac{\partial V}{\partial s} \quad ①$$

② Gravitational force

$$\delta W_s = -\delta W \sin\theta \quad ②$$

③ Pressure force $(\nabla p \neq 0)$; $\delta p_s \simeq \frac{\partial p}{\partial s}\frac{\delta s}{2}$



$$\delta F_{ps} = (p-\delta p_s)\,\delta n\,\delta y - (p+\delta p_s)\,\delta n\,\delta y$$

$$\delta F_{ps} = -2\,\delta p_s\,\delta n\,\delta y = -\frac{\partial p}{\partial s}\,\delta s\,\delta n\,\delta y = -\frac{\partial p}{\partial s}\,\delta \forall \quad ③$$

Viscous forces; $\tau\,\delta s\,\delta y = 0$

Force balance: $\Sigma \delta F_s = \delta W_s + \delta F_{ps} = \left(-\gamma \sin\theta - \frac{\partial p}{\partial s}\right)\delta \forall$

Equation ①

$$\rho V \frac{\partial V}{\partial s}\cancel{\delta \forall} = \left(-\gamma \sin\theta - \frac{\partial p}{\partial s}\right)\cancel{\delta \forall}$$

Change in velocity results from: 1. Gravitational forces] BERNOULLI
2. Pressure gradient

BERNOULLI EQUATION

Rearrange from the form:

$$\overset{\textcircled{2}}{\rho V \frac{\partial V}{\partial s}} = -\overset{\textcircled{1}}{\gamma \sin\theta} - \overset{\textcircled{3}}{\frac{\partial p}{\partial s}}$$

Along streamline: ①

$$\sin\theta = \frac{dz}{ds}$$

② $V\frac{dV}{ds} = \frac{1}{2}d(V^2)/ds$

$$\boxed{\text{i.e. } \frac{d(V^2)}{ds} = \frac{d(V^2)}{dV}\,\frac{dV}{ds} = 2V\,\frac{dV}{ds}}$$

③ n = const. along streamline · dn = 0

$$\therefore\ dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}\cancelto{0}{dn} = \frac{dp}{ds}ds$$

Substituting ①, ② and ③ into Force balance:

$$\frac{1}{2}\rho\,\frac{d(V^2)}{ds} = -\gamma\,\frac{dz}{ds} - \frac{dp}{ds}$$

Simplifying:

$$dp + \frac{1}{2}\rho\, d(V^2) + \gamma\, dz = 0$$

Along streamline.

Integrating:

$$\int\frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C$$

Along streamline

BERNOULLI EQUATION (1738)

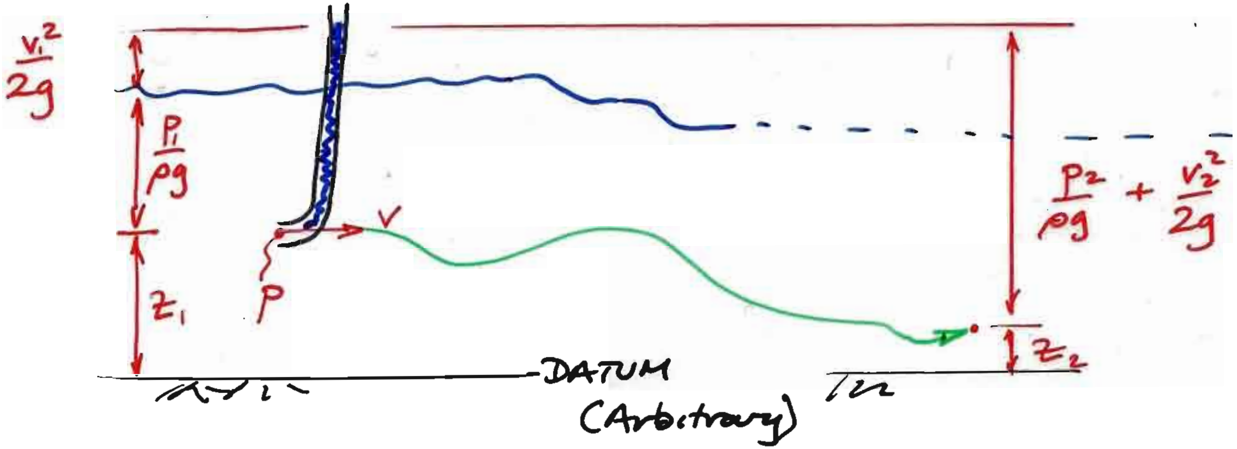
If constant density (ρ) for liquids and "slowly flowing" gases, steady, inviscid, incompressible

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline}$$

OR (dividing by ρg)

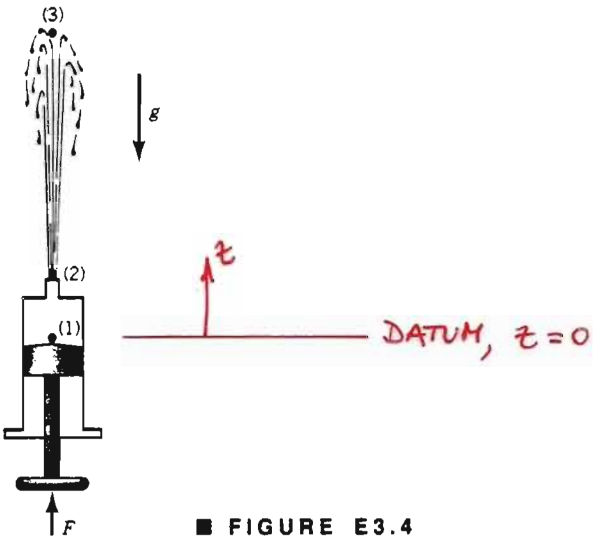
$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{const.}$$

Dimensions of length (heads).



EXAMPLE 3.4

Consider the flow of water from the syringe shown in Fig. E3.4. A force applied to the plunger will produce a pressure greater than atmospheric at point (1) within the syringe. The water flows from the needle, point (2), with relatively high velocity and coasts up to point (3) at the top of its trajectory. Discuss the energy of the fluid at points (1), (2), and (3) by using the Bernoulli equation.



■ FIGURE E3.4

SOLUTION

If the assumptions (steady, inviscid, incompressible flow) of the Bernoulli equation are approximately valid, it then follows that the flow can be explained in terms of the partition of the total energy of the water. According to Eq. 3.13 the sum of the three types of energy (kinetic, potential, and pressure) or heads (velocity, elevation, and pressure) must remain constant. The following table indicates the relative magnitude of each of these energies at the three points shown in the figure.

	Energy Type		
Point	**Kinetic** $\rho V^2/2$	**Potential** γz	**Pressure** p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

(Ans)

The motion results in (or is due to) a change in the magnitude of each type of energy as the fluid flows from one location to another. An alternate way to consider this flow is as follows. The pressure gradient between (1) and (2) produces an acceleration to eject the water from the needle. Gravity acting on the particle between (2) and (3) produces a deceleration to cause the water to come to a momentary stop at the top of its flight.

If friction (viscous) effects were important, there would be an energy loss between (1) and (3) and for the given p_1 the water would not be able to reach the height indicated in the figure. Such friction may arise in the needle (see Chapter 8, pipe flow) or between the water stream and the surrounding air (see Chapter 9, external flow).

[4:3] Fluid Dynamics

Recap

Along streamline

$$\begin{cases} \dfrac{dp}{ds} + \dfrac{1}{2}\rho\dfrac{d(V^2)}{ds} + \gamma\dfrac{dz}{ds} = 0 \text{ (along streamline)} \\ \displaystyle\int\frac{dp}{\rho} + \frac{1}{2}V^2 + gz = \text{constant (along streamline)} \end{cases}$$

Outline

Normal to streamline

$$\begin{cases} \gamma\dfrac{dz}{dn} + \dfrac{\partial p}{\partial n} + \dfrac{\rho V^2}{R} = 0 \text{ (normal to streamline)} \\ p + \rho\displaystyle\int\frac{V^2}{R}dn + \gamma z = \text{constant (normal to streamline)} \end{cases}$$

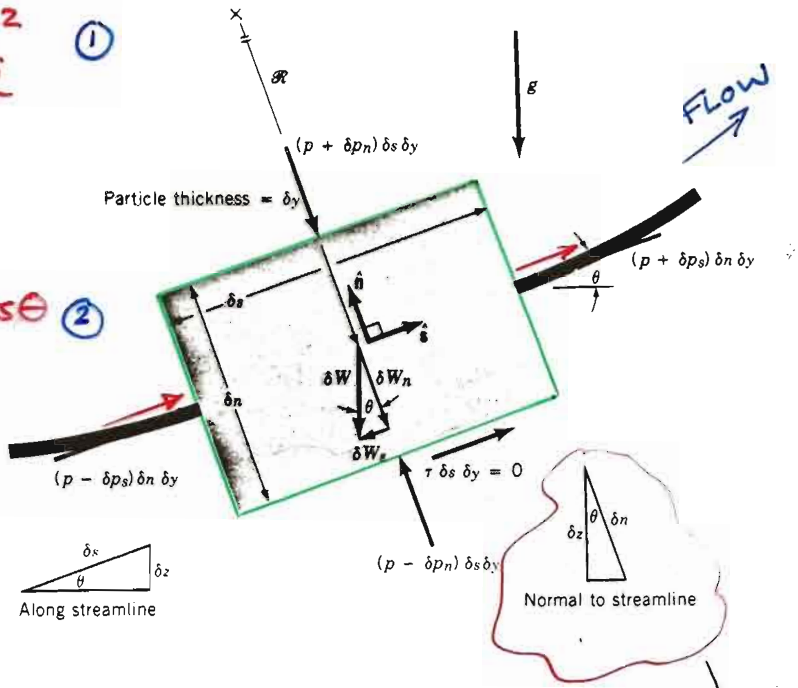
Use of "Bernoulli"

Stagnation

Pressure measurement

Continuity

$\underline{F} = m\,\underline{a}$ NORMAL TO STREAMLINE



① Newton's law of motion ⊥ to streamline

$$\Sigma \delta F_n = \delta m \frac{V^2}{R} = \rho\, \delta\forall \frac{V^2}{R} \qquad \textcircled{1}$$

② Gravitational force

$$\delta W_n = -\delta W \cos\theta = -\gamma\, \delta\forall \cos\theta \qquad \textcircled{2}$$

③ Pressure force

$$\delta p_n \simeq \frac{\partial p}{\partial n} \frac{\delta n}{2}$$

$$\delta F_{p_n} = (p - \delta p_n)\,\delta s\,\delta y - (p + \delta p_n)\,\delta s\,\delta y$$

$$\delta F_{p_n} = -2\,\delta p_n\,\delta s\,\delta y = -\frac{\partial p}{\partial n}\,\delta s\,\delta n\,\delta y$$

$$\delta F_{p_n} = -\frac{\partial p}{\partial n}\,\delta\forall \qquad \textcircled{3}$$

Force balance: $\Sigma \delta F_n = \delta W_n + \delta F_{p_n} = \left(-\gamma\cos\theta - \frac{\partial p}{\partial n}\right)\delta\forall$

Equation ①

$$\rho \frac{V^2}{R} \cancel{\delta\forall} = \left(-\gamma\cos\theta - \frac{\partial p}{\partial n}\right) \cancel{\delta\forall}$$

$$\cos\theta = \frac{dz}{dn}$$

$$\frac{\rho V^2}{R} + \gamma \frac{dz}{dn} + \frac{\partial p}{\partial n} = 0$$

Particle trajectory of radius, R.

Particle weight ⊥ to streamline

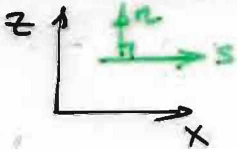
Pressure gradient ⊥ to streamline

OTHER INTERPRETATIONS

If flow in straight line then $R \to \infty$; or $V = 0$

$$\gamma \frac{dz}{dn} + \frac{\partial p}{\partial n} = 0$$

For horizontal flow:

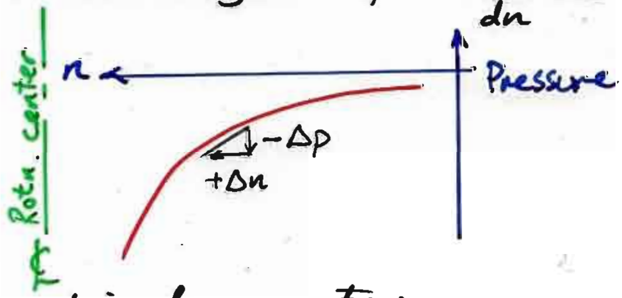


$$\frac{dz}{dn} = +1$$

Consequently $+\gamma + \frac{\partial p}{\partial n} = 0 \Rightarrow \frac{\partial p}{\partial z} = -\gamma$ ✓

For flow of gases $\gamma \to 0$ or in the horizontal plane $\frac{dz}{dn} = 0$

$$\frac{\partial p}{\partial n} = -\rho \frac{V^2}{R}$$



Normal pressure gradient ∝ to radius of curvature.

$\uparrow R \qquad \downarrow \frac{dp}{dn}$

Tornado pressure lower on inside since pressure gradient balances the centripital force of the swirling gas.

FINAL FORM FOR $\underline{F} = m\underline{a}$ $\perp$ TO STREAMLINE

$$\rho \frac{V^2}{R} + \gamma \frac{dz}{dn} + \frac{\partial p}{\partial n} = 0$$

Multiply by dn and note: $\boxed{\frac{\partial p}{\partial n} = \frac{dp}{dn}}$ $\perp$ across streamline

And integrate w.r.t dn

$$\int \frac{V^2}{R} dn + gz + \int \frac{1}{\rho} dp = \text{constant across streamline}$$

Incompressible flows $\rho = \text{const.}$

$$\boxed{\int \frac{V^2}{R} dn + gz + \frac{p}{\rho} = \text{constant across streamline}}$$

Stating in terms of heads: (by dividing through by 'g')

$$\int \frac{V^2}{Rg} dn + z + \frac{p}{\gamma} = \text{constant across streamline.}$$

Pressure or static head

Velocity head

Elevation head

EXAMPLE 3.3

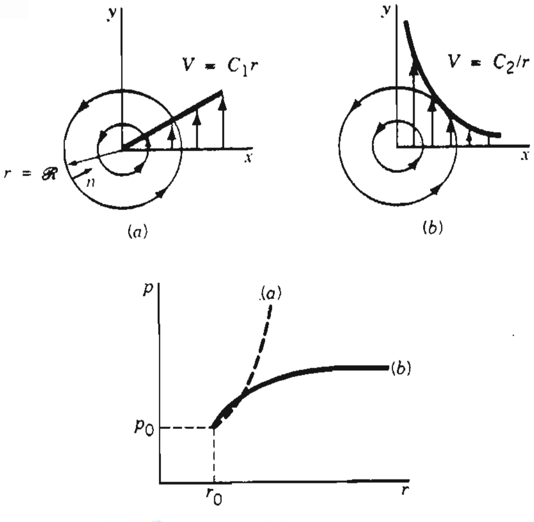
Shown in Figs. E3.3*a*,*b* are two flow fields with circular streamlines. The velocity distributions are

$$V(r) = C_1 r \qquad \text{for case } (a)$$

and

$$V(r) = \frac{C_2}{r} \qquad \text{for case } (b)$$

where C_1 and C_2 are constant. Determine the pressure distributions, $p = p(r)$, for each, given that $p = p_0$ at $r = r_0$.



■ **FIGURE E3.3**

$$\frac{\rho V^2}{R} + \gamma \cancelto{0}{\frac{dz}{dn}} + \frac{\partial p}{\partial n} = 0$$

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$$

$$\frac{\rho V^2}{R} = \frac{\partial p}{\partial r}$$

∴ Substitute for $V(r)$ with $R = r$

SOLUTION

We assume the flows are steady, inviscid, and incompressible with streamlines in the horizontal plane ($dz/dn = 0$). Since the streamlines are circles, the coordinate n points in a direction opposite of that of the radial coordinate, $\partial/\partial n = -\partial/\partial r$, and the radius of curvature is given by $\mathcal{R} = r$. Hence, Eq. 3.10 becomes

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r}$$

For case (*a*) this gives

$$\frac{\partial p}{\partial r} = \rho C_1^2 r \longrightarrow \int_{p_0}^{p} dp = \rho C_1^2 \int_{r_0}^{r} r\, dr$$

while for case (*b*) it gives

$$\frac{\partial p}{\partial r} = \frac{\rho C_2^2}{r^3} \qquad \int_{p_0}^{p} dp = \rho C_2^2 \int_{r_0}^{r} \frac{1}{r^3}\, dr$$

For either case the pressure increases as r increases since $\delta p/\delta r > 0$. Integration of these equations with respect to r, starting with a known pressure $p = p_0$ at $r = r_0$, gives

$$p = \frac{1}{2}\rho C_1^2(r^2 - r_0^2) + p_0 \qquad \textbf{(Ans)}$$

for case (*a*) and

$$p = \frac{1}{2}\rho C_2^2\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + p_0 \qquad \textbf{(Ans)}$$

for case (*b*). These pressure distributions are sketched in Fig. E3.3*c*. The pressure distributions needed to balance the centrifugal accelerations in cases (*a*) and (*b*) are not the same because the velocity distributions are different. In fact for case (*a*) the pressure increases without bound as $r \rightarrow \infty$, while for case (*b*) the pressure approaches a finite value as $r \rightarrow \infty$. The streamline patterns are the same for each case, however.

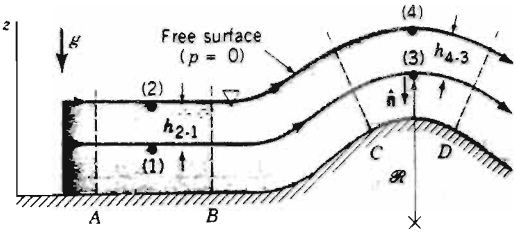
MORAL OF STORY !!

If a gradient $\frac{\partial f}{\partial x}$ defined then limits of integration provide a fix to distribution. i.e. fixes boundary conditions !!

$$p + \rho \int \frac{V^2}{R}\, dn + \gamma z = \text{Constant.}$$

EXAMPLE 3.5

Consider the inviscid, incompressible, steady flow shown in Fig. E3.5. From section A to B the streamlines are straight, while from C to D they follow circular paths. Describe the pressure variation between points (1) and (2) and points (3) and (4).



■ FIGURE E3.5

SOLUTION

With the above assumptions and the fact that $\mathcal{R} = \infty$ for the portion from A to B, Eq. 3.1 becomes

A-B

$$p + \gamma z = \text{constant}$$

The constant can be determined by evaluating the known variables at the two locations using $p_2 = 0$ (gage), $z_1 = 0$, and $z_2 = h_{2-1}$ to give

$$p_1 = p_2 + \gamma(z_2 - z_1) = p_2 + \gamma h_{2-1} \qquad \textbf{(Ans)}$$

Note that since the radius of curvature of the streamline is infinite, the pressure variation in the vertical direction is the same as if the fluid were stationary.

C-D

However, if we apply Eq. 3.14 between points (3) and (4) we obtain (using $dn = -dz$)

$$p_4 + \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}}(-dz) + \gamma z_4 = p_3 + \gamma z_3$$

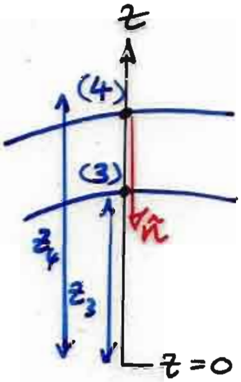
With $p_4 = 0$ and $z_4 - z_3 = h_{4-3}$ this becomes

$$p_3 = \gamma h_{4-3} - \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}}\, dz \qquad \textbf{(Ans)}$$

To evaluate the integral we must know the variation of V and $\mathcal{R}$ with z. Even without this detailed information we note that the integral has a positive value. Thus, the pressure at (3) is less than the hydrostatic value, γh_{4-3}, by an amount equal to $\rho \int_{z_3}^{z_4} (V^2/\mathcal{R})\, dz$. This lower pressure, caused by the curved streamline, is necessary to accelerate the fluid around the curved path.

Note that we did not apply the Bernoulli equation (Eq. 3.13) across the streamlines from (1) to (2) or (3) to (4). Rather we used Eq. 3.14. As is discussed in Section 3.6, application of the Bernoulli equation across streamlines (rather than along them) may lead to serious errors.

Writing Bernoulli ⊥ to streamline.



1. $\bar{n}$ direction *always* points to center of revolution.
∴ in this case $dn = -dz$

$$P_4 + \rho\int_{z_0}^{z_4} \frac{V^2}{R}\,dn + \gamma z_4 = P_3 + \rho\int_{z_0}^{z_3}\frac{V^2}{R}\,dn + \gamma z_3$$

$$\rho\int_{z_0}^{z_4}\frac{V^2}{R}(-dz) - \rho\int_{z_0}^{z_3}\frac{V}{R}(-dz) = \rho\int_{z_3}\frac{V^2}{R}(-dz)$$

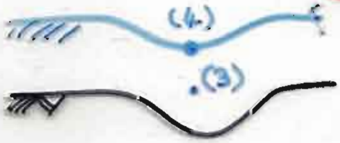
Resubstitute:

$$P_4 - \rho\int_{z_3}^{z_4}\frac{V^2}{R}\,dz + \gamma z_4 = P_3 + \gamma z_3$$

Boundary conditions: $P_4 = 0$ $\qquad z_4 - z_3 = h_{3-4}$

$$P_3 = \gamma h_{3-4} - \rho\int_{z_3}^{z_4}\frac{V^2}{R}\,dz$$

Integral (+ve) ∴ reduced pressure over hydrostatic.



$$P_3 = \gamma h_{3-4} + \rho\int_{z_3}^{z_4}\frac{V^2}{R}\,dz$$ since $dn = +dz$

PHYSICAL INTERPRETATION

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{const.}$$ along streamline

$$p + \rho\int \frac{V^2}{R}\, dn + \gamma z = \text{const.}$$ across streamline

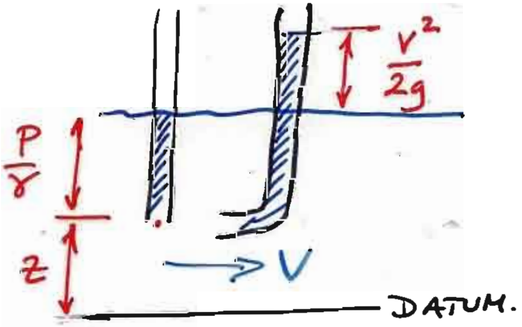
Requirements: Incompressible (liquids)
Inviscid (not porous media/pipes)
Steady

Each term represents "force" needed to provide an acceleration of a fluid particle.

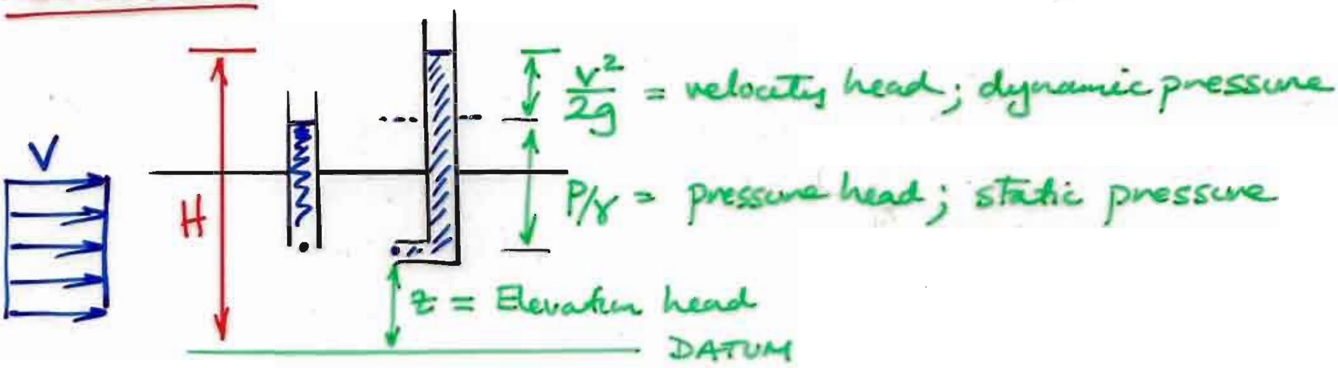
i.e Forces due to:
- pressure, p
- body force/gravity, γz
- kinetic energy, $\frac{1}{2}\rho V^2$; $\rho\int\frac{V^2}{R}\,dn$

In terms of heads: $$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{const.}$$

$\frac{V^2}{2g} \equiv$ vertical distance for freefalling body to reach velocity, V.



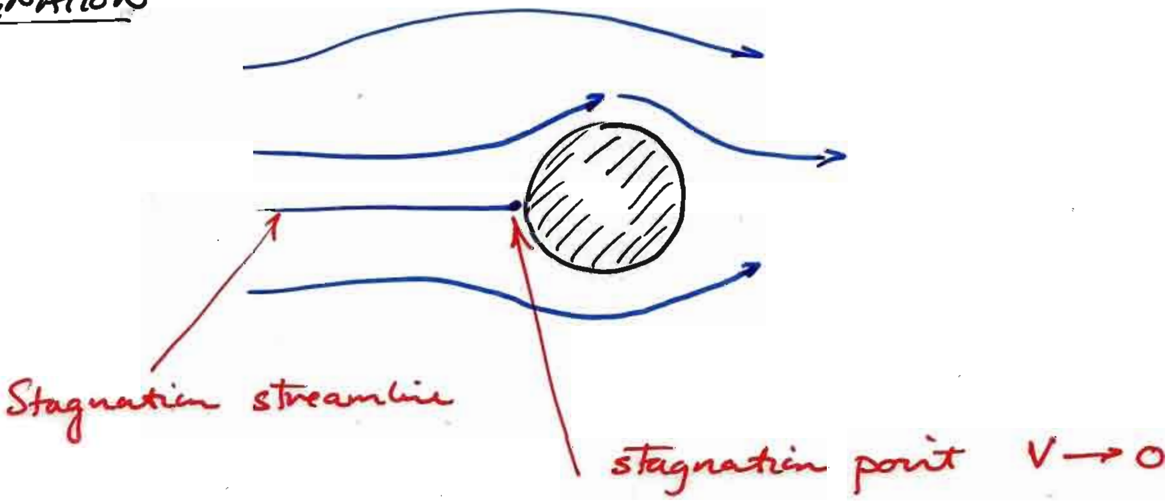
DEFINTIONS



$$H = z + \frac{P}{\gamma} + \frac{v^2}{2g}$$

Total head; total pressure = Hγ.

STAGNATION



Stagnation pressure is max possible along streamline, if z effects are small.

$$H = z + \frac{P}{\gamma} + \frac{v^2}{2g} \qquad H = \text{const.}$$

$V \to 0$ ∴ convert energy to P.

3.31

3.31 A loon is a diving bird equally at home "flying" in the air or water. What swimming velocity under water will produce a dynamic pressure equal to that when it flies in the air at 40 mph?

$$\frac{1}{2}\rho_{air}V_{air}^2 = \frac{1}{2}\rho_{H_2O}V_{H_2O}^2 \quad or \quad V_{H_2O} = \left[\frac{\rho_{air}}{\rho_{H_2O}}\right]^{1/2} V_{air}$$

Thus,

$$V_{H_2O} = \left[\frac{2.38\times10^{-3}\,\frac{slugs}{ft^3}}{1.94\,\frac{slugs}{ft^3}}\right](40\,mph) = \underline{\underline{1.40\,mph}}$$

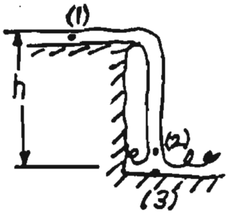
3.32

3.32 Niagara Falls is approximately 167 ft high. If the water flows over the crest of the falls with a velocity of 8 ft/s and viscous effects are neglected, with what velocity does the water strike the rocks at the bottom of the falls? What is the maximum pressure of the water on the rocks? Repeat the calculations for the 1430-ft-high Upper Yosemite Falls in Yosemite National Park. Is it reasonable to neglect viscous effects for these falls? Explain

$$H = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \quad @(2)$$

and

$$H = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + z_3 \quad @(3)$$



with $p_1 = p_2 = 0$, $V_1 = 8\,\frac{ft}{s}$, $V_3 = 0$, $z_2 \approx z_3 = 0$, and $z_1 = h$

Thus,

$$\boxed{V_2 = \sqrt{2g\left(h + \frac{V_1^2}{2g}\right)}} \quad and \quad \boxed{p_3 = \frac{1}{2}\rho V_1^2 + \gamma h}$$

With $h = 167\,ft$,

$$V_2 = \left[2\left(32.2\,\frac{ft}{s^2}\right)\left(167\,ft + \frac{(8\,\frac{ft}{s})^2}{2(32.2\,\frac{ft}{s^2})}\right)\right]^{1/2} = \underline{\underline{104\,\frac{ft}{s}}}$$

and

$$p_3 = \frac{1}{2}\left(1.94\,\frac{slugs}{ft^3}\right)\left(8\,\frac{ft}{s}\right)^2 + 62.4\,\frac{lb}{ft^3}\,(167\,ft) = 10{,}500\,\frac{lb}{ft^2} = \underline{\underline{72.8\,psi}}$$

With $h = 1430\,ft$

$$V_2 = \left[2\left(32.2\,\frac{ft}{s^2}\right)\left(1430\,ft\right) + \frac{(8\,\frac{ft}{s})^2}{2(32.2\,\frac{ft}{s^2})}\right)\right]^{1/2} = \underline{\underline{304\,\frac{ft}{s}}}$$

and

$$p_3 = \frac{1}{2}\left(1.94\,\frac{slugs}{ft^3}\right)\left(8\,\frac{ft}{s}\right)^2 + 62.4\,\frac{lb}{ft^3}\,(1430\,ft) = 89{,}300\,\frac{lb}{ft^2} = \underline{\underline{620\,psi}}$$

Aerodynamic drag on the water would reduce the values of V_2 and p_3 (especially for the $h = 1430$ ft case).

3.3

3.3 An incompressible fluid flows steadily past a circular cylinder as shown in Fig. P3.3. The fluid velocity along the dividing streamline ($-\infty \le x \le -a$) is found to be $V = V_0\,(1 - a^2/x^2)$, where a is the radius of the cylinder and V_0 is the upstream velocity. **(a)** Determine the pressure gradient along this streamline. **(b)** If the upstream pressure is p_0, integrate the pressure gradient to obtain the pressure $p(x)$ for $-\infty \le x \le -a$. **(c)** Show from the result of part **(b)** that the pressure at the stagnation point ($x = -a$) is $p_0 + \rho V_0^2/2$, as expected from the Bernoulli equation.

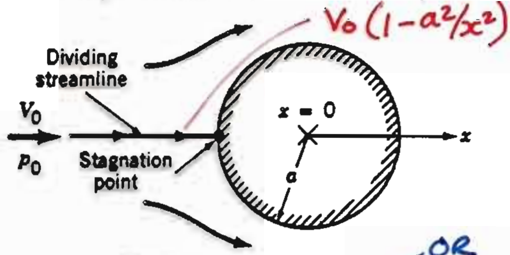


FIGURE P3.3

OR
ds =

(a) $\frac{\partial p}{\partial s} = -\gamma \sin\theta - \rho V \frac{\partial V}{\partial s}$ dx $\quad$ but $\theta = 0$ and $\frac{\partial V}{\partial s} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial s} = \frac{\partial V}{\partial x}$

$$= V_0\left[-a^2\right]\left(\frac{-2}{x^3}\right) = \frac{2a^2V_0}{x^3}$$

Thus,

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial x} = \underline{\underline{-2\rho a^2 V_0^2\left[1-\left(\frac{a}{x}\right)^2\right]/x^3}}$$

(b) $\int_{p_0}^{p} dp = \int_{x=-\infty}^{x} \frac{dp}{dx}\,dx$ or $p - p_0 = -2\rho a^2 V_0^2 \int_{-\infty}^{x}\left[1-\left(\frac{a}{x}\right)^2\right]\frac{dx}{x^3}$

$$= -2\rho a^2 V_0^2 \int_{-\infty}^{x}\left[x^{-3} - a^2 x^{-5}\right]dx$$

Thus,

$$\underline{\underline{p = p_0 + \rho V_0^2\left[\left(\frac{a}{x}\right)^2 - \frac{1}{2}\left(\frac{a}{x}\right)^4\right]}} \quad \text{for} \quad -\infty \le x \le -a$$

(c) For $x = -a$, from part (b):

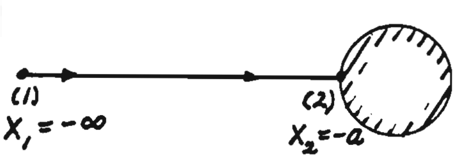
$$p\Big|_{x=-a} = p_0 + \rho V_0^2\left[(-1)^2 - \frac{1}{2}(-1)^4\right] = \underline{\underline{p_0 + \frac{1}{2}\rho V_0^2}}$$

Note: Bernoulli equation from point (1) where $V_1 = V_0$, $p_1 = p_0$ and $z_1 = z_0$ to point (2) where $V_2 = 0$, $z_2 = z_0$ gives

$$p_1 + \frac{1}{2}\rho V_1^2 + \cancel{\gamma z_1} = p_2 + \frac{1}{2}\rho \overset{0}{\cancel{V_2^2}} + \cancel{\gamma z_2}$$

or

$$\underline{\underline{p_2 = p_0 + \frac{1}{2}\rho V_0^2}}$$



[5:1] Fluid Dynamics

Recap

Along streamline

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant (along streamline)}$$

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 + h_P = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 + \sum h_L^{major} + \sum h_L^{minor}$$

Normal to streamline

$$p + \rho\int\frac{V^2}{R}dn + \gamma z = \text{constant (normal to streamline)}$$

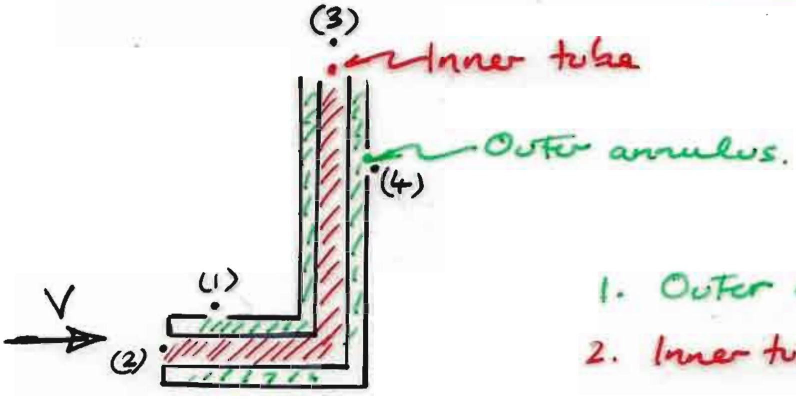
Outline

Pressure measurement

Free jets $V = \sqrt{2gh}$

Continuity $A_1V_1 = A_2V_2$

MEASUREMENT OF BERNOULLI TERMS



PITOT – STATIC TUBE.

1. Outer annulus measures static pressure.
2. Inner tube measures $v^2/2g$.

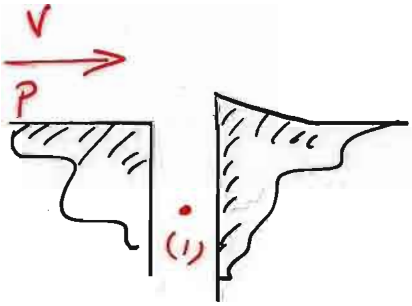
Pitot tube: $p_3 = p_2 + \frac{1}{2}\rho v^2$ ⓐ $p_2 \equiv p$

if elev. differences small.

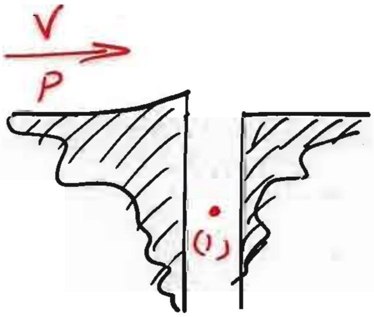
Static tube: $p_4 = p_1 = p$ ⓑ

Combining ⓐ and ⓑ $p_3 - p_4 = \frac{1}{2}\rho v^2$

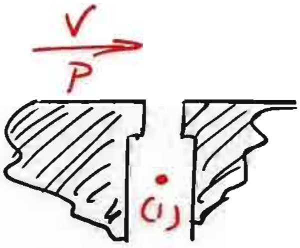
$$v = \sqrt{2(p_3 - p_4)/\rho}$$



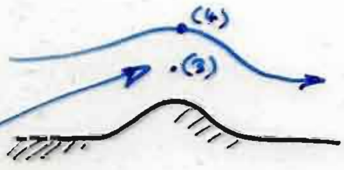
$P_1 > P$



$P_1 < P$



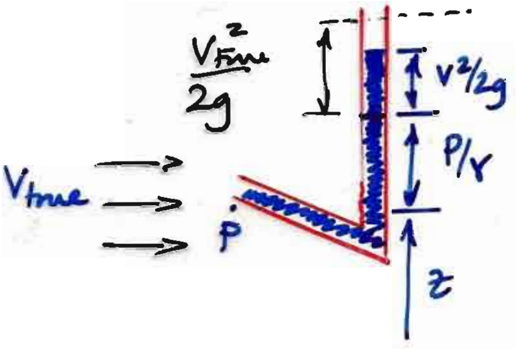
$P_1 = P$

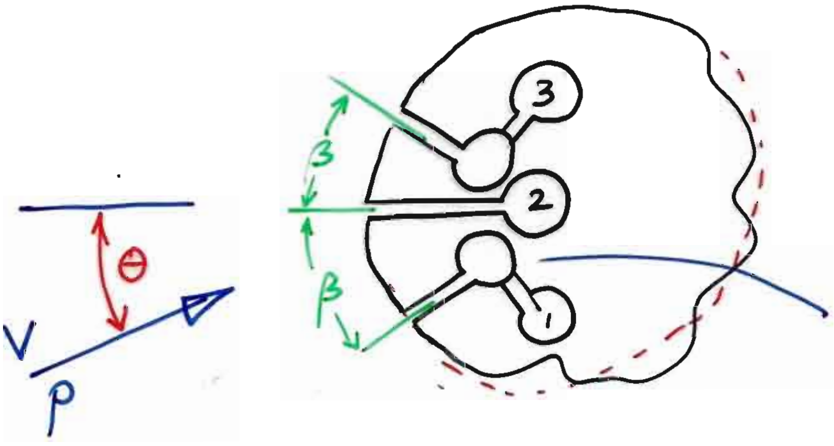


Pressure reduced due to angular accn.

DIRECTION FINDING PITOT TUBES

- Eases problems of directing tube 'directly' into airstream





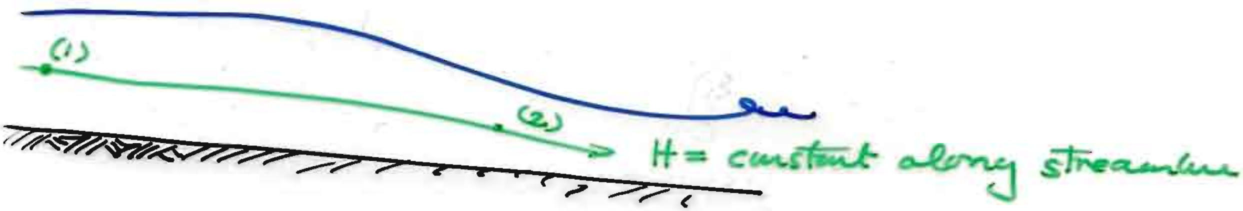
Need ⊥ to 'stagnati' flow.

If $\theta = 0$

$$p_1 = p_3 = p$$

$$p_2 = p + \frac{1}{2}\rho V^2$$

APPLICATION OF "BERNOULLI PRINCIPLE"

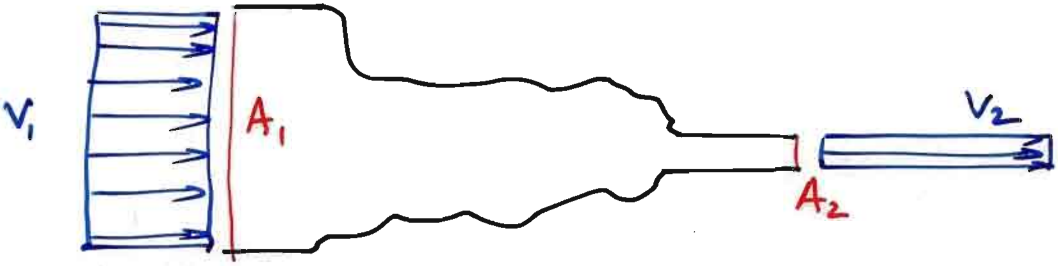


$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

☐ Steady, incompressible, inviscid : Requirements

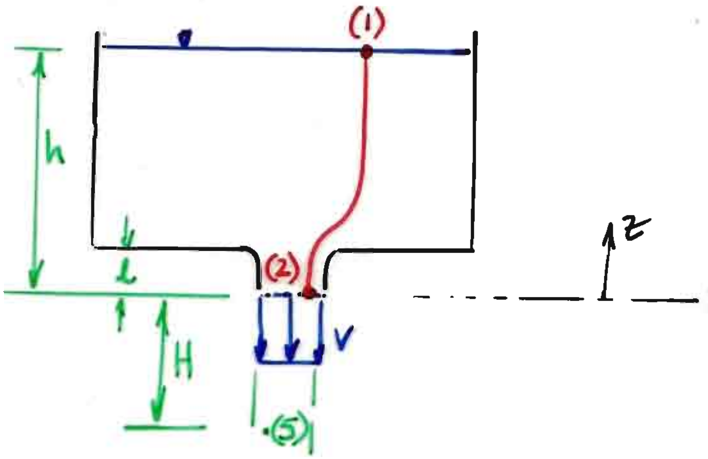
☐ Solve problems if 5 variables known and 1 unknown.

☐ May require to introduce other concepts. e.g.
Conservation of Mass.
Volume in = volume out.



$$V_1 A_1 = V_2 A_2$$

FREE JETS



Meets Bernoulli Requirements?

1. Tank is 'large' compared to jet outflow ∴ steady
2. Fluid is water ∴ incompressible
3. Inviscid – V is large compared to μ and ∴ τ.

(1) & (2) on streamline ∴ $p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$

Point (1) $p_1 = 0$; $V_1 = 0$ and $\gamma z_1 = \gamma h$

Point (2) $p_2 = 0$; $V_2 \neq 0$ and $\gamma z_2 = 0$

$$\therefore \quad \gamma h = \frac{1}{2}\rho V_2^2$$

Free jet, implies $p_2 = 0$.

$$V = \sqrt{\frac{2\gamma h}{\rho}} = \sqrt{2gh}$$

If $p = 0$ @ point (2) and $p = 0$ @ point (5) then fluid falls as a "free" jet. With zero pressure throughout (along streamline).

$$p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 = p_5 + \frac{1}{2}\rho V_5^2 + \gamma z_5$$

$p_2 = p_5 = 0$

$z_2 = 0$

$z_5 = -H$

$V_2 = \sqrt{2gh}$

Resubstituting:

$$0 + \frac{1}{2}\rho(2gh) + 0 = 0 + \frac{1}{2}\rho V_5^2 - \gamma H$$

$$\cancel{\frac{1}{2}}\cancel{\rho}(2gh) + \cancel{\rho}gH2 = \cancel{\frac{1}{2}}\cancel{\rho}V_5^2$$

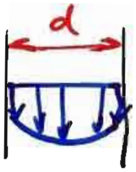
$$\sqrt{2g(h+H)} = V_5$$

i.e. additional velocity is due to free fall $\Delta V = \sqrt{2gH}$

i.e. All potential energy → kinetic energy.

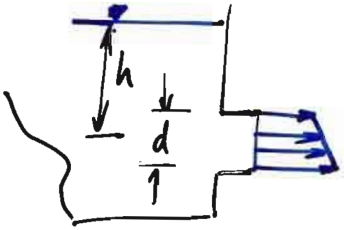
Other issues

1. Flow velocity profile in pipe – smaller @ edges.



ok if $d \ll h$

2. Horizontal nozzle:

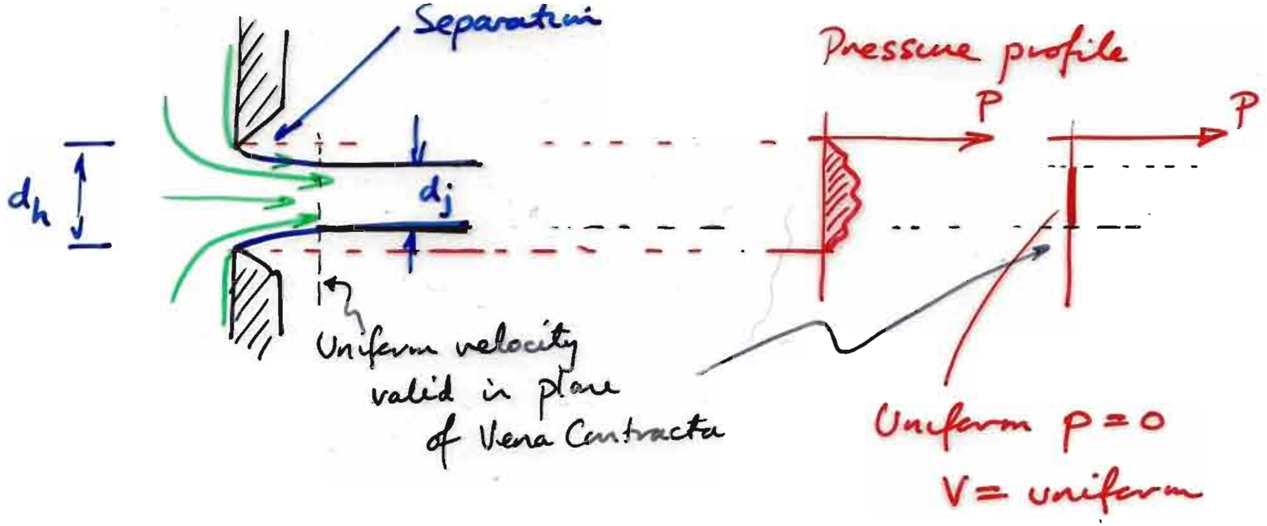


z not uniform over nozzle unless $h \gg d$.

3. Vena Contracta effect.

Effect in sharp nozzles give separation between fluid and nozzle wall.

i.e. $R \rightarrow 0$.



Use correction factor C_c $\qquad C_c = A_j / A_h$

See Figure 3.14.

Defines an "effective" area

[5:2] Fluid Dynamics

Recap

Along streamline

$$\frac{p}{\gamma}+\frac{V^2}{2g}+z=\text{constant (along streamline)}$$

$$\frac{p_1}{\gamma}+\frac{v_1^2}{2g}+z_1+h_P=\frac{p_2}{\gamma}+\frac{v_2^2}{2g}+z_2+\sum h_L^{major}+\sum h_L^{minor}$$

Normal to streamline $$p+\rho\int\frac{V^2}{R}dn+\gamma z=\text{constant (normal to streamline)}$$

Pressure measurement

Free jets $$V=\sqrt{2gh}$$

Outline

Continuity $A_1V_1=A_2V_2$

Compressible fluids

Flow measurements

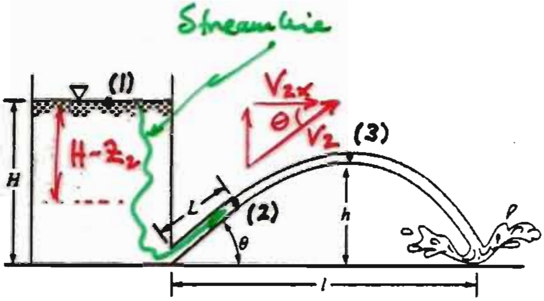
- Confined (pipe) flows
- Open-channel flows

Spatial reference frame

Unsteady flows

3.37

3.37 Water flows from a large tank of depth H, through a pipe of length L, and strikes the ground as shown in Fig. P3.37. Viscous effects are negligible. Determine the distance h as a function of θ.



■ FIGURE P3.37

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2\,,$$ where $p_1=p_2=0$, $z_1=H$, $z_2=L\sin\theta$, and $V_1=0$

Hence, $H=\dfrac{V_2^2}{2g}+L\sin\theta$

or $\quad V_2=\sqrt{2g(H-L\sin\theta)}$ = Same as free jet $(H-z_2)$ (1)

Also since from (2) to (3) the only acceleration the particle feels is that of gravity, it follows that $a_x=0$. Thus, $V_3=V_{2x}=V_2\cos\theta$ (2)

From the Bernoulli equation between (1) and (3),

$$\frac{p_3}{\gamma}+\frac{V_3^2}{2g}+z_3=\overset{0}{\cancel{\frac{p_1}{\gamma}}}+\overset{0}{\cancel{\frac{V_1^2}{2g}}}+\overset{H}{\cancel{z_1}}\,,$$ where $p_1=p_3=0$, $V_1=0$, $z_1=H$, and $z_3=h$

or

$$H=\frac{V_3^2}{2g}+h$$

V_2

By using Eqs. (1) and (2) this gives

$$H=\frac{V_2^2\cos^2\theta}{2g}+h=\frac{2g(H-L\sin\theta)\cos^2\theta}{2g}+h$$

Thus,

$$h=H(1-\cos^2\theta)+L\sin\theta\cos^2\theta$$

or since $1-\cos^2\theta=\sin^2\theta$, $\quad \underline{h=H\sin^2\theta+L\sin\theta\cos^2\theta}$

Note: 1) If $\theta=0$, then $h=0$

2) If $\theta=90°$, then $h=H$

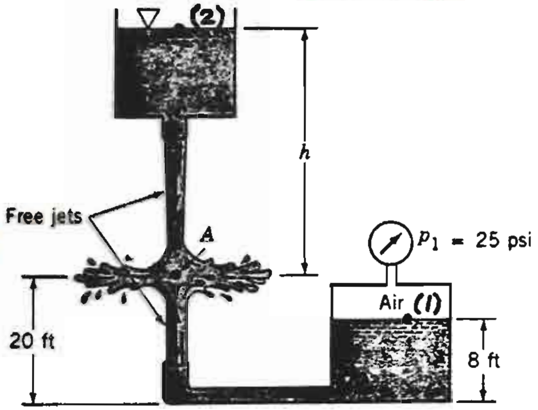
3) If $L\sin\theta>H$, then the above is not valid since $V_2=\sqrt{\text{negative number}}$ (see Eq. 1), which is not possible. Why is this so?

3.38

$P_2=0;\ V_2=0;\ z_2=h+20\,ft$

3.38 Streams of water from two tanks impinge upon each other as shown in Fig. P3.38. If viscous effects are negligible and point A is a stagnation point, determine the height h.

$V_A=0 \quad \begin{cases} P_A=? \\ h=? \end{cases}$



$P_1=25\,psi$
$V_1=0$
$z_1=8\,ft$

FIGURE P3.38

$$\cancelto{0}{\frac{p_2}{\gamma}} + \cancelto{0}{\frac{V_2^2}{2g}} + z_2 = \frac{p_A}{\gamma} + \cancelto{0}{\frac{V_A^2}{2g}} + z_A$$ where $p_2=0$, $V_2=0$, $z_2=h+20ft$

$V_A=0$, and $z_A=20\,ft$

Thus,

$$h+20ft = \frac{p_A}{\gamma} + 20\,ft$$

or

$$h = \frac{p_A}{\gamma} \quad \text{(i.e. Same as standpipe piezometer!)} \qquad (1)$$

Also,

$$\frac{p_1}{\gamma} + \cancelto{0}{\frac{V_1^2}{2g}} + z_1 = \frac{p_A}{\gamma} + \cancelto{0}{\frac{V_A^2}{2g}} + z_A$$ where $p_1=25psi$, $V_1=0$ and $z_1=8ft$

Thus,

$$\frac{p_A}{\gamma} = \frac{p_1}{\gamma} + z_1 - z_A$$ which when combined with Eq. (1) gives

$$h = \frac{p_1}{\gamma} + z_1 - z_A = \frac{25\,\frac{lb}{in.^2}\left(144\,\frac{in.^2}{ft^2}\right)}{62.4\,\frac{lb}{ft^3}} + 8\,ft - 20\,ft = \underline{\underline{45.7\,ft}}$$

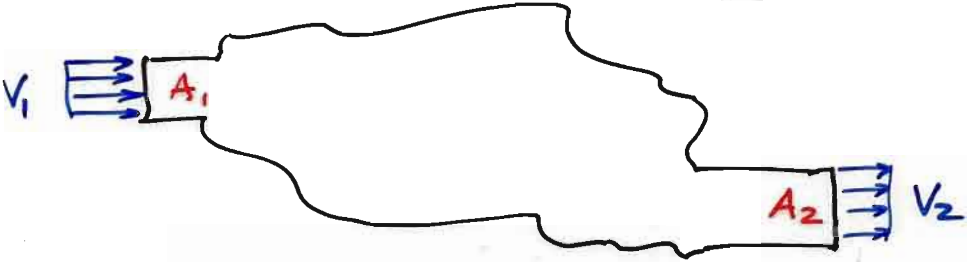
CONFINED FLOWS

- Sometimes extra "constraint" is needed to give an extra "equation" to the Bernoulli expression.

- Typically apply "conservation of mass"

  Mass flowrate in = Mass flowrate out

  If constant density ⟶ Volume in = volume out.



$$\rho_1 A_1 V_1 \cancel{dt} = \rho_2 A_2 V_2 \cancel{dt}$$

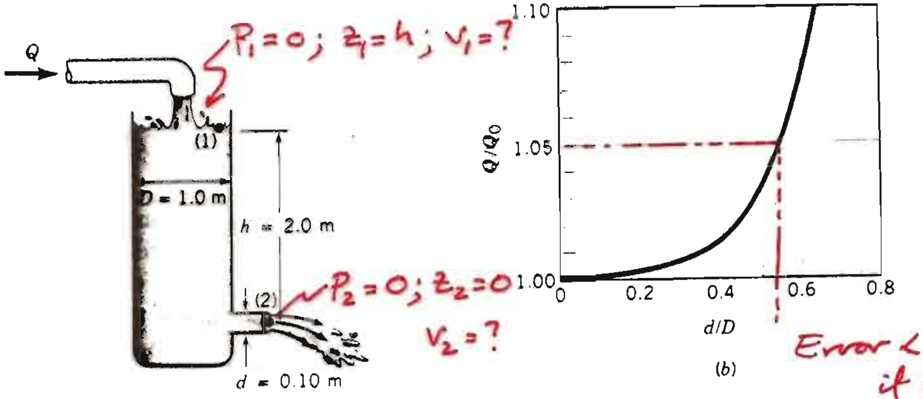
$$Q_1 = Q_2$$

EXAMPLE 3.7

FLOW INTO TOP OF TANK ∴ $V_1 \neq 0$

Need to apply continuity, $\dot{m}_{in} = \dot{m}_{out}$

A stream of water of diameter $d = 0.1$ m flows steadily from a tank of diameter $D = 1.0$ m as shown in Fig. E3.7*a*. Determine the flowrate, Q, needed from the inflow pipe if the water depth remains constant, $h = 2.0$ m.



$P_1 = 0;\ z_1 = h;\ V_1 = ?$

$P_2 = 0;\ z_2 = 0$

$V_2 = ?$

Error < 5% if d/D < .52

(*a*)

(*b*)

SOLUTION

For steady, inviscid, incompressible flow the Bernoulli equation applied between points (1) and (2) is

$$\cancel{p_1} + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = \cancel{p_2}^{\,0} + \tfrac{1}{2}\rho V_2^2 + \cancel{\gamma z_2}^{\,0} \tag{1}$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2 \tag{2}$$

Although the water level remains constant (h = constant), there is an average velocity, V_1, across section (1) because of the flow from the tank. From Eq. 3.19 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\boxed{\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2}$$

CONTINUITY.

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2 \tag{3}$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1 - (d/D)^4}}$$

Thus, with the given data

$$V_2 = \sqrt{\frac{2(9.81\ \text{m/s}^2)(2.0\ \text{m})}{1 - (0.1\text{m}/1\text{m})^4}} = 6.26\ \text{m/s}$$

and

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1\ \text{m})^2(6.26\ \text{m/s}) = 0.0492\ \text{m}^3/\text{s} \tag{Ans}$$

If Q_0 = flow rate if V_1 set to zero.

Then Q/Q_0 is ratio of complex and simplified calculations.

In this example we have not neglected the kinetic energy of the water in the tank ($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D \gg d$), Eq. 3 indicates that $V_1 \ll V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error associated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq 0$, denoted Q, to that assuming $V_1 = 0$, denoted Q_0. This ratio, written as

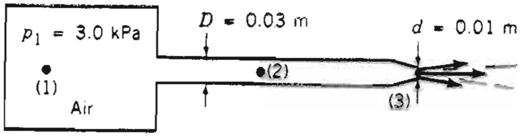
$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{D=\infty}} = \frac{\sqrt{2gh/[1 - (d/D)^4]}}{\sqrt{2gh}} = \frac{1}{\sqrt{1 - (d/D)^4}}$$

is plotted in Fig. E3.7*b*. With $0 < d/D < 0.4$ it follows that $1 < Q/Q_0 \lesssim 1.01$, and the error in assuming $V_1 = 0$ is less than 1%. Thus, it is often reasonable to assume $V_1 = 0$.

COMPRESSIBLE FLOW ?

EXAMPLE 3.8

Air flows steadily from a tank, through a hose of diameter $D = 0.03$ m and exits to the atmosphere from a nozzle of diameter $d = 0.01$ m as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure. Determine the flowrate and the pressure in the hose.



■ **FIGURE E3.8**

SOLUTION

If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline shown as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet) this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2 \tag{1}$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

$$\begin{aligned}\rho &= \frac{p_1}{RT_1} \\ &= [(3.0 + 101)\ \mathrm{kN/m^2}] \\ &\quad \times \frac{10^3\ \mathrm{N/kN}}{(286.9\ \mathrm{N{\cdot}m/kg{\cdot}K})(15 + 273)\mathrm{K}} \\ &= 1.26\ \mathrm{kg/m^3}\end{aligned}$$

Thus, we find that

$$V_3 = \sqrt{\frac{2(3.0 \times 10^3\ \mathrm{N/m^2})}{1.26\ \mathrm{kg/m^3}}} = 69.0\ \mathrm{m/s}$$

or

$$\begin{aligned}Q &= A_3V_3 = \frac{\pi}{4}d^2V_3 = \frac{\pi}{4}(0.01\ \mathrm{m})^2(69.0\ \mathrm{m/s}) \\ &= 0.00542\ \mathrm{m^3/s}\end{aligned}$$

(Ans)

Note that the value of V_3 is determined strictly by the value of p_1 (and the assumptions involved in the Bernoulli equation), independent of the "shape" of the nozzle. The pressure head within the tank, $p_1/\gamma = (3.0 \text{ kPa})/(9.81 \text{ m/s}^2)(1.26 \text{ kg/m}^3) = 243 \text{ m}$, is converted to the velocity head at the exit, $V_2^2/2g = (69.0 \text{ m/s})^2/(2 \times 9.81 \text{ m/s}^2) = 243 \text{ m}$. Although we used gage pressure in the Bernoulli equation ($p_3 = 0$), we had to use absolute pressure in the perfect gas law when calculating the density.

The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 3.19)

$$A_2V_2 = A_3V_3$$

Hence,

$$V_2 = A_3V_3/A_2 = \left(\frac{d}{D}\right)^2 V_3 = \left(\frac{0.01 \text{ m}}{0.03 \text{ m}}\right)^2 (69.0 \text{ m/s})$$

$$= 7.67 \text{ m/s}$$

and from Eq. 1

$$p_2 = 3.0 \times 10^3 \text{ N/m}^2 - \tfrac{1}{2}(1.26 \text{ kg/m}^3)(7.67 \text{ m/s})^2$$

$$= (3000 - 37.1)\text{N/m}^2 = 2963 \text{ N/m}^2 \qquad \text{(Ans)}$$

In the absence of viscous effects the pressure throughout the hose is constant and equal to p_2. Physically, the decreases in pressure from p_1 to p_2 to p_3 accelerate the air and increase its kinetic energy from zero in the tank to an intermediate value in the hose and finally to its maximum value at the nozzle exit. Since the air velocity in the nozzle exit is nine times that in the hose, most of the pressure drop occurs across the nozzle ($p_1 = 3000 \text{ N/m}^2$, $p_2 = 2963 \text{ N/m}^2$ and $p_3 = 0$).

Since the pressure change from (1) to (3) is not too great [that is, in terms of absolute pressure $(p_1 - p_3)/p_1 = 3.0/101 = 0.03$], it follows from the perfect gas law that the density change is also not significant. Hence, the incompressibility assumption is reasonable for this problem. If the tank pressure were considerably larger or if viscous effects were important, the above results would be incorrect.

[5:3] Fluid Dynamics

Recap

Along streamline
$$\frac{p}{\gamma}+\frac{V^2}{2g}+z=\text{constant (along streamline)}$$
$$\frac{p_1}{\gamma}+\frac{v_1^2}{2g}+z_1+h_P=\frac{p_2}{\gamma}+\frac{v_2^2}{2g}+z_2+\sum h_L^{major}+\sum h_L^{minor}$$

Normal to streamline
$$p+\rho\int\frac{V^2}{R}dn+\gamma z=\text{constant (normal to streamline)}$$

Pressure measurement

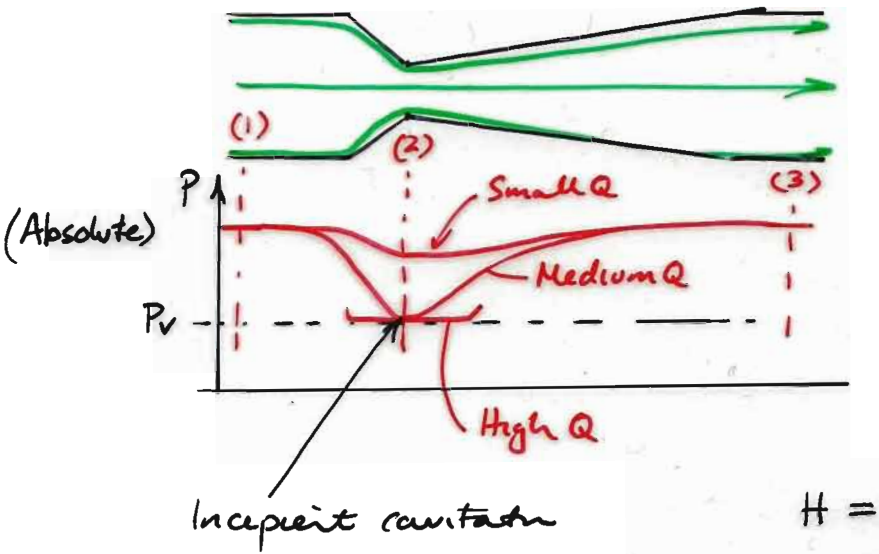
Free jets $V=\sqrt{2gh}$

Continuity $A_1V_1=A_2V_2$

Outline

- Flow measurements
  - Confined (pipe) flows
  - Open-channel flows
- Compressible fluids
- Spatial reference frame
- Unsteady flows

CAVITATION (REVISITED)



$$H = \overset{\downarrow}{p} + \overset{\uparrow}{\tfrac{1}{2}\rho V^2} + \gamma z$$

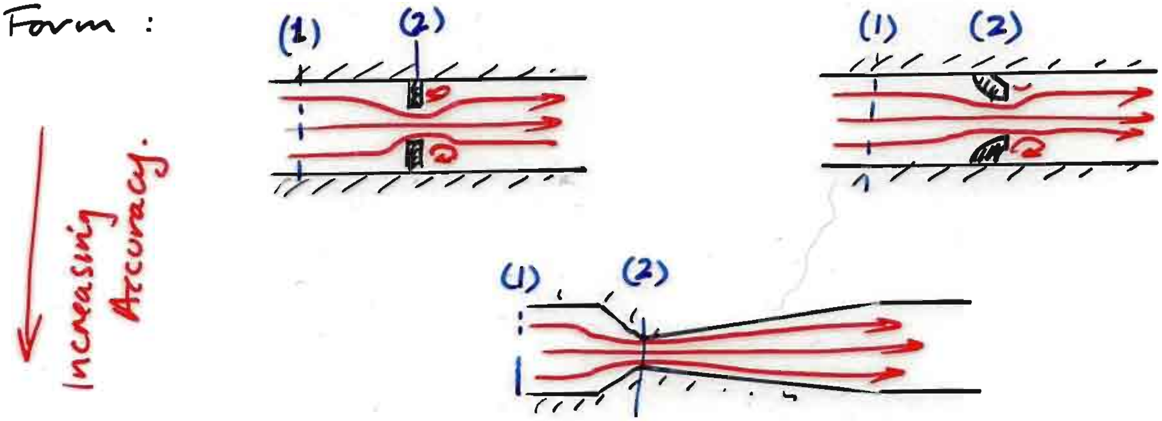
Water $p_v = 14.7$ psia @ 212° F

$p_v = .507$ psia @ 80° F

FLOWRATE MEASUREMENT

☐ Principle : Obstruct flow ↑V ∴ ↓p.
and measure pressure diff.

☐ Form :



☐ Solution : Assume $z_1 = z_2$

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

Apply continuity $V_1 A_1 = V_2 A_2 = Q$

$$Q = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho[1-(A_2/A_1)^2]}} \qquad A_2 < A_1$$

Measure $(p_1 - p_2)$.
Know : A_1 ; A_2 ; assume ρ.

∴ determine Q.

3.53

3.53 Oil of specific gravity 0.83 flows in the pipe shown in Fig. P3.53. If viscous effects are neglected, what is the flowrate?

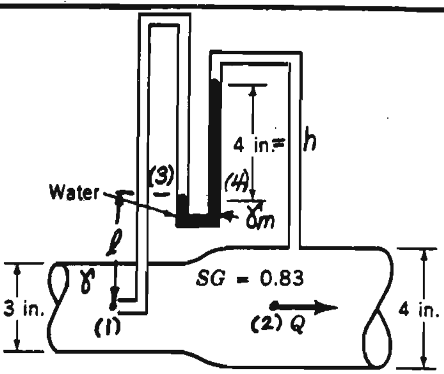


FIGURE P3.53

WRITE BERNOULLI @ 1 & 2

$$\frac{p_1}{\gamma}+\overset{0}{\cancel{\frac{V_1^2}{2g}}}+\cancel{z_1}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+\cancel{z_2} \quad \text{where } z_1=z_2 \text{ and } V_1=0$$

Thus,

$$\boxed{\frac{V_2^2}{2g}=\frac{p_1-p_2}{\gamma}} \qquad (1)$$

but,

$$p_1=p_3+\gamma\ell=p_4+\gamma\ell$$

MANOMETER EQUATION from (3) or (4).

and

$$p_2=\gamma(\ell+h)-\gamma_m h+p_4$$

MANOMETER EQUATION from (4)

Thus,

$$\boxed{p_1-p_2=(\gamma_m-\gamma)h} \qquad (2)$$

Combine Eqs. (1) and (2) to obtain

$$V_2=\sqrt{2g\left(\frac{p_1-p_2}{\gamma}\right)}=\sqrt{2g\left(\frac{\gamma_m}{\gamma}-1\right)h}=\sqrt{2\left(32.2\,\frac{ft}{s^2}\right)\left(\frac{62.4\,\frac{lb}{ft^3}}{0.83\left(62.4\,\frac{lb}{ft^3}\right)}-1\right)\left(\frac{4}{12}ft\right)}$$

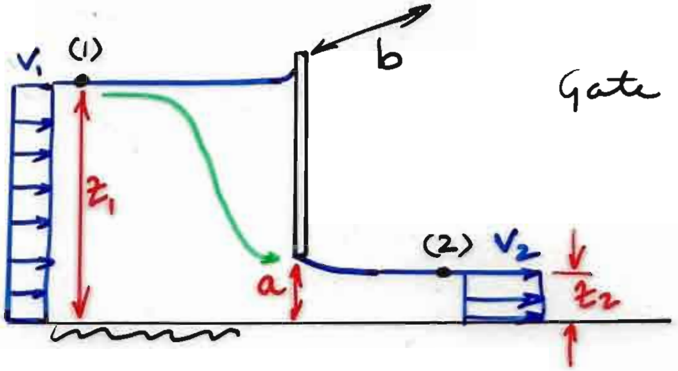
or

$$V_2=2.10\,\frac{ft}{s}$$

Thus,

$$Q=A_2V_2=\frac{\pi}{4}\left(\frac{4}{12}ft\right)^2\left(2.10\,\frac{ft}{s}\right)=\underline{\underline{0.183\,\frac{ft^3}{s}}}$$

SLUICE GATE



Gate width, b

Apply Bernoulli:
$$\cancelto{0}{p_1} + \frac{1}{2}\rho V_1^2 + \gamma z_1 = \cancelto{0}{p_2} + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

Continuity:
$$A_1 V_1 = A_2 V_2$$
$$z_1 b V_1 = z_2 b V_2 = Q$$

Combining:
$$Q = z_2 b \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}} \qquad (1)$$

In the limit, $z_1 \gg z_2$
$$Q = z_2 b \sqrt{2 g z_1}$$

Equation (1) also available from basal streamline with $p = \gamma z_1$ and γz_2.

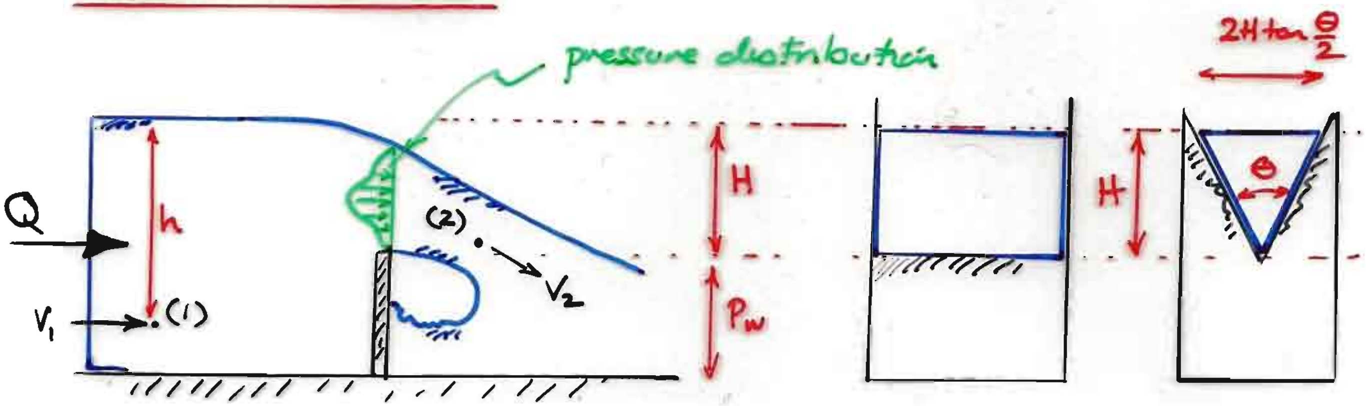
Vena contracta effect

$$C_c = z_2/a$$

$C_c \doteq 0.61$ $\qquad 0 < a/z_1 < 0.2$

C_c increases above $a/z_1 = 0.2$.

SHARP CRESTED WEIR



RECTANGULAR

V-NOTCH

ANALOG TO FREE JET

$$V \propto \sqrt{2gh}$$

$$V = C_1\sqrt{2gH}$$

$$Q = V\,bH = C_1\,bH\sqrt{2gH}$$

$$Q = C_1\,b\sqrt{2g}\;H^{3/2}$$

Determine C_1 experimentally. $C_1 = f\,(H_w/p_w\,;\,b^{?}\ldots)$

3.105

3.105 Water flows over the weir plate which has a semicircular opening as shown in Fig. P3.105. Determine the functional dependence of the flowrate on the head, $Q = Q(H)$, for $0 \leq H \leq R$. Illustrate this dependence with a graph of Q/Q_{max} as a function of H/R for $0 \leq H/R \leq 1$, where Q_{max} is the flowrate when $H = R$.

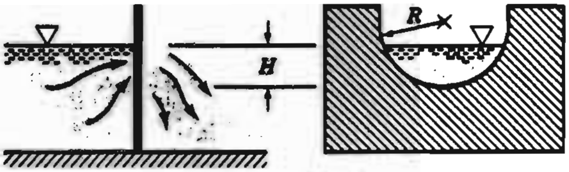
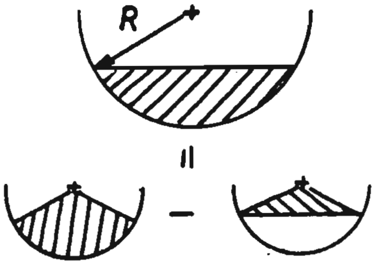


FIGURE P3.105

$Q = AV$ where it is expected that V is a function of the head, H.
That is, $V \sim \sqrt{2gH}$ ← Free jet!! $V = C_1\sqrt{2gH}$

Also, from geometry it follows that

$$A = R^2 \cos^{-1}\frac{(R-H)}{R} - (R-H)\sqrt{2RH - H^2}$$



Thus,

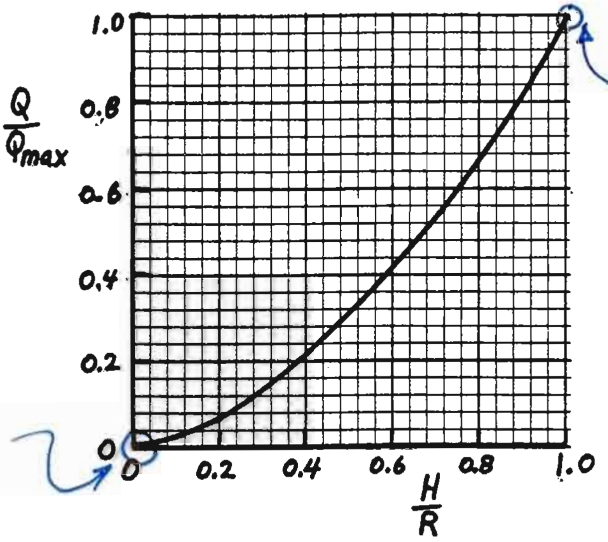
$$Q = C_1\sqrt{2g}\sqrt{H}\left[R^2\cos^{-1}\frac{(R-H)}{R} - (R-H)\sqrt{2RH-H^2}\right] \text{ where } C_1 = \text{constant}$$

With $H = R$ this gives $Q = Q_{max} = C_1\sqrt{2g}\sqrt{R}\,\frac{\pi}{2}R^2$ $\cos^{-1}(0) = \pi/2$

Thus,

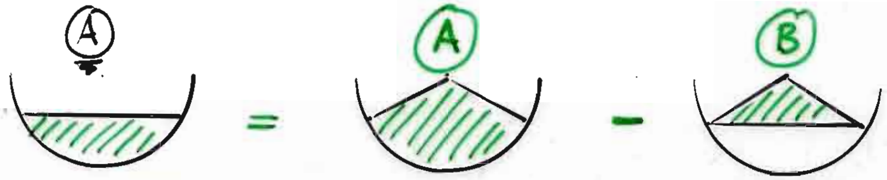
$$\boxed{\frac{Q}{Q_{max}} = \frac{2}{\pi}\sqrt{\frac{H}{R}}\left[\cos^{-1}\left(1-\frac{H}{R}\right) - \left(1-\frac{H}{R}\right)\sqrt{2\frac{H}{R} - \left(\frac{H}{R}\right)^2}\right]} \text{ for } 0 \leq \frac{H}{R} \leq 1$$

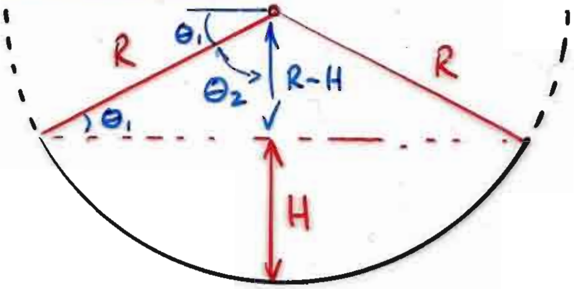
This dependence is illustrated below.



Check reality?

Check reality?

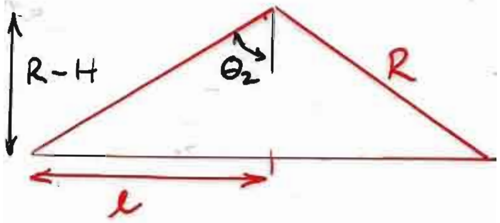




$$R - H = R\cos\theta_2$$

$$\frac{R-H}{R} = \cos\theta_2 \; ; \quad \theta_2 = \cos^{-1}\left(\frac{R-H}{R}\right)$$

For Ⓐ $\quad \text{Area}_A = \pi R^2 \frac{2\theta_2}{2\pi} = R^2\theta_2$



$$l^2 + (R-H)^2 = R^2$$

$$l = \sqrt{R^2 - (R-H)^2}$$

For Ⓑ $\quad \text{Area}_B = \frac{1}{2}(R-H)\,l \times 2$

c/s area of flow $\quad \underline{A} = Ⓐ - Ⓑ$

$$= R^2\cos^{-1}\left(\frac{R-H}{R}\right) - (R-H)\sqrt{\underbrace{R^2-(R-H)^2}}$$

and $R^2 - (R-H)^2 = 2RH - H^2$

QED.

3.107

3.107 Water flows over the spillway shown in Fig. P3.107. If the velocity is uniform at sections (1) and (2) and viscous effects are negligible, determine the flowrate per unit width of the spillway.

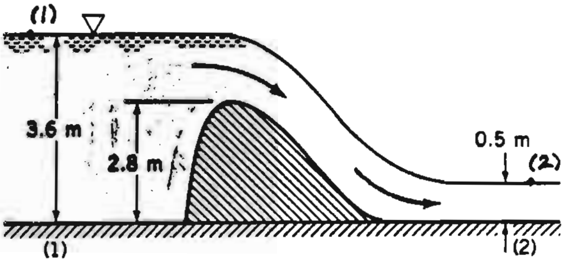


FIGURE P3.107

$$\frac{p_1}{\gamma} + \overset{?}{\frac{V_1^2}{2g}} + z_1 = \frac{p_2}{\gamma} + \overset{?}{\frac{V_2^2}{2g}} + z_2$$

where if points (1) and (2) are located on the free surface $p_1 = 0$, $p_2 = 0$, $z_1 = 3.6m$, and $z_2 = 0.5m$

1 equation & 2 unknowns!!

$\therefore$ Also $A_1V_1 = A_2V_2$

or

$$V_1 = \frac{z_2}{z_1}V_2 = \frac{0.5m}{3.6m}V_2 = 0.139\,V_2$$

Thus, Eq. (1) becomes

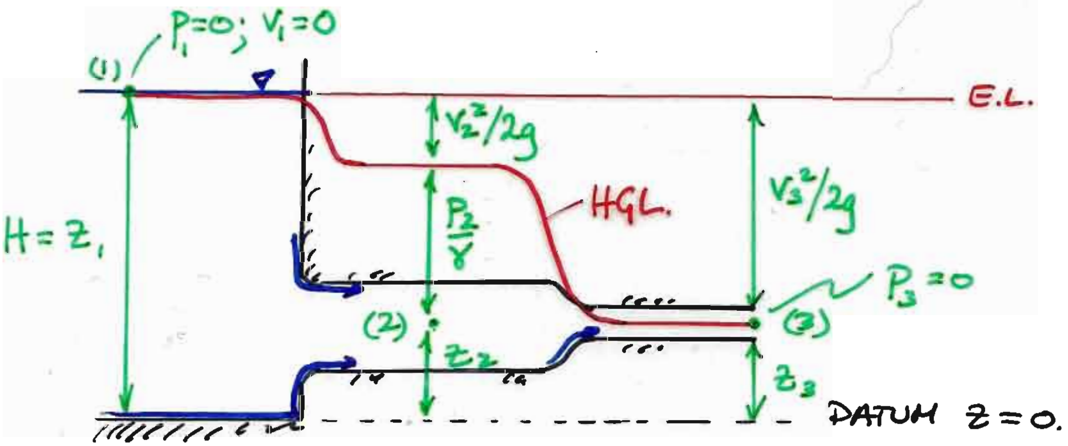
$$\frac{V_2^2}{2(9.81\frac{m}{s^2})}\left[1-(0.139)^2\right] = 3.6m - 0.5m \quad \text{or} \quad V_2 = 7.88\frac{m}{s}$$

Hence,

$$q = V_2 z_2 = (7.88\tfrac{m}{s})(0.5m) = \underline{\underline{3.94\frac{m^2}{s}}}$$

ENERGY LINE AND HYDRAULIC GRADE LINE

$$\underbrace{\frac{V^2}{2g} + \underbrace{\frac{P}{\gamma} + z}_{\text{Hydraulic head or piezometric head (HGL)}}}_{\text{Total head, H. Energy head, (EL).}} = \text{const along streamline} = H$$



(1) Write Bernoulli @ (1) and (3) $\qquad z_1 = z_3 + \underline{V_3}^2/2g$

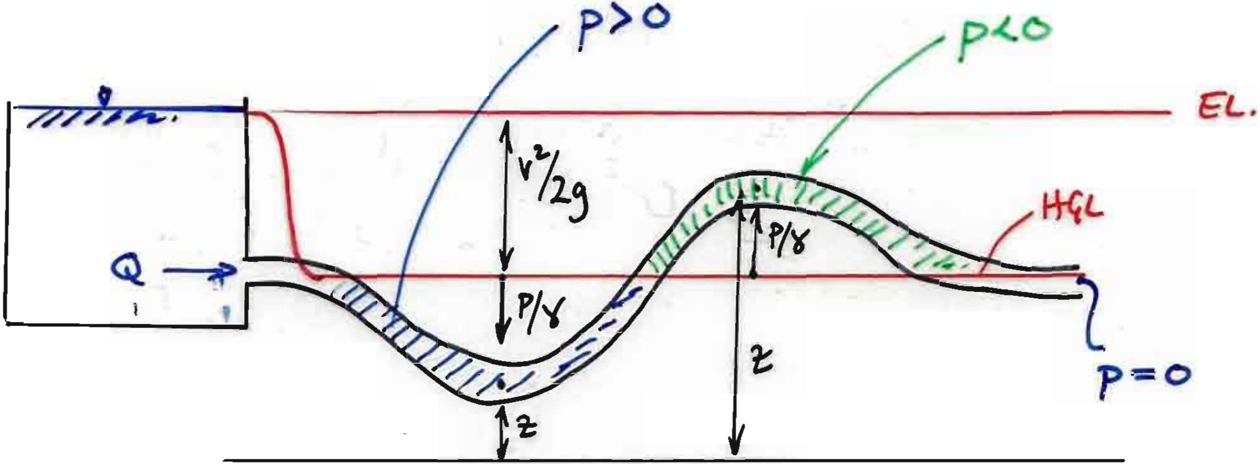
(2) Continuity: $\qquad V_2 A_2 = V_3 A_3$

$$\underline{V_2} = V_3(A_3/A_2)$$

(3) Write Bernoulli @ (1) and (2).

$$z_1 = \frac{V_2^2}{2g} + \overset{?}{\frac{P_2}{\gamma}} + z_2$$

Solve for P_2.



☐ V is constant along pipe if constant area, A
i.e. Q = const = VA

☐ -ve and +ve pressure portions along pipe.

3.109

3.109 The flowrate in a water channel is sometimes determined by use of a device called a Venturi flume. As shown in Fig. P3.109, this device consists simply of a hump on the bottom of the channel. If the water surface dips a distance of 0.07 m for the conditions shown, what is the flowrate per width of the channel? Assume the velocity is uniform and viscous effects are negligible.

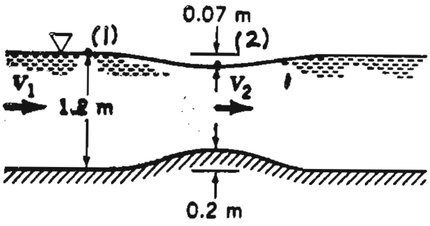


FIGURE P3.109

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \qquad (1)$$

with $p_1 = 0$, $p_2 = 0$, $z_1 = 1.2m$, and $z_2 = 1.2m - 0.07m = 1.13m$

Also, $A_1 V_1 = A_2 V_2$

or

$$V_2 = \frac{h_1}{h_2} V_1 = \frac{1.2m \quad V_1}{(1.2-0.07-0.2)m} = 1.29 V_1$$

Thus, from Eq. (1):

$$\frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 \quad \text{or} \quad \left[(1.29)^2 - 1\right] V_1^2 = 2\left(9.81\frac{m}{s^2}\right)(1.2-1.13)m$$

or $V_1 = 1.438 \frac{m}{s}$

Hence,

$$q = h_1 V_1 = \left(1.438\frac{m}{s}\right)(1.2m) = \underline{\underline{1.73\frac{m^2}{s}}}$$

RESTRICTIONS ON USE OF BERNOULLI EQUATION

1 INCOMPRESSIBILITY

Original derivation, assumed $\int \frac{1}{\rho} dp \rightarrow \frac{1}{\rho} \int dp$

Not true for gas $p = \rho RT$ $\rho = \frac{P}{RT}$

Evaluate Bernoulli for Isentropic and Isothermal.

Incompressibility ok for gas up to $V = 0.3$ Mach No.

$V = 335$ ft/s $= 230$ mph.

---

2 STEADY

Original derivation along streamline $V = V(s)$; $a_s = V\,\partial V/\partial s$

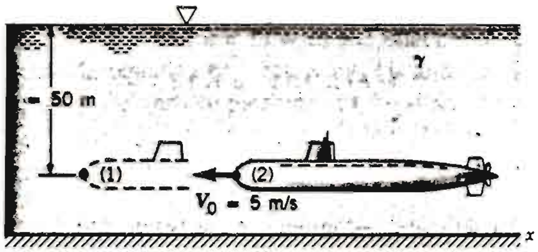
unsteady $V = V(s,t)$; $a_s = V\,\partial V/\partial s + \frac{\partial V}{\partial t}$

Require to have steady flow.

Change coordinate system to meet This.

EXAMPLE 3.17

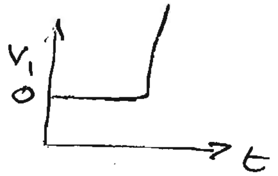
A submarine moves through the seawater ($SG = 1.03$) at a depth of 50 m with velocity $V_0 = 5.0$ m/s as shown in Fig. E3.17. Determine the pressure at the stagnation point (2).



■ **FIGURE E3.17**

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

$z_1 = z_2$



SOLUTION

In a coordinate system fixed to the ground, the flow is unsteady. For example, the water velocity at (1) is zero with the submarine in its initial position, but at the instant when the nose, (2), reaches point (1) the velocity there becomes $\mathbf{V}_1 = -V_0\,\hat{\mathbf{i}}$. Thus, $\partial \mathbf{V}_1/\partial t \neq 0$ and the flow is unsteady. Application of the steady Bernoulli equation between (1) and (2) would give the incorrect result that "$p_1 = p_2 + \rho V_0^2/2$." According to this result the static pressure is greater than the stagnation pressure—an incorrect use of the Bernoulli equation.

We can either use an unsteady analysis for the flow (which is outside the scope of this text) or redefine the coordinate system so that it is fixed on the submarine, giving steady flow with respect to this system. The correct method would be

$$p_2 = \frac{\rho V_1^2}{2} + \gamma h = [(1.03)(1000)\text{ kg/m}^3]\,(5.0\text{ m/s})^2/2$$
$$+\ (9.80 \times 10^3\text{ N/m}^3)(1.03)(50\text{ m})$$
$$= (12{,}900 + 505{,}000)\text{ N/m}^2 = 518\text{ kPa} \qquad \textbf{(Ans)}$$

similar to that discussed in Example 3.2.

If the submarine were accelerating, $\partial V_0/\partial t \neq 0$, the flow would be unsteady in either of the above coordinate systems and we would be forced to use an unsteady form of the Bernoulli equation.

Static frame of ref

$V_1 = 0$ $\quad V_2 \neq 0$

$P_1 = \gamma h$ $\quad P_2 = ?$

Cannot solve for P_2 since not steady state.

i.e. $\frac{\partial p_2}{\partial t} \neq 0$

Moving frame of ref.

$V_1 = 5$ m/s $\quad V_2 = 0$

$P_1 = \gamma h$ $\quad P_2 = ?$

$$P_1 + \frac{\rho V_1^2}{2} = P_2 + \frac{\rho V_2^2}{2}$$

$$\underset{\underset{\gamma h}{\uparrow}}{P_1} + \frac{\rho V_1^2}{2} = P_2$$

∴ solve for P_2

If static reference used then → $P_1 + \overset{0}{\frac{\rho V_1^2}{2}} = P_2 + \overset{\times}{\frac{\rho V_2^2}{2}}$

$$P_2 = P_1 - \frac{\rho V_2^2}{2}$$

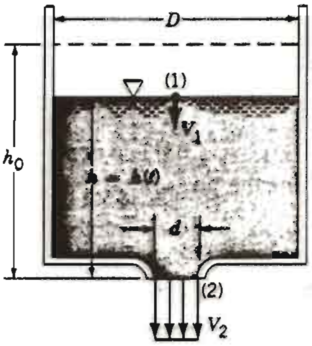
i.e. stagnation pressure (P_2) is less than the static pressure!!

Not True.

UNSTEADY FLOWS (Contd)

EXAMPLE 3.18

A stream of liquid of diameter d drains from a circular tank of diameter D as is shown in Fig. E3.18. The depth of the water was h_0 at time $t = 0$. Determine the water depth as a function of time, $h = h(t)$.



■ FIGURE E3.18

SOLUTION

Clearly this is an unsteady flow—the deeper the water, the faster it flows from the tank. However, if the hole in the tank is not too big, the water will drain slowly, and the unsteady effect, $\partial V/\partial t$, at any point in the flow will be smaller than the steady effect, $V\,\partial V/\partial s$. Under these conditions it is reasonable to consider the flow as "quasisteady" and to apply the steady Bernoulli equation as follows.

As was shown in Example 3.7, the velocity of the water leaving the tank can be written as

$$\boxed{V_2 = \sqrt{\frac{2gh}{1 - (d/D)^4}}}$$

Flowrate assuming $V_1 \neq 0$.

Apply continuity again.

$$\frac{dh}{dt} = -V_1$$

$$V_1A_1 = V_2A_2$$

or → $V_1 = \frac{V_2A_2}{A_1}$

$$\therefore \frac{dh}{dt} = -\frac{V_2A_2}{A_1}$$

Hence, by equating the flowrate from the tank, V_2A_2, and the rate at which the amount of water in the tank changes with time, $-(dh/dt)A_1$, we obtain

$$\frac{dh}{dt} = -\frac{V_2A_2}{A_1} = -\left(\frac{d}{D}\right)^2\sqrt{\frac{2gh}{1 - (d/D)^4}} \tag{1}$$

This result can be integrated from the initial time and depth, $t = 0$ when $h = h_0$, to an arbitrary time and depth as follows.

$$\int_{h_0}^{h} \frac{1}{\sqrt{h}}\,dh = -\left(\frac{d}{D}\right)^2\sqrt{\frac{2g}{1 - (d/D)^4}}\int_0^t dt$$

or

$$2(\sqrt{h} - \sqrt{h_0}) = -\left(\frac{d}{D}\right)^2\sqrt{\frac{2g}{1 - (d/D)^4}}\,t$$

This can be arranged into the form

$$\frac{h}{h_0} = \left[1 - \frac{t\sqrt{g/2h_0}}{\sqrt{(D/d)^4 - 1}}\right]^2 \qquad \textbf{(2)} \quad \textbf{(Ans)}$$

Correlates OK with expt.

Also $d \ll D$

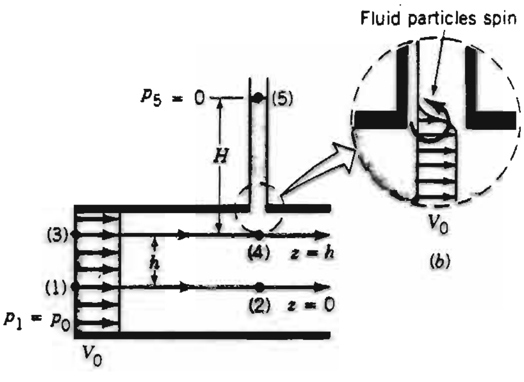
$h/h_0 \rightarrow 1$

The results of Eq. 2 correlate quite well with experiments, provided d/D is not too large, even though we have used a steady flow analysis for an unsteady flow. This is another way of saying $\partial V/\partial t \ll V\,\partial V/\partial s$. For larger values of d/D the unsteady Bernoulli equation gives a nonlinear, second-order differential equation that, unlike Eq. 1, is not easy to integrate.

ROTATIONAL EFFECTS

EXAMPLE 3.19

Consider the uniform flow in the channel shown in Fig. E3.19*a*. Discuss the use of the Bernoulli equation between points (1) and (2), points (3) and (4), and points (4) and (5). The liquid in the vertical piezometer tube is stationary.



■ **FIGURE E3.19**

SOLUTION

If the flow is steady, inviscid, and incompressible, Eq. 3.7 written between points (1) and (2) gives

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2 = \text{constant} = C_{12}$$

Since $V_1 = V_2 = V_0$ and $z_1 = z_2 = 0$, it follows that $p_1 = p_2 = p_0$ and the Bernoulli constant for this streamline, C_{12}, is given by

$$C_{12} = \tfrac{1}{2}\rho V_0^2 + p_0$$

Along the streamline from (3) to (4) we note that $V_3 = V_4 = V_0$ and $z_3 = z_4 = h$. As was shown in Example 3.5, application of $\mathbf{F} = m\mathbf{a}$ across the streamline (Eq. 3.12) gives $p_3 = p_1 - \gamma h$ because the streamlines are straight and horizontal. The above facts combined with the Bernoulli equation applied between (3) and (4) show that $p_3 = p_4$ and that the Bernoulli constant along this streamline is the same as that along the streamline between (1) and (2). That is, $C_{34} = C_{12}$, or

$$p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3 = p_4 + \tfrac{1}{2}\rho V_4^2 + \gamma z_4 = C_{34} = C_{12}$$

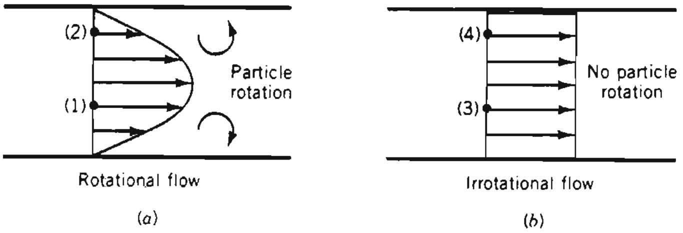
Similar reasoning shows that the Bernoulli constant is the same for any streamline in Fig. E3.19. Hence,

$$p + \tfrac{1}{2}\rho V^2 + \gamma z = \text{constant throughout the flow in the channel}$$

Again from Example 3.5 we recall that

$$p_4 = p_5 + \gamma H = \gamma H$$

If we apply the Bernoulli equation across streamlines from (4) to (5) we obtain the incorrect result "$H = p_4/\gamma + V_4^2/2g$." The correct result is $H = p_4/\gamma$.



■ **FIGURE 3.26** **Rotational and irrotational flows.**

[6]

Control Volumes

Reynolds’ Transport Theorem [6]

$$\text{Material derivative: } \frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\mathbf{V} \cdot \nabla)()$$

$$\mathbf{V} \cdot \nabla() = u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + w\frac{\partial()}{\partial z}$$

$$\text{Streamline acceleration: } \mathbf{a} = V\frac{\partial V}{\partial s}\hat{s} + \frac{V^2}{R}\hat{n}$$

$$\text{Transport Theorem: } \frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b dV - + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} dA \text{ for } b = \frac{B}{m}$$

[6:1] Control Volumes – Reference Frames

Recap

Along streamline

$$\frac{p}{\gamma}+\frac{V^2}{2g}+z=\text{constant (along streamline)}$$

$$\frac{p_1}{\gamma}+\frac{v_1^2}{2g}+z_1+h_P=\frac{p_2}{\gamma}+\frac{v_2^2}{2g}+z_2+\sum h_L^{major}+\sum h_L^{minor}$$

Normal to streamline

$$p+\rho\int\frac{V^2}{R}dn+\gamma z=\text{constant (normal to streamline)}$$

Continuity

$$A_1V_1=A_2V_2$$

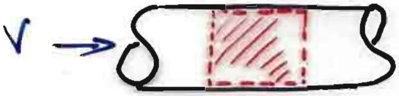
Outline

Control volumes

Static and moving reference frames

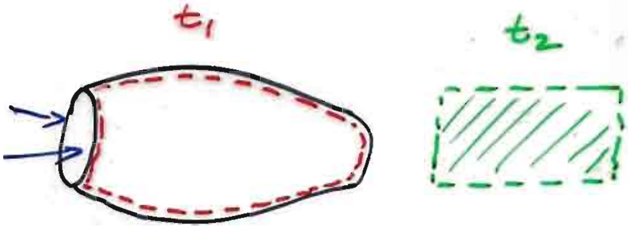
Reynolds' transport theorem

CONTROL VOLUMES



FIXED

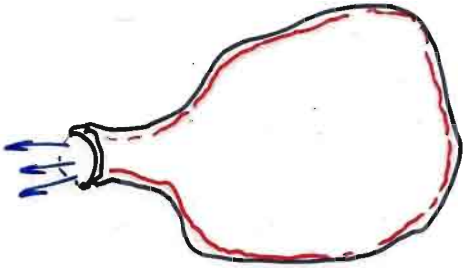
- ☐ Static
- ☐ Constant volume
- ☐ Extensive quantity (m, mv etc.) tracked through the volume.



FIXED w.r.t aircraft

MOVING w.r.t. ground

- ☐ Static or moving depends on ref. frame.
- ☐ Constant volume (does not deform).

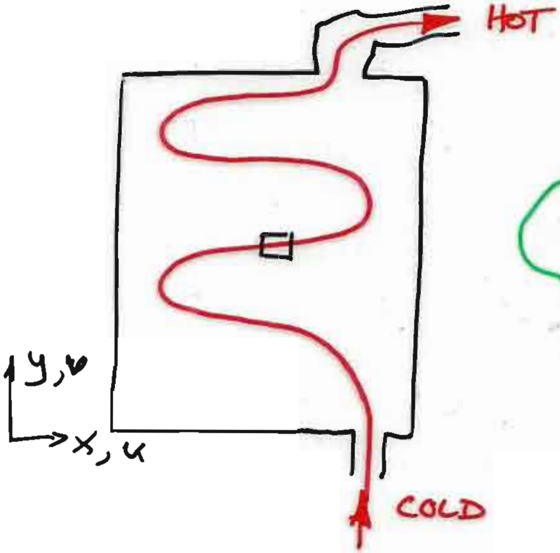


- ☐ Static or moving in space
- ☐ Control volume changes with time.

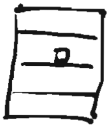
DEFORMING

Reynold's Transport Theorem provides a means of unifying these concepts in a single form.

WATER HEATER EXAMPLE



$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

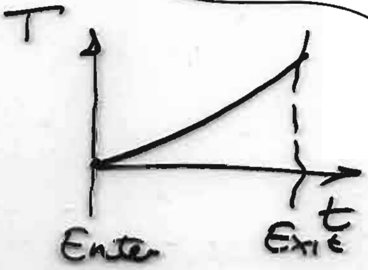


☐ **EULERIAN**

- Static control volume (the tank)
- $\frac{\partial T}{\partial t} = 0$ everywhere
- $v\frac{\partial T}{\partial y} \neq 0$ Convective term

☐ LAGRANGIAN

- Follows 'a single particle through system
- $\frac{DT}{Dt} \neq 0$ since particle warms from bottom to top.



RELEVANCE OF SUBSTANTIAL DERIVATIVE

Temporal term

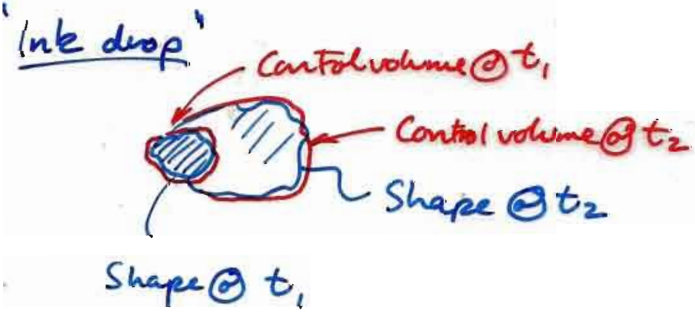
Convective term (derivative)

$$\underbrace{\frac{D(\underline{m})}{Dt}}_{} = \underbrace{\frac{\partial(\underline{m})}{\partial t} + (\underline{V}\cdot\underline{\nabla})(\underline{m})}_{}$$

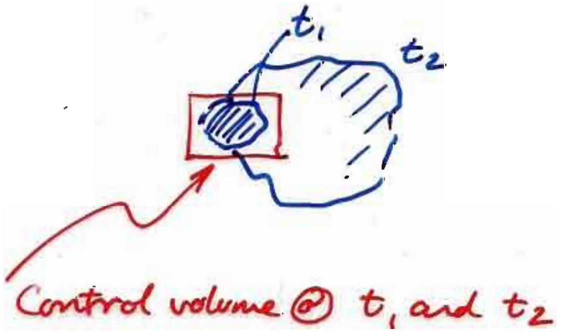
Moving coordinate system that moves with particles moving through the system

Static coordinate system, fixed in space that the particles move through.

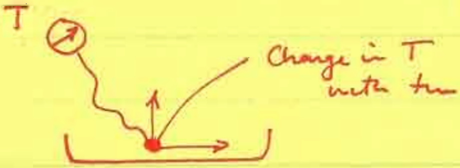
"LAGRANGIAN"



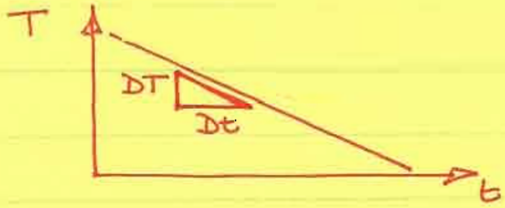
"EULERIAN"



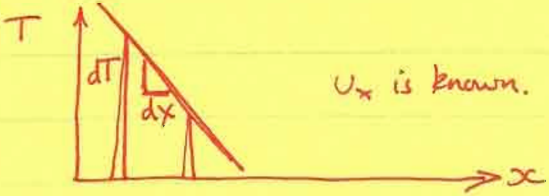
Static and Moving Coordinate Systems



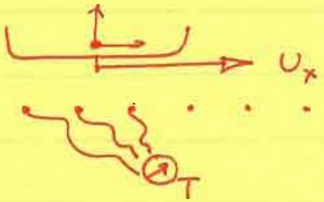
Moving coordinates



Static coordinates



U_x is known.



In this simplified example:

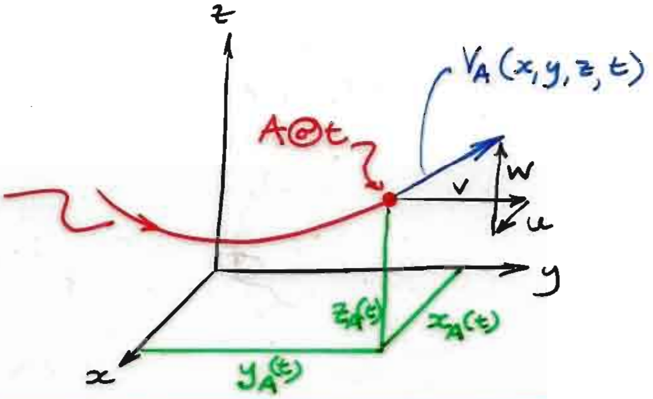
1) Ignore that $\partial T/\partial t$ is not zero.

Moving coord. Static coord

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial T}{\partial t}\frac{\partial x}{\partial x}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{U_x}$$

THE MATERIAL DERIVATIVE



In general; $\underline{V}_A = \underline{V}_A[x, y, z, t]$

non-steady

Use chain rule (Remember $\underline{V}$ is a vector $\underline{V} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$) !!

Accn.
$$\underline{a}(t) = \frac{d\underline{V}_A}{dt} = \frac{\partial \underline{V}_A}{\partial t} + \frac{\partial \underline{V}_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial \underline{V}_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial \underline{V}_A}{\partial z}\frac{dz_A}{dt}$$

Components of velocity: $u = \frac{dx_A}{dt}$; $v = \frac{dy_A}{dt}$; $w = \frac{dz_A}{dt}$

Resubstituting:

$$\underline{a} = \underbrace{\frac{\partial \underline{V}}{\partial t}}_{} + \underbrace{u\frac{\partial \underline{V}}{\partial x} + v\frac{\partial \underline{V}}{\partial y} + w\frac{\partial \underline{V}}{\partial z}}_{}$$

$$a = \frac{D\underline{V}}{Dt} \Rightarrow$$ Substantial/Material Derivative

---

$$\frac{D(\_)}{Dt} = \frac{\partial(\_)}{\partial t} + (\underline{V}\cdot\underline{\nabla})(\_)$$

$$\underline{V}\cdot\underline{\nabla}(\_) = u\frac{\partial(\_)}{\partial x} + v\frac{\partial(\_)}{\partial y} + w\frac{\partial(\_)}{\partial z}$$

Tensor, $\underline{a}$

$$\left.\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} = \frac{D\underline{V}}{Dt}$$

Scalar, T

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

EXAMPLE

A 3-d velocity field is given by $\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 2x \\ -y \\ z \end{Bmatrix}$

Determine the acceleration vector.

$$\begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \frac{\partial}{\partial t}\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \begin{Bmatrix} u\boxed{\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\boxed{\frac{\partial v}{\partial y}} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial z} + w\boxed{\frac{\partial w}{\partial z}} \end{Bmatrix}$$

① Steady $\therefore \cancelto{0}{\frac{\partial(\ )}{\partial t}}$

② Only remaining derivatives are $\frac{\partial u}{\partial x} \nearrow 2$; $\frac{\partial v}{\partial y} \nearrow -1$; $\frac{\partial w}{\partial z} \nearrow +1$

all others are zero.

$$\begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{Bmatrix} u\frac{\partial u}{\partial x} \\ v\frac{\partial v}{\partial y} \\ w\frac{\partial w}{\partial z} \end{Bmatrix} = \begin{Bmatrix} 2x(2) \\ -y(-1) \\ z(1) \end{Bmatrix} = 4x\hat{i} + y\hat{j} + z\hat{k}$$

Q.E.D.

[6:2] Fluid Dynamics

Recap

Control volumes

Static and moving reference frames

$$\text{Material derivative: } \frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\mathbf{V}\cdot\nabla)()$$

$$\mathbf{V}\cdot\nabla() = u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + w\frac{\partial()}{\partial z}$$

Outline

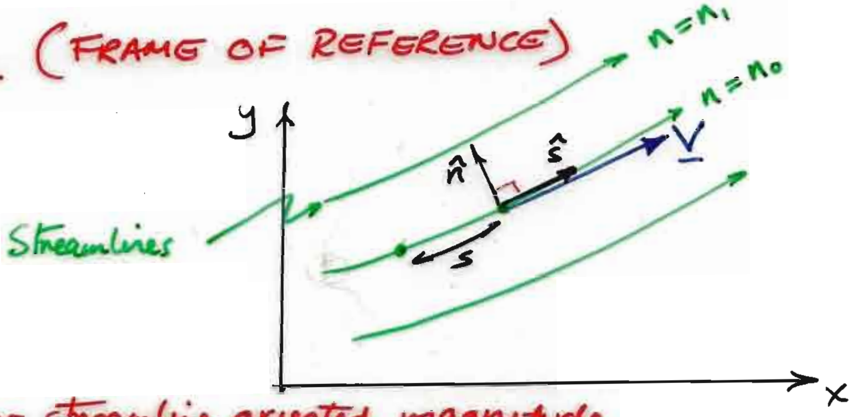
Streamline acceleration $\mathbf{a} = V\frac{\partial V}{\partial s}\hat{s} + \frac{V^2}{R}\hat{n}$

Reynolds' transport theorem

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv}\rho b d\forall + \int_{cs}\rho b\mathbf{V}\cdot\hat{n}dA \text{ for } b = \frac{B}{m}$$

STREAMLINE COORDINATES (FRAME OF REFERENCE)

Choose frame of reference // and ⊥ to streamlines.



scalar streamline oriented magnitude

$$\underline{V} = V\hat{s}$$

$$\underline{a} = \frac{D\underline{V}}{Dt} = a_s\hat{s} + a_n\hat{n} \qquad \left(\text{if steady } \frac{\partial V}{\partial t} = 0\right)$$

Note that: $V = f(s;$ but $n =$ constant since flow along streamline).

∴ For a particle, s changes with time; but n is constant.

$$\underline{a} = \frac{D(V\hat{s})}{Dt} = \frac{DV}{Dt}\hat{s} + V\frac{D\hat{s}}{Dt}$$

or:

$$\underline{a} = \overbrace{\left(\cancelto{0}{\frac{\partial V}{\partial t}} + \frac{\partial V}{\partial s}\frac{ds}{dt} + \cancelto{0}{\frac{\partial V}{\partial n}\frac{\partial n}{\partial t}}\right)}\hat{s} + V\overbrace{\left(\cancelto{0}{\frac{\partial \hat{s}}{\partial t}} + \frac{\partial \hat{s}}{\partial s}\frac{ds}{dt} + \cancelto{0}{\frac{\partial \hat{s}}{\partial n}\frac{dn}{dt}}\right)}$$

$V = \frac{ds}{dt}$

Steady flow

$$\underline{a} = \left(V\frac{\partial V}{\partial s}\right)\hat{s} + V\left(V\frac{\partial \hat{s}}{\partial s}\right)$$

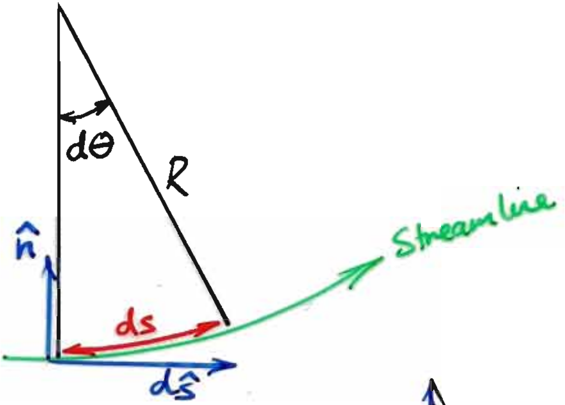
$$\frac{\partial \hat{s}}{\partial s} = \frac{\hat{n}}{R}$$

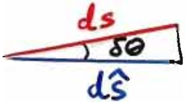
$$\underline{a} = \underbrace{V\frac{\partial V}{\partial s}}_{a_s}\hat{s} + \underbrace{\frac{V^2}{R}}_{a_n}\hat{n}$$

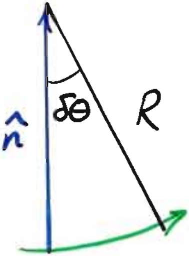
Convective accn. Centrifugal accn.

Recall: $F_s = m a_s = m V\frac{\partial V}{\partial s}$ $\qquad$ $F_n = m a_n = m\frac{V^2}{R}$

Why $\frac{\partial \hat{s}}{\partial s} = \frac{\hat{n}}{R}$?

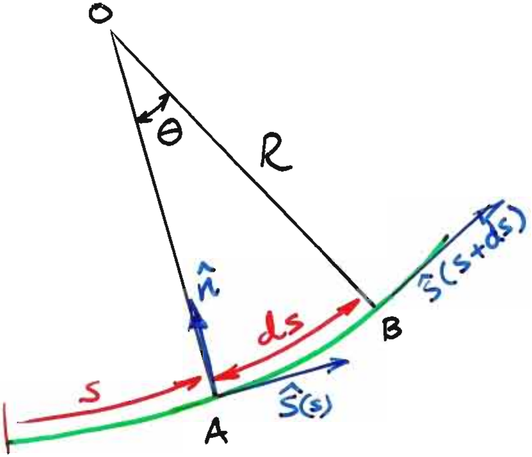


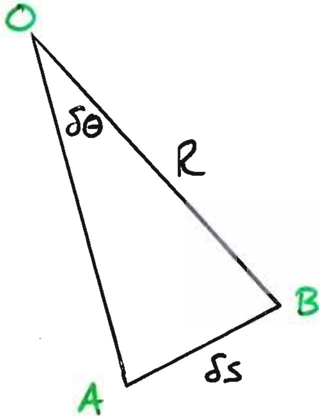


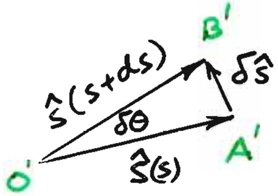


Why $\frac{\partial \hat{s}}{\partial s} = \frac{\hat{n}}{R}$?

Note similar Δs.







From similar Δs :

$$\frac{\delta s}{R} = \frac{|\delta \hat{s}|}{|\hat{s}|}$$

4.30

4.30 A nozzle is designed to accelerate the fluid from V_1 to V_2 in a linear fashion. That is, $V = ax + b$, where a and b are constants. If the flow is constant with $V_1 = 10$ m/s at $x_1 = 0$ and $V_2 = 25$ m/s at $x_2 = 1$ m, determine the local acceleration, the convective acceleration, and the acceleration of the fluid at points (1) and (2).

With $u = ax + b$, $v = 0$, and $w = 0$ the acceleration $\vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}$ can be written as

$$\vec{a} = a_x \hat{i} \quad \text{where} \quad a_x = u \frac{\partial u}{\partial x}. \qquad (1)$$

Since $u = V_1 = 10 \frac{m}{s}$ at $x = 0$ and $u = V_2 = 25 \frac{m}{s}$ at $x = 1$ we obtain

$$10 = 0 + b$$

$$25 = a + b \quad \text{so that } a = 15 \text{ and } b = 10$$

That is, $u = (15x + 10) \frac{m}{s}$, where $x \sim m$, so that from Eq.(1)

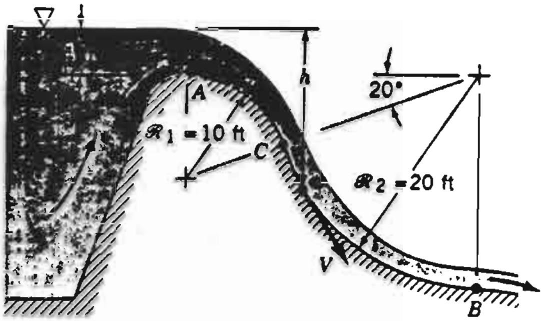
$$a_x = (15x + 10) \frac{m}{s} \left(15 \frac{1}{s}\right) = \underline{\underline{(225x + 150) \frac{m}{s^2}}}$$

Note: The local acceleration is zero, $\underline{\underline{\frac{\partial \vec{V}}{\partial t} = 0}}$, and the convective acceleration is $u \frac{\partial u}{\partial x} \hat{i} = \underline{\underline{(225x + 150) \hat{i} \frac{m}{s^2}}}$

At $x = 0$, $\vec{a} = \underline{\underline{150 \hat{i} \frac{m}{s^2}}}$; at $x = 1\,m$, $\vec{a} = \underline{\underline{375 \hat{i} \frac{m}{s^2}}}$

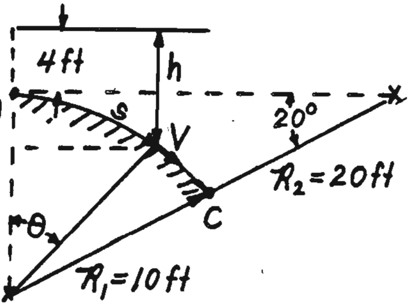
4.41

4.41 For the flow given in Problem 4.40, plot the streamwise acceleration, a_s, as a function of distance, s, along the surface of the dam from A to C.



Define accn: $a_s = V\frac{\partial V}{\partial s}$, where $V=\sqrt{2gh} = \sqrt{2(32.2\,\frac{ft}{s^2})h} = 8.02\sqrt{h}\ \frac{ft}{s}$, where h~ft

From the figure it follows that



Define, h

$h = \mathcal{R}_1 + 4 - \mathcal{R}_1\cos\theta$
$= [4+10(1-\cos\theta)]$ ft

(functn of geometry) Θ

Also, $s = \mathcal{R}_1\theta = 10\,\theta$ with θ~rad

so that

$V \propto \sqrt{2g\theta}$

$\frac{\partial V}{\partial s} = \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial s}$, with $\frac{\partial \theta}{\partial s} = \frac{1}{10}$

$\frac{\partial V}{\partial s} = \frac{\partial V}{\partial \theta}\ \frac{\partial \theta}{\partial s}$

and $V = 8.02\sqrt{h} = 8.02[4+10(1-\cos\theta)]^{1/2}$ or

$\frac{\partial V}{\partial \theta} = 8.02\,(\tfrac{1}{2})[4+10(1-\cos\theta)]^{-1/2}(10\sin\theta)$

QED

Hence

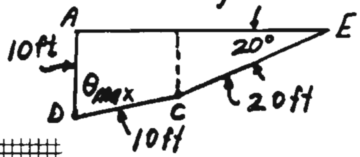
$a_s = V\frac{\partial V}{\partial s} = V\frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial s} = 8.02\,[4+10(1-\cos\theta)]^{1/2}(\tfrac{80.2}{2})\sin\theta[4-10(1-\cos\theta)]^{-1/2}(\tfrac{1}{10})$

which simplifies to

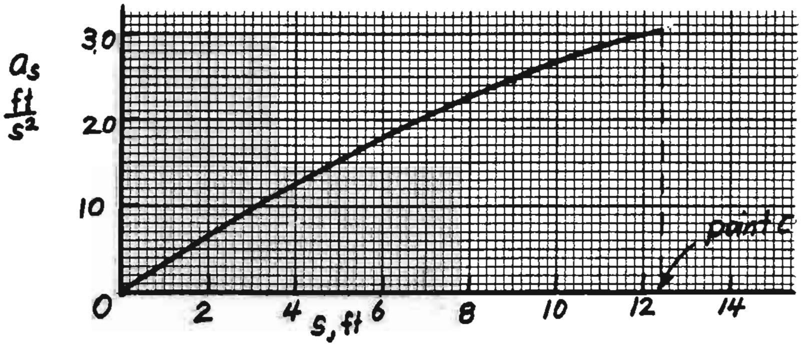
$a_s = \underline{\underline{32.2\sin\theta\ \frac{ft}{s^2}}}$ Note: $a_s = g\sin\theta$, the same as if a particle were sliding down a frictionless hill.

with $s = 10\,\theta$

This result is plotted below. The result is valid from A to C, or from $\theta = 0$ to $\theta = \theta_{max}$, where $\theta_{max} = 71.6°$



$20\sin 20° = 10 - 10\cos\theta_{max}$



REYNOLD'S TRANSPORT THEOREM

Allows control volume "accounting" to be completed for arbitrarity
- Stationary
- Moving
- Deforming

} control volumes.

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\forall + \int_{cs} \rho b \underline{V}\cdot\hat{n} \, dA$$

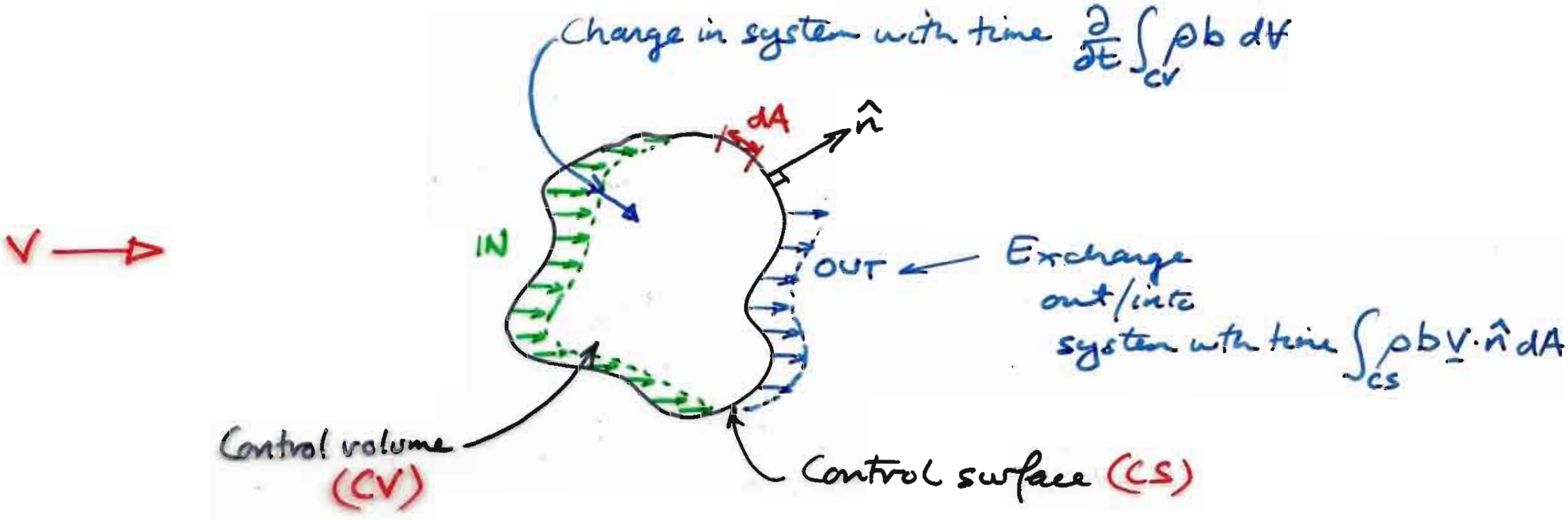
Change in extensive property 'B' within the system, with time = Change within the control volume with time + Exchange into/out of control volume across control surface.

$$B = m$$
$$B = m\underline{V}$$
$$B = \frac{1}{2} m \underline{V}^2$$

anything!

$$b = \frac{B}{m}$$

b (intensive) equals B (ext.) per unit mass (m).



REYNOLD'S TRANSPORT THEOREM (Cont'd).

Define arbitrary conservation equations $\quad b = \frac{B}{m}$

$B =$ conserved (extensive) parameter

$b =$ amount of parameter per unit mass.

		B	$b = B/m$
Conserve:	Mass	m	1
	Kinetic energy	$\frac{1}{2} m V^2$	$\frac{1}{2} V^2$
	Momentum	$m \underline{V}$	$\underline{V}$

Conservation of Mass (Continuity equation). $\quad b = 1 \; ; \; B_{sys} = M_{sys}$

$$\frac{D\cancelto{M_{sys}}{B_{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho \cancelto{1}{b} \, d\forall + \int_{cs} \rho \cancelto{1}{b} \, \underline{V} \cdot \hat{n} \, dA$$

Statement for entire system: Time rate of change of system mass $= 0$. $\quad \frac{DM_{sys}}{Dt} = 0$

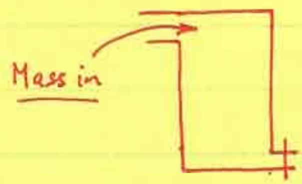
Rewrite as $\quad M_{sys} = \int_{sys} \rho \, d\forall$

$$\cancelto{0}{\frac{D}{Dt}\int_{sys} \rho \, d\forall} = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \underbrace{\int_{cs} \rho \underline{V} \cdot \hat{n} \, dA}_{\Sigma \dot{m}_{out} - \Sigma \dot{m}_{in}}$$

Fixed non-deforming control volume $\therefore \rightarrow 0$

$$\boxed{\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \underline{V} \cdot \hat{n} \, dA = 0}$$

For Simplified System



Mass in; $\Delta \forall = 0$

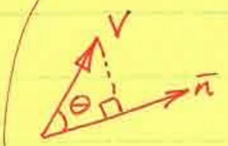
$\therefore \Delta\rho$ with time

Assume V is constant on area segments ⊥

Assume ρ is uniform in space!

~~Control volume~~

$$\frac{\partial}{\partial t}\int_{cv}\rho\, d\forall + \int_{cs}\rho\, \underline{V}\cdot\underline{n}\, dA = 0$$



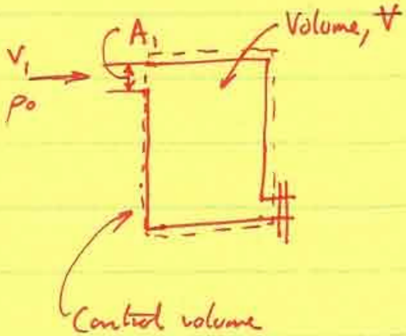
$\underline{V}\cdot\bar{n} = Vn\cos\theta$

$$\frac{\partial}{\partial t}(\rho\forall) + (\pm)\rho VA = 0$$

+ve out
-ve in

$$\forall\frac{\partial\rho}{\partial t} + \rho\frac{\partial\forall}{\partial t} + \Sigma\dot{m} = 0$$

Eg 1: Compressor pumps water into closed rigid vessel at constant velocity, V_1 and area, A.



$$\forall\frac{d\rho}{dt} + \rho\cancelto{0}{\frac{d\forall}{\partial t}} + \Sigma\dot{m} = 0$$

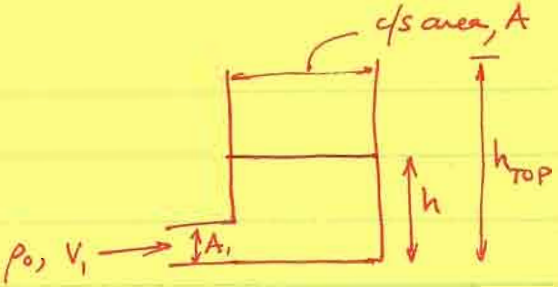
$$\forall\frac{d\rho}{dt} - \rho_0\forall_1 A_1 = 0$$

$$\forall\int_{\rho_0}^{\rho_f} d\rho = \rho_0 V_1 A\int_0^t dt$$

$$\forall(\rho_f-\rho_0) = \rho_0 V_1 A\, t$$

Solve for ρ_f.

Eg 2: Water flows into tank of constant cross sectional area, A, at flowrate $V_1 A_1$



$$\cancel{\forall \frac{d\rho}{dt}} + \rho \frac{d\forall}{dt} + \Sigma \dot{m} = 0$$

$$\rho_0 \frac{d\forall}{dt} - \rho_0 V_1 A_1 = 0$$

$$d\forall = V_1 A_1 \, dt$$

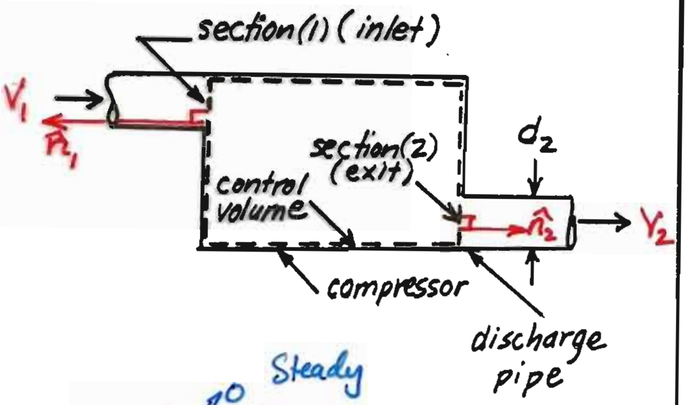
and $\forall = Ah$; $d\forall = A\,dh$

$$A \int_0^{h_{TOP}} dh = V_1 A_1 \int_0^{t} dt$$

$$A \, h_{TOP} = V_1 A_1 \, t$$

5.13

5.13 Air at standard atmospheric conditions is drawn into a compressor at the steady rate of 30 m³/min. The compressor pressure ratio, p_{exit}/p_{inlet}, is 10 to 1. Through the compressor p/ρ^n remains constant with $n = 1.4$. If the average velocity in the compressor discharge pipe is not to exceed 30 m/s, calculate the minimum discharge pipe diameter required.



For steady flow

$$\dot{m}_2 = \dot{m}_1$$

or

$$\rho_2 A_2 \overline{V}_2 = \rho_1 Q_1$$

Thus

$$\rho_2 \frac{\pi d_2^2}{4} \overline{V}_2 = \rho_1 Q_1$$

and

$$d_2 = \sqrt{\frac{\rho_1}{\rho_2} \frac{Q_1}{\frac{\pi}{4}\overline{V}_2}}$$

However

$$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{n}}$$

so

$$d_2 = \sqrt{\left(\frac{p_1}{p_2}\right)^{\frac{1}{n}} \frac{Q_1}{\frac{\pi}{4}\overline{V}_2}} = \sqrt{\left(\frac{1}{10}\right)^{\frac{1}{1.4}} \frac{30\,\frac{m^3}{min}}{\frac{\pi}{4}\left(30\,\frac{m}{s}\right)\,60\,\frac{s}{min}}}$$

Finally

$$d_2 = \underline{\underline{0.064 \quad m}}$$

Steady

$$\cancel{\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall}^{\,0} + \int_{cs} \rho \, \underline{V} \cdot \hat{n} \, dA = 0$$

$$\cancel{\rho}[\rho_1 V_1 \hat{n}_1 A_1 + \rho_2 V_2 \hat{n}_2 A_2] = 0$$

$$\boxed{-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0}$$

$$\rho_2 V_2 A_2 = \rho_1 V_1 A_1$$

$$\dot{m}_2 = \dot{m}_1$$

$$\frac{\partial m_2}{\partial t} = \frac{\partial m_1}{\partial t}$$

5.14

All the air, some of wate

5.14 An evaporative cooling tower (see Fig. P5.14) is used to cool water from 110 to 80 °F. Water enters the tower at a rate of 250,000 lbm/hr. Dry air (no water vapor) flows into the tower at a rate of 151,000 lbm/hr. If the rate of wet air flow out of the tower is 156,900 lbm/hr, determine the rate of water evaporation in lbm/hr and the rate of cooled water flow in lbm/hr.

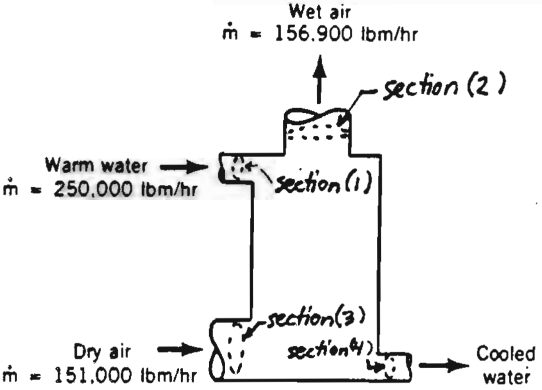


FIGURE P5.14

We know $\dot{m}_1 + \dot{m}_3 = \dot{m}_2 + \dot{m}_4$

For steady flow of dry air

$$\dot{m}_3 = \dot{m}_{2,\,dry\ air} \qquad (1)$$

For steady flow of water

$$\dot{m}_1 = \dot{m}_{2,\,water} + \dot{m}_4 \qquad (2)$$

Also

$$\dot{m}_2 = \dot{m}_{2,\,dry\ air} + \dot{m}_{2,\,water} \qquad (3)$$

Combining Eqs. 1 and 3 we get

$$\dot{m}_{2,\,water} = \dot{m}_2 - \dot{m}_3 = \text{rate of water evaporation}$$

So

$$\dot{m}_{2,\,water} = 156,900\ \frac{lbm}{hr} - 151,000\ \frac{lbm}{hr} = \underline{\underline{5900\ \frac{lbm}{hr}}}$$

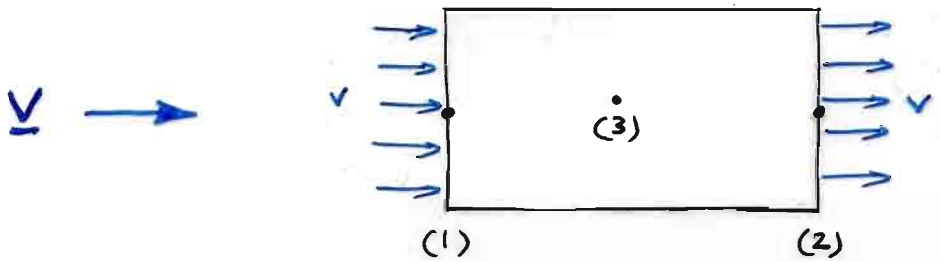
From Eq. 2 we get

$$\dot{m}_4 = \dot{m}_1 - \dot{m}_{2,\,water} = \text{rate of cooled water flow}$$

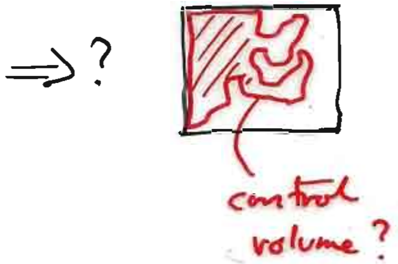
or

$$\dot{m}_4 = 250,000\ \frac{lbm}{hr} - 5900\ \frac{lbm}{hr} = \underline{\underline{244,\overset{1}{\cancel{0}}00\ \frac{lbm}{hr}}}$$

HOW ARBITRARY MAY THE CONTROL SURFACE/VOLUME BE?



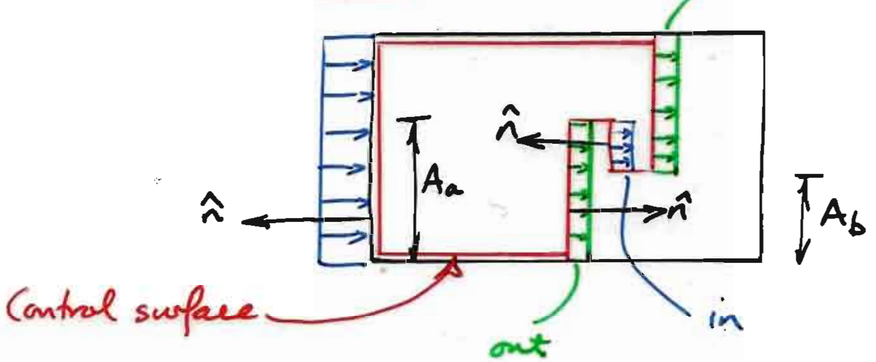
⇒ ?



$$A_1 = A_2$$

$$\therefore \; \bar{V}_1 = \bar{V}_2$$

CHOOSE ANY CONTOUR !!



WRITE CONSERVATION FOR 

$$\rho \int \underline{V}\cdot\hat{n}\, dA = \rho\, \underline{V}\cdot\hat{n} \int dA$$

① Entry @ L.H.S.

$$\rho\, v\cdot n \int dA = \rho(-v)A \qquad ①$$

② Entry/exit @ R.H.S.

$$\rho\, v\cdot n \int dA = \rho\left[v\cancel{A_a} + (-v)(\cancel{A_a} - \cancel{A_b}) + v(A - \cancel{A_b}) \right]$$

$$= \rho(v)A \qquad ②$$

Equating total contour $① + ② = \rho A(v - v) = 0$ $\therefore$ QED

[6:3] Control Volumes

Recap

Reynolds' transport theorem $\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\rlap{-}V + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} dA \text{ for } b = \frac{B}{m}$

Outline

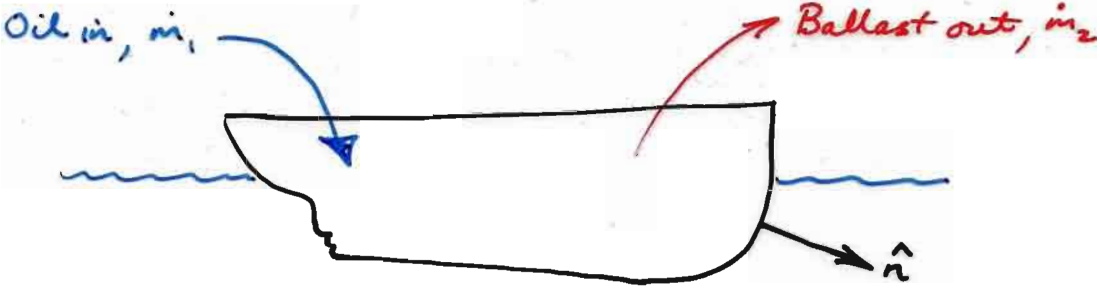
$$\frac{\partial}{\partial t}\int\limits_{cv} \rho \, d\rlap{-}V + \int\limits_{cs} \rho W \cdot n \, dA = 0$$

$$\frac{\partial}{\partial t}(\rho \rlap{-}V) + \sum \rho W A = 0$$

	Vcs	dVol/dt	
Static – Non-deforming	0	0	Vstatic=W+Vcs W=Vstatic-Vcs
Moving – Non-deforming	Vcv	0	
Moving - Deforming	Vcs	Not 0	

CONTROL VOLUMES/SURFACES

Non-deforming control volume, static.



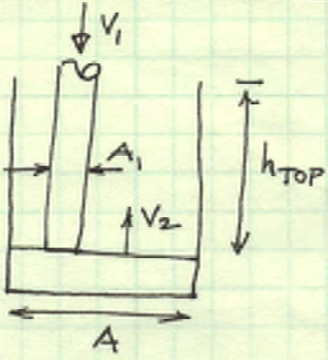
$$\cancelto{0}{\frac{DM_{sys}}{Dt}} = \frac{\partial}{\partial t}\int_{CV}\rho \, d\forall + \int_{CS}\rho \, \underline{V}\cdot\hat{n} \, dA$$

☐ CV remains constant

☐ CS remains fixed in space

☐ Steady, changes with time are zero. (Dt and dt → 0)

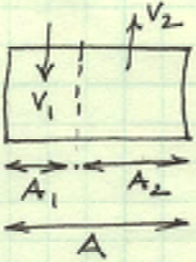
NON-DEFORMING & STATIONARY



$$\frac{\partial(\rho \forall)}{\partial t} + \int \rho V \cdot n \, dA = 0$$

$$\rho \cancel{\frac{\partial \forall}{\partial t}} + \forall \cancel{\frac{\partial \rho}{\partial t}} - \rho V_1 A_1 + \rho V_2 A_2 = 0$$

$$\therefore \quad V_2 = \frac{V_1 A_1}{A_2}$$



$$t_f = \frac{\forall_{TOTAL}}{Q} = \frac{A_2 h_{TOP}}{V_2 A_2} = \frac{A_2 h_{TOP}}{V_1 A_1}$$

QED.

Static control volume:

5.22

5.22 Flow of a viscous fluid over a flat plate surface results in the development of a region of reduced velocity adjacent to the wetted surface as depicted in Fig. P5.22. This region of reduced flow is called a boundary layer. At the leading edge of the plate, the velocity profile may be considered uniformly distributed with a value U. All along the outer edge of the boundary layer, the fluid velocity component parallel to the plate surface is also U. If the x direction velocity profile at section (2) is

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \Rightarrow u = U\left(\frac{y}{\delta}\right)^{1/7}$$

develop an expression for the volume flowrate through the edge of the boundary layer from the leading edge to a location downstream at x where the boundary layer thickness is δ.

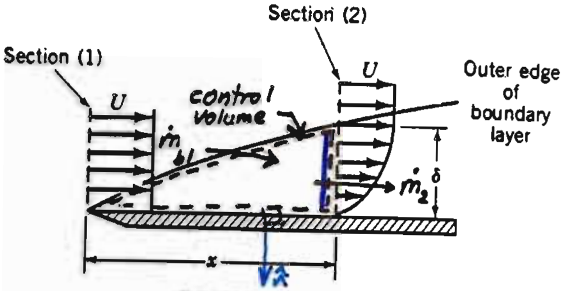


FIGURE P5.22

$$\int \rho \underline{V}\cdot\hat{n}\,dA = \rho\int u\,dA$$
$$= \rho l \int_0^{\delta} U\left(\frac{y}{\delta}\right)^{1/7} dy$$

From the conservation of mass principle applied to the flow through the control volume shown in the figure we have

$$\dot{m}_{bl} = \dot{m}_2 = \int_{A_2} \rho\,\vec{V}\cdot\hat{n}\,dA$$

For incompressible, flow

$$\rho Q_{bl} = \rho U l \delta \int_0^1 \left(\frac{y}{\delta}\right)^{\frac{1}{7}} d\left(\frac{y}{\delta}\right)$$

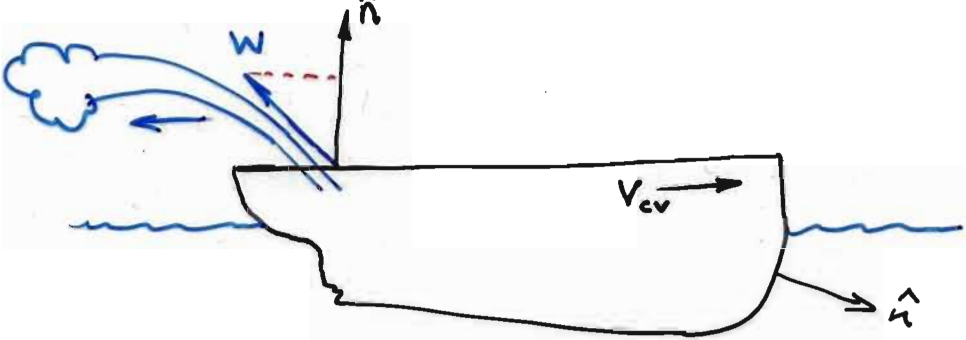
where

l = width of the plate

and thus

$$\underline{\underline{Q_{bl} = \rho \frac{7}{8} U l \delta}}$$

Non-deforming moving control volume



W = velocity of plume relative to observer on the moving control volume (ship).

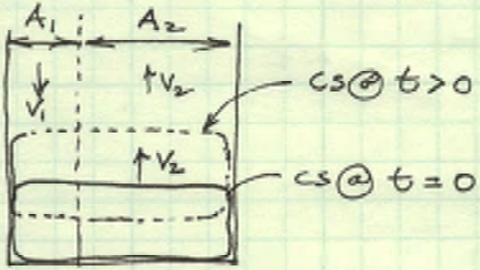
V_{cv} = velocity of control volume

V = true velocity of flow relative to static

$$\underline{V} = \underline{W} + \underline{V}_{cv}$$

$$\cancelto{0}{\frac{DM_{sys}}{Dt}} = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \, \underline{W} \cdot \hat{n} \, dA$$

NON-DEFORMING & NON-STATIONARY



$$\rho \cancelto{0}{\frac{\partial \forall}{\partial t}} + \forall \cancelto{0}{\frac{\partial \rho}{\partial t}} - \rho w_1 A_1$$

$$+ \rho \cancelto{0}{w_2} A_2 + \rho A V_2 = 0$$

W = flow velocity relative to moving control surface.

$$\therefore \quad w_1 = V_1 - (-V_{cs}) = V_1 + V_2$$

$$\therefore \quad \rho A V_2 = \rho (V_1 + V_2) A_1$$

$$V_2 = \frac{A_1 V_1}{(A - A_1)} = \frac{A_1 V_1}{A_2}$$

$$t_f = \frac{h_{TOP}}{V_2} = \frac{A_2 \, h_{TOP}}{A_1 V_1}$$

QED.

Deforming Control Volume

Deforms ∴ $\int dV$ changes with time

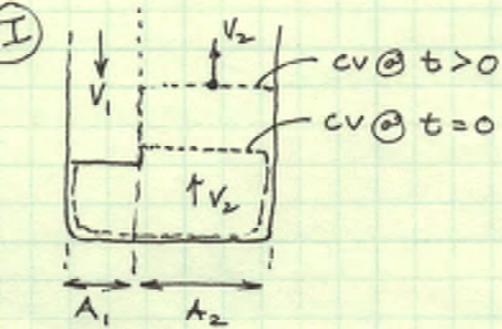
$$\underline{V} = \underline{W} + \underline{V}_{cs}$$

note control surface.
not volume stile

$$\frac{DM_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \underline{W} \cdot \underline{\hat{n}} \, dA$$

This changes with time!!

DEFORMING CONTROL VOLUME



$V_{cs} = V_2$ on A_2

$V_{cs} = 0$ on A_1

$$\rho \frac{\partial \forall}{\partial t} + \forall \cancelto{0}{\frac{\partial \rho}{\partial t}} - \rho V_1 A_1 + \rho \cancelto{0}{V_2} A_2 = 0$$

$$\forall = V_0 + A_2 h = V_0 + A_2 V_2 t$$

$h = V_2 t$

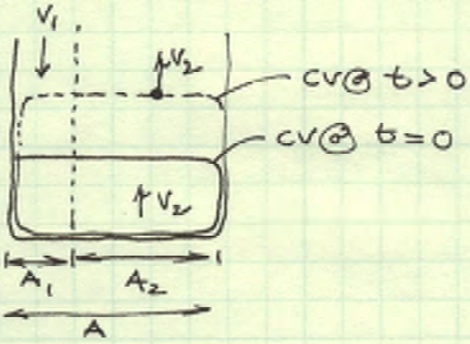
$$\frac{\partial \forall}{\partial t} = A_2 V_2$$

$$\therefore \quad \rho A_2 V_2 - \rho V_1 A_1 = 0 \qquad \therefore \quad V_2 = V_1 \frac{A_1}{A_2}$$

$$t_F = \frac{h_{TOP}}{V_2} = \frac{A_2 h_{TOP}}{V_1 A_1}$$

QED

DEFORMING CONTROL VOLUME



$V_{cs} = V_2$ on A_2

$V_{cs} = V_2$ on A_1

$$\rho \frac{\partial \forall}{\partial t} + \cancelto{0}{\forall \frac{\partial \rho}{\partial t}} - \rho w_1 A_1 + \cancelto{0}{\rho w_2 A_2} = 0$$

$$\frac{\partial \forall}{\partial t} = A V_2$$ (note: not A_2)

$$w_1 = V_1 - (-V_{cv}) = V_1 + V_2$$

$$\therefore \quad \rho A V_2 - \rho A_1 (V_1 + V_2) = 0$$

$$V_2 (A - A_1) = A_1 V_1$$

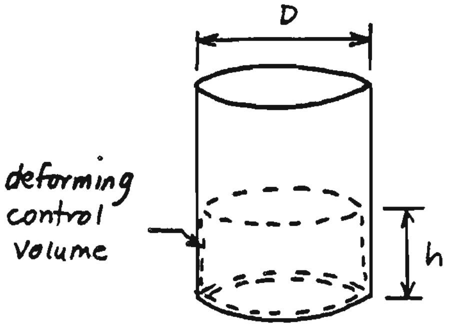
$$V_2 = \frac{A_1 V_1}{A_2}$$

$$t_F = \frac{h_{TOP}}{V_2} = \frac{A_2 h_{TOP}}{V_1 A_1}$$

QED.

5.24

5.24 How long would it take to fill a cylindrical shaped swimming pool having a diameter of 5 m to a depth of 1.5 m with water from a garden hose if the flowrate is 1.0 liter/s?



From application of the conservation of mass principle to the control volume containing water only as shown in the figure we have

$$\frac{\partial}{\partial t}\int_{cv}\rho\, d\forall + \underbrace{\int_{cs}\rho\, \vec{V}\cdot\hat{n}\, dA}_{} = 0$$

Assume ρ = constant

$$Q = \int_{cs}\underline{V}\cdot n\, dA = VA$$

For incompressible flow

$$\frac{\partial \forall}{\partial t} - Q = 0$$

or

$$\int_0^{\forall} d\forall = Q\int_0^t dt \implies \forall = Qt \quad \text{or} \quad t = \frac{Q}{\forall}$$

Thus

$$t = \frac{\forall}{Q} = \frac{\pi D^2 h}{4\, Q} = \frac{\pi\, (5\ m)^2 (1.5\ m)\left(1000\ \frac{liters}{m^3}\right)}{4\ \left(1.0\ \frac{liter}{s}\right)\left(60\ \frac{s}{min}\right)}$$

or

$$t = \underline{\underline{491\ min}}$$

5.27

5.27 A hypodermic syringe (see Fig. P5.27) is used to apply a vaccine. If the plunger is moved forward at the steady rate of 20 mm/s and if vaccine leaks pass the plunger at 0.1 of the volume flowrate out the needle opening, calculate the average velocity of the needle exit flow. The inside diameters of the syringe and the needle are 20 mm and 0.7 mm.

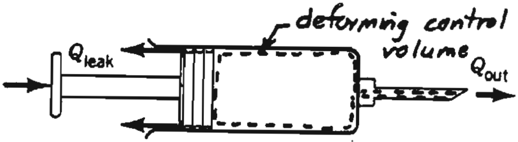


FIGURE P5.27

Using a deforming control volume and the conservation of mass principle (Eq. 5.17) as outlined in Example 5.8, we obtain (see Eq. 8 of Example 5.8)

$$-\rho A_1 V_p + \rho Q_2 + \rho Q_{leak} = 0 \qquad (1)$$

Since $\rho =$ constant, $Q_{leak} = 0.1\, Q_2$ and $Q_2 = A_2 V_2$ we obtain from Eq. 1

$$1.1\, A_2 V_2 = A_1 V_p$$

or

$$V_2 = \left(\frac{A_1}{A_2}\right)\frac{V_p}{1.1} = \left(\frac{d_1^2}{d_2^2}\right)\frac{V_p}{1.1} = \frac{(20\,mm)^2 (20\,mm/s)}{(0.7\,mm)^2 (1.1) \left(1000\,\frac{mm}{m}\right)}$$

and

$$V_2 = \underline{\underline{14.8}}\ \frac{m}{s}$$

Deforming control volume

$$\frac{\partial}{\partial t}\int_{cv}\cancel{\rho}\,d\forall \quad + \quad \int_{cs}\cancel{\rho}\,\underline{W}\cdot\hat{n}\,dA \quad = 0$$

1. Constant density, ρ

Rearrange: $$\frac{\partial \forall}{\partial t} + W_{out}A_{out} + W_{leak}A_{leak} = 0$$

2. Know that $$\underbrace{W_{out}A_{out}}_{Q_{leak}} = 10\times W_{leak}A_{leak}$$

Therefore: $$\frac{d\forall}{dt} + \frac{11}{10}Q_{leak} = 0$$

and solve since $$\frac{d\forall}{dt} = 20\ \text{mm/s.} \times \text{Area.}$$

and $$Q = W_{out}A_{out}$$

<u>QED.</u>

4.56

4.56 A plunger is pushed into a closed, air-filled pipe of diameter D with velocity V_p as shown in Fig. P4.56. The flow within the control volume consists of two distinct regions—one of zero velocity and constant density ρ_1; and the other of uniform velocity $\mathbf{V} = V_p\hat{\mathbf{i}}$ and density ρ_2. The two regions are separated by an interface (a shock wave) that travels to the right with constant speed, V_s (different than the fluid speed; $V_s > V_p$). Determine the mass within the control volume as a function of time from $t = 0$ (when the shock wave is at $x = 0$ as indicated) until the shock wave reaches the end of the pipe ($x = \ell$).

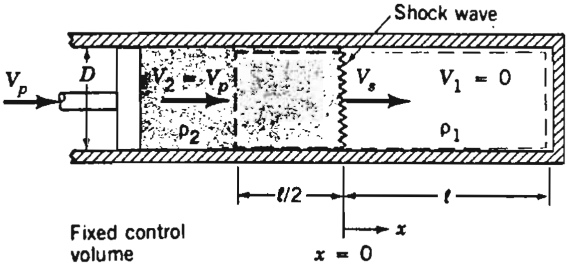
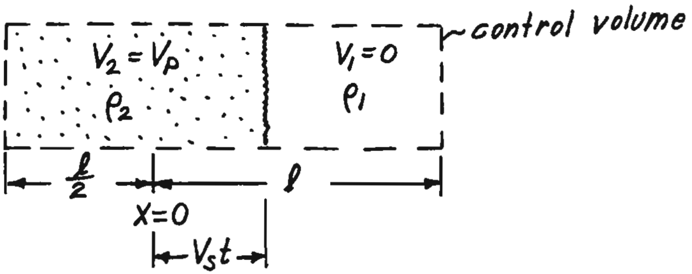


FIGURE P4.56

At time $t>0$ (but before the shock wave hits the end of the tube at $x=\ell$) the flow is as shown below.



Thus,

$$M = \text{mass in control volume} = M_1 + M_2 = \rho_1 \forall_1 + \rho_2 \forall_2$$

$$= \frac{\pi}{4} D^2 (\ell - V_s t)\rho_1 + \frac{\pi}{4} D^2 \left(\frac{\ell}{2} + V_s t\right)\rho_2$$

or

$$\underline{\underline{M = \frac{\pi}{4} D^2 \left[\rho_1 \ell + \rho_2 \frac{\ell}{2} + V_s t(\rho_2 - \rho_1)\right]}}$$

Note: This is valid from $t=0$ until $t = \frac{\ell}{V_s}$

[7]

Newton’s Law & Conservation of Momentum

Conservation Laws [7,8]

Relative velocities: $\mathbf{V}=\mathbf{W}+\mathbf{V_{cs}}$

Mass (continuity): $b=1$ and $\frac{D}{Dt}M_{sys}=\frac{D}{Dt}\int_{sys}\rho d\cancel{V}=\frac{\partial}{\partial t}\int_{cv}\rho d\cancel{V}+\int_{cs}\rho\mathbf{W}\cdot\hat{n}dA=0$

Linear Momentum:

Static: $b=\mathbf{V}$ and $\frac{D}{Dt}F_{sys}=\frac{D}{Dt}\int_{sys}\mathbf{V}\rho d\cancel{V}=\frac{\partial}{\partial t}\int_{cv}\mathbf{V}\rho d\cancel{V}+\int_{cs}\mathbf{V}\rho\mathbf{V}\cdot\hat{n}dA=\sum\mathbf{F}$

Moving and steady: $\int_{cs}(\mathbf{W}+\mathbf{V_{cs}})\rho\mathbf{W}\cdot\hat{n}dA=\sum\mathbf{F}$

Moment-of-Momentum:

Steady: $b=(\mathbf{r}\times\mathbf{V})$ and $\int_{cs}(\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{n}dA=\sum(\mathbf{r}\times\mathbf{F})$

$$T_{shaft}=\pm rV_{\theta}\dot{m};\quad \dot{W}_{shaft}=T_{shaft}\omega;\quad w_{shaft}=\frac{\dot{W}_{shaft}}{\dot{m}}$$

First Law of Thermodynamics: $b=e$ and

$$\frac{\partial}{\partial t}\int_{cv}e\rho d\cancel{V}+\int_{cs}e\rho\mathbf{V}\cdot\hat{n}dA=\dot{Q}_{netin}+\dot{W}_{netin}$$

$$\frac{p_{out}}{\gamma}+\frac{\alpha_o V_{out}^2}{2g}+z_{out}+h_L=\frac{p_{in}}{\gamma}+\frac{\alpha_i V_{in}^2}{2g}+z_{in}+h_P$$

$$\dot{m}[(\breve{h}_{out}-\breve{h}_{in})+\tfrac{1}{2}(v_{out}^2-v_{in}^2)+g(z_{out}-z_{in})]=\dot{Q}_{netin}+\dot{W}_{netin}$$

$$h_P=\frac{w_{shaftin}}{g};\quad w_{shaftin}=\frac{\dot{W}_{shaftin}}{\dot{m}}$$

$\alpha=1$ for uniform flow.

Differential Analysis of Fluid Flow [7,8]

Euler's Equation: $\rho\mathbf{g}-\nabla p=[\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}]$

Parallel plate flow: $q=-\frac{(2b)^3}{12\mu}\frac{\partial p}{\partial x};\quad \hat{U}=-\frac{(2b)^2}{12\mu}\frac{\partial p}{\partial x}$

Circular pipe flow: $q=-\pi\frac{(2R)^4}{128\mu}\frac{\partial p}{\partial x};\quad \hat{U}=-\frac{(2R)^2}{32\mu}\frac{\partial p}{\partial x}$

[7:1] Conservation of Momentum

Recap

Reynolds' transport theorem $$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\cancel{V} + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} dA \text{ for } b = \frac{B}{m}$$

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\cancel{V} + \int_{cs} \rho W \cdot n \, dA = 0$$

$$\frac{\partial}{\partial t}(\rho \cancel{V}) + \sum \rho W A = 0; \quad V_{static} = V_{cs} + W$$

Outline

Linear momentum

- Static
- Moving

CONSERVATION OF LINEAR MOMENTUM

Newton's Second Law

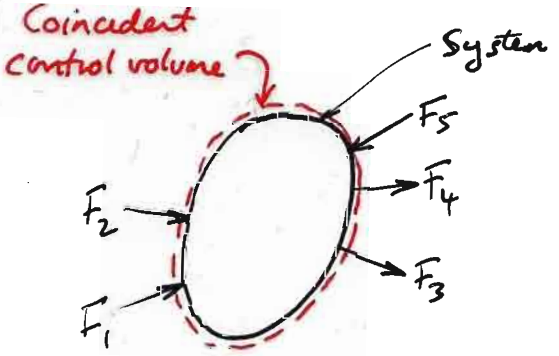
Time rate of change of linear momentum of the system = Sum of external forces acting on the system.

$$\frac{D}{Dt}\int_{sys}\underbrace{\underline{V}\rho}\, d\forall = \sum F_{sys}$$

$\int \rho\, d\forall$ = mass

Control volume is coincedent with a system at any time then

$$\sum F_{sys} = \sum F_{\text{contents of the coincident control volume}}$$



Reynold's Transport Theorem

$B = m\underline{V}$ $\qquad$ $b = \frac{m\underline{V}}{m} = \underline{V}$

$$\underbrace{\frac{D}{Dt}\int_{sys}\underline{V}\rho\, d\forall}_{\sum F_{\text{contentents of coincident contol vol}}} = \boxed{\frac{\partial}{\partial t}\int_{cv}\underline{V}\rho\, d\forall + \int_{cs}\underline{V}\rho\underline{V}\cdot\hat{n}\, dA = \sum \underline{F}_{cont}}$$

Fixed and non-deforming control volume or any other form of moving or deforming control volume may be included.

WHAT DOES THE FULL EQUATION LOOK LIKE ?

$$\frac{\partial}{\partial t}\int_{cv} \underline{V}\rho \, d\forall + \int_{cs} \underline{V}\rho \, \underline{V}\cdot\hat{n} \, dA = \Sigma \underline{F}$$

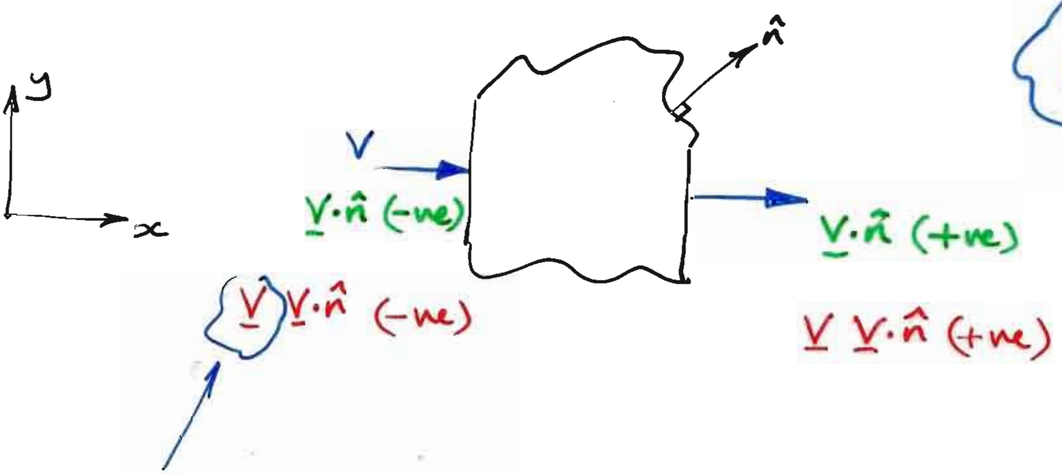
$$\frac{\partial}{\partial t}\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}\int_{cv}\rho \, d\forall + \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}\int_{cs}\rho \, \underbrace{\underline{V}\cdot\hat{n}}_{\text{Scalar magnitude}} \, dA = \begin{Bmatrix} \Sigma F_x \\ \Sigma F_y \\ \Sigma F_z \end{Bmatrix}$$

N.B. Can only remove the integrations if velocity components are piecewise constant !!

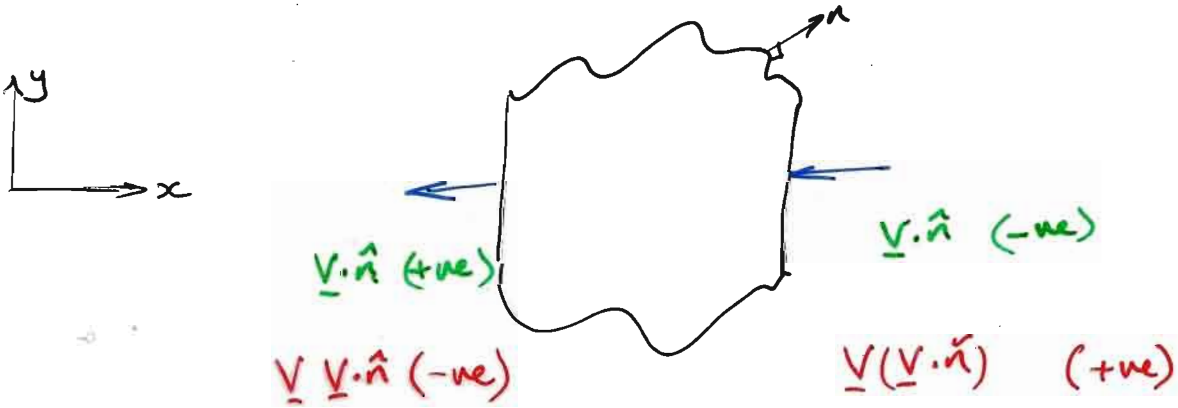
IMPORTANT !

Page 233 conditions

$$\rho\int_{cv}\underline{V}(\underline{V}\cdot\hat{n})\,dA$$



This is relative to the global coordinate directions.



i.e Momentum balance depends on:

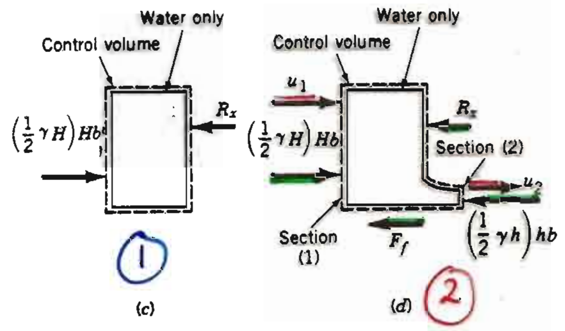
1. Direction of velocity w.r.t. global coords
2. Inflow or outflow from volume.
   (gaining or loosing).

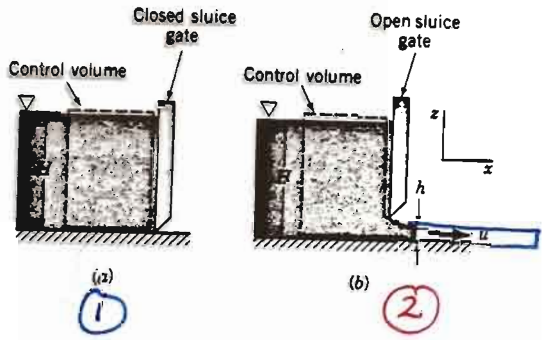
EXAMPLE 5.15

A sluice gate across a channel of width b is shown in the closed and open positions in Figs. E5.15*a* and E5.15*b*. Is the anchoring force required to hold the gate in place larger when the gate is closed or when it is open?

SOLUTION

We will answer this question by comparing expressions for the horizontal reaction force, R_x, between the gate and the water when the gate is closed and when the gate is open. The control volume used in each case is indicated with dashed lines in Figs. E5.14*a* and E5.14*b*.





① GATE CLOSED

When the gate is closed, the horizontal forces acting on the contents of the control volume are identified in Fig. E5.15*c*. Application of Eq. 5.22 to the contents of this control volume yields

$$\boxed{\int_{\text{cs}} \overset{0 \text{ (no flow)}}{\cancel{u\rho\mathbf{V}}} \cdot \hat{\mathbf{n}}\, dA = \tfrac{1}{2}\gamma H^2 b - R_x} \qquad (1)$$

Note that the hydrostatic pressure force, $\gamma H^2 b/2$, is used. From Eq. 1, the force exerted on the water by the gate (which is equal to the force necessary to hold the gate stationary) is

$$\underline{R_x = \tfrac{1}{2}\gamma H^2 b} \qquad (2)$$

which is equal in magnitude to the hydrostatic force exerted on the gate by the water.

② GATE OPEN

When the gate is open, the horizontal forces acting on the contents of the control volume are shown in Fig. E5.15*d*. Application of Eq. 5.22 to the contents of this control volume leads to

$$\boxed{\int_{\text{cs}} u\rho\mathbf{V} \cdot \hat{\mathbf{n}}\, dA = \tfrac{1}{2}\gamma H^2 b - R_x - \tfrac{1}{2}\gamma h^2 b - F_f} \qquad (3)$$

Note that we have assumed that the pressure distribution is hydrostatic in the water at sections (1) and (2). Also, the frictional force between the channel bottom and the water is specified as F_f. The surface integral in Eq. 3 is nonzero only where there is flow across the control surface. With the assumption of uniform velocity distributions

$$\int_{\text{cs}} u\rho\mathbf{V} \cdot \hat{\mathbf{n}}\, dA = (u_1)\rho(-u_1)Hb + (+u_2)\rho(+u_2)hb \qquad (4)$$

Note ⑥ is less than ②.

With $H \gg h$, the upstream velocity, u_1, is much less than u_2 so that the contribution of the incoming momentum flow to the control surface integral can be neglected. Thus, Eqs. 3 and 4 combine to form

Makes sense ??

$$\rho u_2^2 hb = \tfrac{1}{2}\gamma H^2 b - R_x - \tfrac{1}{2}\gamma h^2 b - F_f \qquad (5)$$

Solving Eq. 5 for the reaction force, R_x, we obtain

$u_1 \rightarrow 0$

$$R_x = \tfrac{1}{2}\gamma H^2 b - \tfrac{1}{2}\gamma h^2 b - F_f - \rho u_2^2 hb + \rho u_1^2 Hb \qquad (6)$$

Comparing the expressions for R_x (Eqs. 2 and 6) we conclude that the reaction force between the gate and the water (and therefore the anchoring force required to hold the gate in place) is smaller when the gate is open than when it is closed. (Ans)

CONSERVATION OF LINEAR MOMENTUM

ALTERNATIVE APPROACH (REPRESENTATION).

In any of the coordinate directions, say x:

$$\frac{\partial}{\partial t}\int_{cv} u\rho\, d\forall + \int_{cs} u\rho\, \underline{V}\cdot\hat{n}\, dA = \Sigma F_x$$

$$\dot{m} = \rho\int_{cs} \underline{V}\cdot\hat{n}\, dA$$

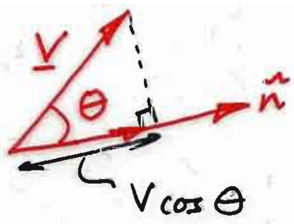
$$\dot{m} = \rho V A$$

$$\boxed{u\dot{m} = \Sigma F_x}$$

mass rate of flow $\Rightarrow$ $\dot{m}$ $\begin{cases} +\text{ve for outflow from body} \\ -\text{ve for inflow into body} \end{cases}$

flow velocity $\Rightarrow$ u positive/negative in coordinate directions.

Note; $\underline{V}\cdot\hat{n} = V n \cos\theta$



5.34

5.34 Determine the magnitude and direction of the x and y components of the anchoring force required to hold in place the horizontal 180° elbow and nozzle combination shown in Fig. P5.34. Also determine the magnitude and direction of the x and y components of the reaction force exerted by the 180° elbow and nozzle on the flowing water.

$$\left.\begin{aligned} V\cdot n &= U_2 \\ VV\cdot n &= -U_2^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} V\cdot n &= -U_1 \\ VV\cdot n &= -U_1^2 \end{aligned}\right\}$$

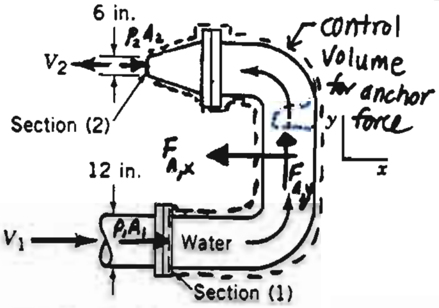


FIGURE P5.34 $p_1 = 15$ psi
$V_1 = 5$ ft/s

$p_2 = 0 =$ Atmospheric!!

For determining the x and y direction components of the anchoring force a control volume that contains the elbow, nozzle and water between sections (1) and (2) is used. Th control volume and the forces involved are shown in the sketch above. Application of the y direction component of the linear momentum equation (Eq. 5.22) leads to

$$F_{A,y} = \underline{\underline{0}}$$

$$\underbrace{\rho\int_{cs} V(V\cdot n)\,dA}$$

Application of the x direction component of the linear momentum equation yields

CONS of MOMENTUM

$$\boxed{-u_1\underbrace{\rho u_1 A_1} - u_2\underbrace{\rho u_2 A_2} = p_1A_1 - F_{A,x} + p_2A_2} \qquad (1)$$

From the conservation of mass equation

CONS of MASS

$$\boxed{\dot{m} = \rho u_1 A_1 = \rho u_2 A_2} \quad \therefore \text{ use to relate } U_1 \text{ to } U_2 \qquad (2)$$

Thus Eq. 1 may be expressed as

$$-\rho u_1 A_1\left(u_1 + u_2\right) = p_1A_1 - F_{A,x} + p_2A_2$$

and

$$F_{A,x} = \rho u_1 A_1(u_1+u_2) + p_1A_1 + p_2A_2 = \rho u_1 \frac{\pi D_1^2}{4}(u_1+u_2) + p_1\frac{\pi D_1^2}{4} + (0)A_2$$

Also from Eq. 2

$$u_2 = \frac{A_1}{A_2}u_1 = \frac{D_1^2}{D_2^2}u_1$$

Thus

$$F_{A,x} = \rho_1 u_1\frac{\pi D_1^2}{4}\left(u_1 + \frac{D_1^2}{D_2^2}u_1\right) + p_1\frac{\pi D_1^2}{4}$$

(con't)

5.34 (con't)

$$F_{A,x} = \left(1.94\ \frac{\text{slugs}}{\text{ft}^3}\right)\left(5\ \frac{\text{ft}}{\text{s}}\right)\frac{\pi(12\ \text{in.})^2}{4\left(144\ \frac{\text{in.}^2}{\text{ft}^2}\right)}\left[5\ \frac{\text{ft}}{\text{s}} + \frac{(12\ \text{in.})^2}{(6\ \text{in.})^2}\,5\ \frac{\text{ft}}{\text{s}}\right]\left(\frac{1\ \text{lb}}{\text{slug}\cdot\frac{\text{ft}}{\text{s}^2}}\right)$$

$$+\left(15\ \frac{\text{lb}}{\text{in.}^2}\right)\pi\,\frac{(12\ \text{in.})^2}{4}$$

$$F_{A,x} = \underline{\underline{1890\ \text{lb}}}$$

For determining the x and y components of the reaction force a control volume that contains only the water between sections (1) and (2) is used. Application of the y direction component of the linear momentum equation yields

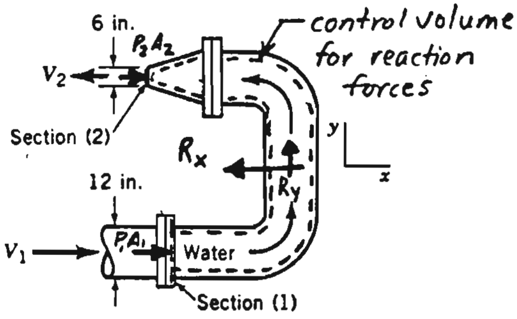
$$R_y = \underline{\underline{0}}$$

Application of the x direction component of the linear momentum equation leads to

$$R_x = \rho_1 u_1 \frac{\pi D_1^2}{4}\left(u_1 + \frac{D_1^2}{D_2^2}u_1\right) + p_1\frac{\pi D_1^2}{4}$$

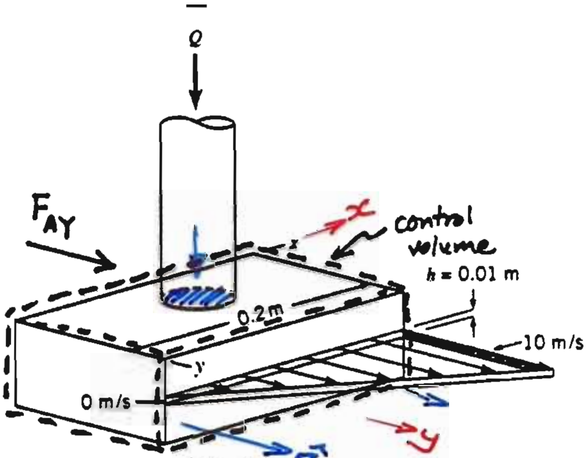
or

$$R_x = \underline{\underline{1890\ \text{lb}}}$$



5.41

5.41 A sheet of water of uniform thickness (h = 0.01 m) flows from the device shown in Fig. P5.41. The water enters vertically through the inlet pipe and exits horizontally with a speed that varies linearly from 0 to 10 m/s along the 0.2 m length of the slit. Determine the y component of anchoring force necessary to hold this device stationary.



■ FIGURE P5.41

$$\int_{cs} \underline{V}\rho(\underline{V}\cdot\hat{n})\,dA = \Sigma F_y$$

$$V_y = 50x$$

A control volume that contains the box portion of the device and the water in the box as shown in the sketch above is used. Application of the y-direction component of the linear momentum equation yields

$$F_{AY} = \int_{A_{slit}} v\rho\vec{V}\cdot\hat{n}\,dA = \rho\int_0^{0.2} v^2\,\underbrace{h\,dx}_{dA} \qquad dA = h\,dx$$

The variation of v with x is linear or

$$v = 50x \ \frac{m}{s}$$

Thus

$$F_{AY} = \rho\int_0^{0.2}(50x)^2 h\,dx = \rho\,(50)^2 h\,\frac{x^3}{3}\Big|_0^{0.2}$$

or

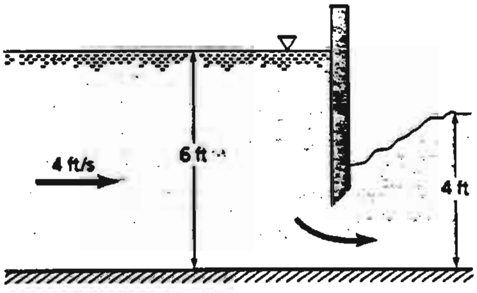
$$F_{AY} = \left(999\,\frac{kg}{m^3}\right)\left(50\,\frac{1}{s}\right)^2(0.01m)\,\frac{(0.2m)^3}{3}\left(1\,\frac{N\cdot s^2}{kg\cdot m}\right)$$

and

$$F_{AY} = \underline{\underline{66.6N}}$$

5.50

5.50 Determine the magnitude of the horizontal component of the anchoring force per unit width required to hold in place the sluice gate shown in Fig. P5.50. Compare this result with the size of the horizontal component of the anchoring force required to hold in place the sluice gate when it is closed and the depth of water upstream is 6 ft.

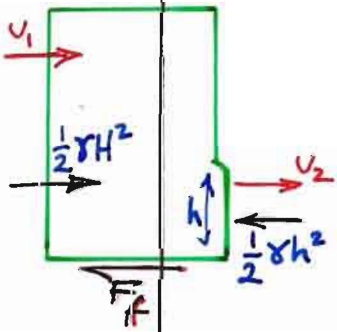


■ FIGURE P5.50

This analysis is similar to the one of Example 5.15. The control volumes of Fig. E 5.15 are appropriate for use in solving this problem. When the sluice gate is closed (see Figs. E 5.15 a and E 5.15 c) application of the x direction component of the linear momentum equation leads to

$$R_x = \frac{1}{2}\gamma H^2 = \frac{1}{2}\left(62.4\,\frac{lb}{ft^3}\right)(6ft)^2 = \underline{\underline{1120}}\ \frac{lb}{ft}$$

When the sluice gate is open (see Figs. E 5.15b and E5.15d) application of the x direction component of the linear momentum equation leads to



$$R_x = \frac{1}{2}\gamma H^2 - \frac{1}{2}\gamma h^2 - F_f + \boxed{\rho u_1^2 H - \rho u_2^2 h}\ -\rho\int_{cv} \underline{V}\,\underline{V}\cdot\hat{\underline{n}}\,dA$$

The exit velocity u_2 may be expressed in terms of the inlet velocity u_1 with the conservation of mass equation as follows

$$u_2 = u_1\frac{H}{h}$$

Thus

$$R_x = \frac{1}{2}\gamma H^2 - \frac{1}{2}\gamma h^2 - F_f + \rho u_1^2 H - \rho u_1^2\frac{H^2}{h}$$

Assuming F_f is negligibly small, we obtain

$$R_x = \frac{1}{2}\left(62.4\,\frac{lb}{ft^3}\right)(6ft)^2 - \frac{1}{2}\left(62.4\,\frac{lb}{ft^3}\right)(4ft)^2$$

$$+\left(1.94\,\frac{slugs}{ft^3}\right)\left(4\,\frac{ft}{s}\right)^2(6ft)\left(\frac{1\,lb}{slug\cdot\frac{ft}{s^2}}\right) - \left(1.94\,\frac{slugs}{ft^3}\right)\left(4\,\frac{ft}{s}\right)^2\frac{(6ft)^2}{(4ft)}\left(\frac{1\ lb}{slug\cdot\frac{ft}{s^2}}\right)$$

$$R_x = \underline{\underline{531}}\ \frac{lb}{ft}$$

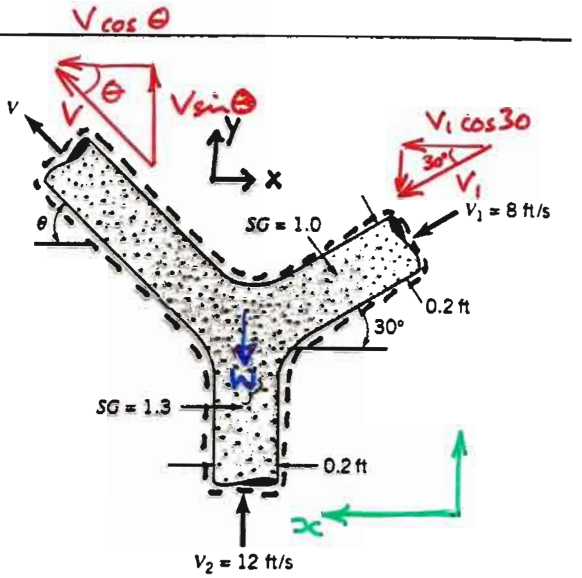
Thus it takes considerably less force to hold in place the sluice gate when it is opened as compared to when it is closed.

5.59

5.59 Two jets of liquid, one with specific gravity 1.0 and the other with specific gravity 1.3, collide and form one homogeneous jet as shown in Fig. P5.59. Determine the speed, V, and the direction, θ, of the combined jet. Gravity is negligible.

Note: open to atmosphere

$\therefore p = 0.$



■ FIGURE P5.59

For the control volume shown in the sketch above the linear momentum equations for the x and y directions are, for the x direction

2. equns – but 3 unknowns i.e V, A, θ

$$(V_1 \cos 30^\circ)\rho_1 V_1 A_1 - (V\cos\theta)\rho V A = 0 \qquad (1)$$

and the y direction

$$-V_2 \rho_2 V_2 A_2 + (V\sin\theta)\rho V A + (V_1 \sin 30)\rho V_1 A = -\cancelto{0}{W} \qquad (2)$$

Also, for conservation of mass

Use conservation of mass to constrain (VA).

⊛
$$-\rho_1 V_1 A_1 - \rho_2 V_2 A_2 + \rho V A = 0 \qquad (3)$$

From Eqs. 1 and 2 we get

$$\frac{V_1^2 \cos 30^\circ\,(1.0\,\rho_w)A_1}{V_2^2\,(1.3\,\rho_w)A_2} = \cot\theta$$

So

$$\theta = \cot^{-1}\left[\frac{V_1^2 \cos 30^\circ \left(\frac{\pi d_1^2}{4}\right)}{V_2^2 (1.3)\left(\frac{\pi d_2^2}{4}\right)}\right]$$

or

$$\theta = \cot^{-1}\left[\frac{\left(8\,\frac{ft}{s}\right)^2 \cos 30^\circ\ \pi\frac{(0.2\,ft)^2}{4}}{\left(12\,\frac{ft}{s}\right)^2 (1.3)\ \pi\frac{(0.2\,ft)^2}{4}}\right] = 73.5^\circ$$

(con't)

Now, combining Eqs. 2 and 3 we get

$$-V_2^2 \rho_2 A_2 + V \sin\theta \left(\rho_1 V_1 A_1 + \rho_2 V_2 A_2 \right) = 0$$

or

$$V = \frac{V_2^2 \rho_2 A_2}{\sin\theta \left(\rho_1 V_1 A_1 + \rho_2 V_2 A_2 \right)}$$

$$V = \frac{V_2^2 (1.3 \rho_w) \frac{\pi d_2^2}{4}}{\sin\theta \left(1.0 \rho_w V_1 \frac{\pi d_1^2}{4} + 1.3 \rho_w V_2 \frac{\pi d_2^2}{4} \right.}$$

and

$$V = \frac{\left(12 \frac{ft}{s}\right)^2 (1.3)\, \pi \frac{(0.2\, ft)^2}{4}}{(\sin 73.5^\circ) \left[\left(8 \frac{ft}{s}\right) \pi \frac{(0.2 ft)^2}{4} + \left(12 \frac{ft}{s}\right) 1.3\, \pi \frac{(0.2\, ft)^2}{4} \right]}$$

$$V = \underline{\underline{8.27}}\ \frac{ft}{s}$$

[7:2] Conservation of Momentum

Recap

Linear Momentum:

Static: $b = \mathbf{V}$ and $\dfrac{D}{Dt}B_{sys} = \dfrac{D}{Dt}\int_{sys}\mathbf{V}\rho\, d\mathcal{V} = \dfrac{\partial}{\partial t}\int_{cv}\mathbf{V}\rho\, d\mathcal{V} + \int_{cs}\mathbf{V}\rho\mathbf{V}\cdot\hat{n}dA = \sum\mathbf{F}$

Outline

Undershot weir

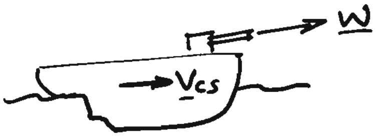
Vane Moving and steady: $\int_{cs}(\mathbf{W}+\mathbf{V_{cs}})\rho\mathbf{W}\cdot\hat{n}dA = \sum\mathbf{F}$

Jet-pack

Decapitated cylinder

MOVING CONTROL VOLUME (NON-DEFORMING)

Recall: $\underline{V} = \underline{W} + \underline{V}_{cs}$



Reynold's Transport Theorem

$$\frac{D}{Dt}\int_{sys}\underline{V}\rho\,d\forall = \frac{\partial}{\partial t}\int_{cv}\underline{V}\rho\,d\forall + \int_{cs}\underline{V}\rho\,\underline{W}\cdot\hat{n}\,dA = \Sigma F_{\text{contents of control volume.}}$$

$\underline{W}$ since this is relative to the control volume only.

Substituting $\underline{V} = \underline{W} + \underline{V}_{cs}$ yields:

$$\frac{\partial}{\partial t}\int_{cv}(\underline{W}+\underline{V}_{cs})\rho\,d\forall + \int_{cs}(\underline{W}+\underline{V}_{cs})\rho\,\underline{W}\cdot\hat{n}\,dA = \Sigma \underline{F}_{\text{contents of}\ \dots}$$

How to simplify??

1. Constant $\underline{V}_{cs}$ then $\frac{\partial}{\partial t}\underline{V}_{cs} = 0$
2. Steady flow: $\frac{\partial}{\partial t}\underline{W} = 0$
3. Non-deforming control volume, $\underline{V}_{cs} = \underline{V}_{cv}$

   $\therefore$ Split surface integral as

$$\int_{cs}(\underline{W}+\underline{V}_{cs})\rho\,\underline{W}\cdot\hat{n}\,dA = \int_{cs}\underline{W}\rho\,\underline{W}\cdot\hat{n}\,dA + \underline{V}_{cv}\int_{cs}\underbrace{\rho\,\underline{W}\cdot\hat{n}\,dA}_{}$$

Steady flow, $\int\rho\,\underline{W}\cdot\hat{n}\,dA = 0$

Steady; Non-deforming; Inertial

$$\boxed{\int_{cs}\underline{W}\rho\,\underline{W}\cdot\hat{n}\,dA = \Sigma F_{\text{contents of cv}}}$$

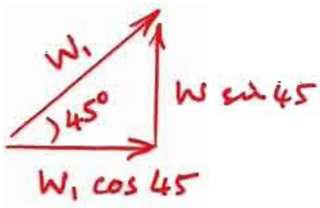
EXAMPLE 5.16

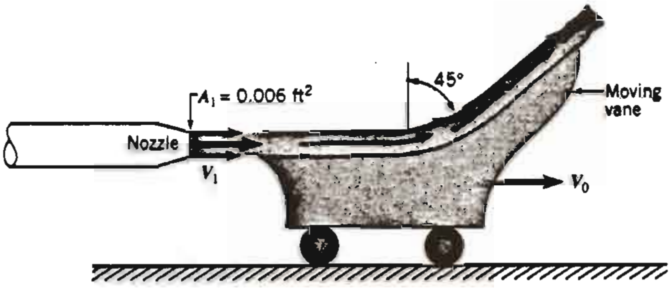
A vane on wheels moves with constant velocity $\mathbf{V}_0$ when a stream of water having a nozzle exit velocity of $\mathbf{V}_1$ is turned 45° by the vane as indicated in Fig. E5.16*a*. Note that this is the same moving vane considered in Section 4.4.6 earlier. Determine the magnitude and direction of the force, **F**, exerted by the stream of water on the vane surface. The speed of the water jet leaving the nozzle is 100 ft/s and the vane is moving to the right with a constant speed of 20 ft/s.

Relative velocities:

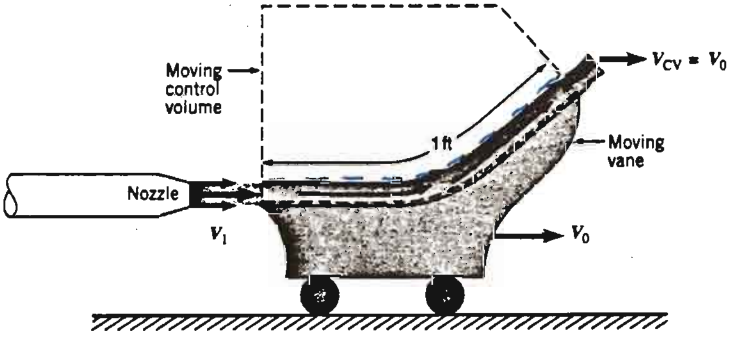
$\underline{V} = \underline{W} + \underline{V}_{cs}$

$\underline{W} = \underline{V} - \underline{V}_{cs}$

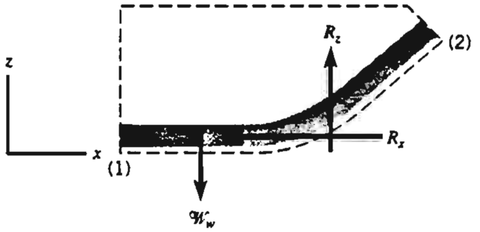




(*a*)



(*b*)



(*c*)

$$\boxed{\int_{cs} \mathbf{W}\rho\mathbf{W}\cdot\hat{\mathbf{n}}\,dA = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}}$$

x direction:

$$(W_1)\rho(-W_1)A_1 + (W_1\cos 45)\rho(W_1)A_1 = -R_x$$

direction:

$$0\rho(-W_1)A_1 + (W_1 \sin 45)\rho(W_1)A_1 = R_z - W_w$$

$$W_w = \rho g A_1 \ell$$

[7:3] Conservation of Momentum

Recap

Moving and steady: $\int_{cs}(\mathbf{W}+\mathbf{V_{cs}})\rho\mathbf{W}\cdot\hat{n}dA=\sum\mathbf{F}$

Outline

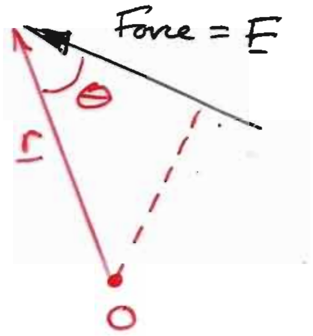
Moment-of-Momentum:

Steady: $b=(\mathbf{r}\times\mathbf{V})$ and $\int_{cs}(\mathbf{r}\times\mathbf{V})\rho\mathbf{V}\cdot\hat{n}dA=\sum(\mathbf{r}\times\mathbf{F})$

$$T_{shaft}=\pm rV_\theta\dot{m};\quad \dot{W}_{shaft}=T_{shaft}\omega;\quad w_{shaft}=\frac{\dot{W}_{shaft}}{\dot{m}}$$

MOMENT - OF - MOMENTUM

☐ Conservation laws for torques.



Moment about O is given as:

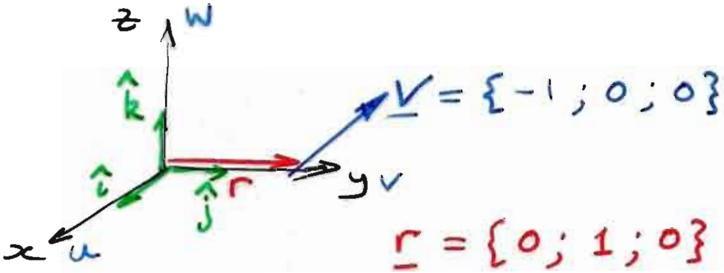
Cross product:

$\underline{r} \times \underline{F} =$ moment

Definition of cross product:

$$\underline{r} \times \underline{F} = \underbrace{F\, r}_{\text{Scalar magnitudes}} \sin\theta$$

REMEMBER CROSS PRODUCTS, $(\underline{r} \times \underline{V})$?



$$\underline{r} \times \underline{V}$$

$$\underline{r} \times \underline{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_i & \enclose{circle}{r_j} & r_k \\ \enclose{circle}{u} & v & w \end{vmatrix}$$

$$\underline{r} \times \underline{V} = \hat{i}(r_j w - r_k v) - \hat{j}(r_i w - r_k u) + \hat{k}(r_i v - \overset{\overset{1}{\downarrow}}{r_j}\underset{\underset{-1}{\uparrow}}{u})$$

Substituting values

$$r = \{0; \overset{\overset{r_j}{\downarrow}}{1}; 0\}$$

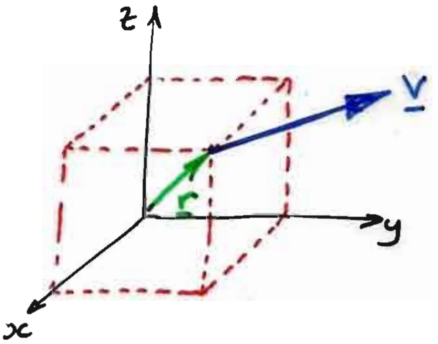
$$\underline{V} = \{\underset{\underset{u}{\uparrow}}{-1}; 0; 0\}$$

$$\underline{r} \times \underline{V} = +1\hat{k} = \text{lever arm} \times V$$

Torque about
z-axis.

Actually moment-of-
velocity about z axis.

MOMENT-OF-MOMENTUM, CONSERVATION OF



$\underline{r}$ = position vector from the origin to the fluid particle

$\underline{V}$ = velocity vector of fluid particle.

Torque or moment, $\underline{M}$, $= \underline{r} \times \underline{V}$

Reynolds' Transport Theorem

$$\underline{B} = m(\underline{r} \times \underline{V})$$

$$\underline{b} = \frac{\underline{B}}{m} = \underline{r} \times \underline{V}$$

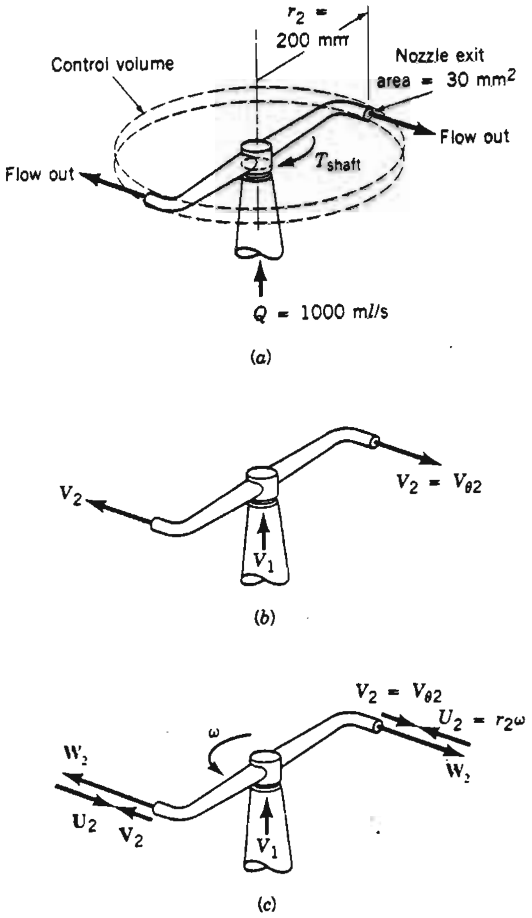
Substituting

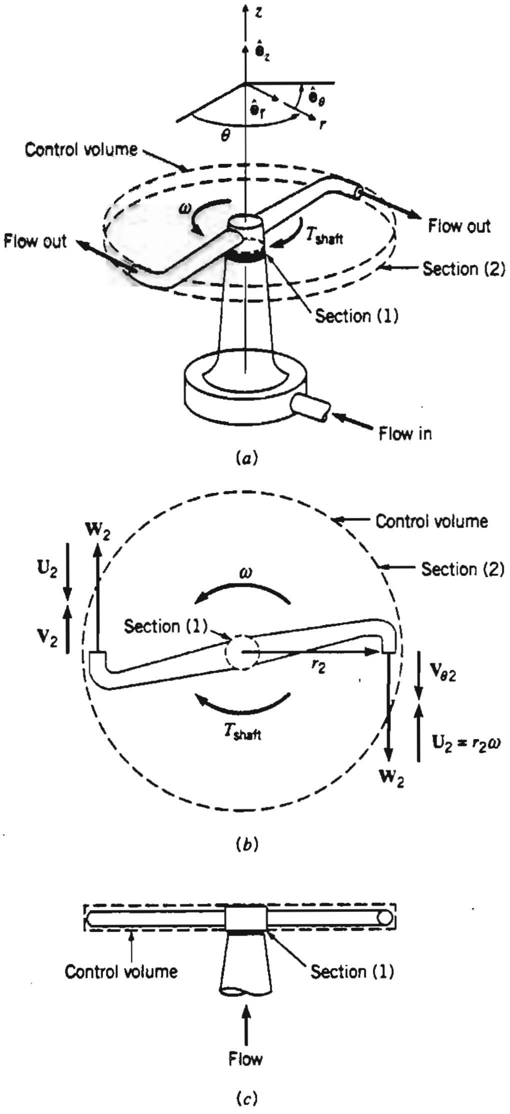
$$\underbrace{\frac{D}{Dt}\int_{sys}(\underline{r}\times\underline{V})\rho\, d\forall}_{\Sigma(\underline{r}\times\underline{F})\text{ contents of control volume}} = \frac{\partial}{\partial t}\int_{cv}(\underline{r}\times\underline{V})\rho\, d\forall + \int_{cs}(\underline{r}\times\underline{V})\rho\, \underline{V}\cdot\hat{n}\, dA$$

since $\underline{V}\rho\, d\forall = \delta\underline{F}$

Steady behaviour

$$\boxed{\Sigma(\underline{r}\times\underline{F}) = \int_{cs}(\underline{r}\times\underline{V})\rho\, \underline{V}\cdot\hat{n}\, dA}$$



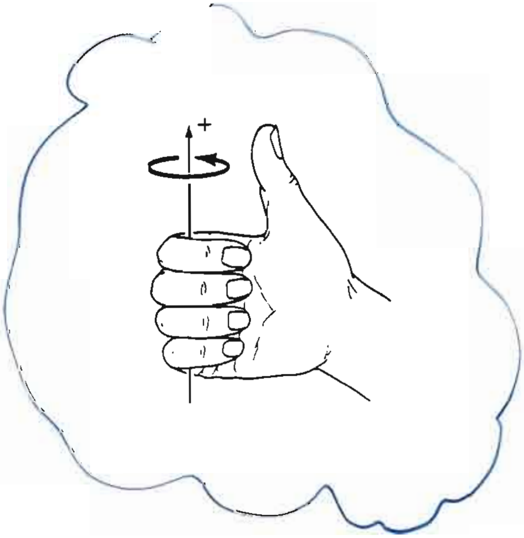


W_2 = relative velocity

V_2 = velocity relative to fixed frame of ref.

U_2 = velocity of control surface

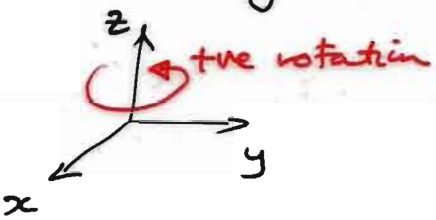
$$\boxed{V_2 = W_2 + U_2}$$



RIGHT - HAND RULE

PHYSICAL MEANING OF M-o-M EQUATION

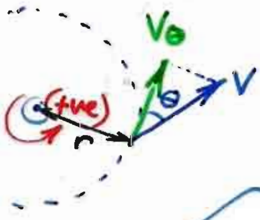
- ☐ Consider only rotation around a single axis. z.



- ☐ Steady behavior ∴ $\frac{\partial}{\partial t} \to 0$
- ☐ Fixed control surface.

$$\int_{cs} \underbrace{(\underline{r}\times\underline{V})}\,\underbrace{\rho\,\underline{V}\cdot\hat{n}}\,dA = \overbrace{\Sigma(\underline{r}\times\underline{F})}^{\text{Torque}}$$

$\underline{r}\times\underline{V} = r V_\theta$



Mass rate of flow $\dot{m} \doteq vA\rho$ leaving the control volume (+ve) or entering (−ve).

Note also that V is relative to the static control volume, and as previous

$$\underline{V} = \underline{W} + \underline{u}$$

Relative to static = Relative to control volume + Velocity of control volume

---

Rewriting

$$\boxed{T_{shaft} = r V_\theta \dot{m}}$$

Shaft power, $\dot{W}_{shaft} = T_{shaft}\,\omega$ ω = rotational speed.

Shaft work per unit mass, $\dot{w}_{shaft} = \frac{T_{shaft}\,\omega}{m}$

5.67

5.67 Five liters/s of water enters the rotor shown in Fig. P5.67 along the axis of rotation. The cross section area of each of the three nozzle exits normal to the relative velocity is 18 mm². How large is the resisting torque required to hold the rotor stationary? How fast will the rotor spin steadily if the resisting torque is reduced to zero and: **(a)** $\theta = 0°$; **(b)** $\theta = 30°$; **(c)** $\theta = 60°$?

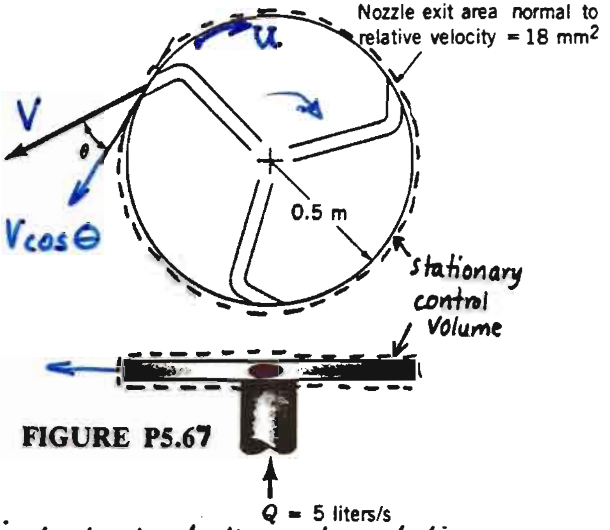


FIGURE P5.67

Hold the rotor static $V = W + \cancel{u}^{0}$

To determine the torque required to hold the rotor stationary we use the moment-of-momentum torque equation (Eq. 5.50) to obtain

$$T_{shaft} = \dot{m}\, r_{out}\, V_{out} \cos\theta \qquad (1)$$

We note that

$$\dot{m} = \rho Q \qquad (2)$$

and

$$V_{out} = \frac{Q}{3A_{\substack{nozzle\\exit}}} \qquad (3)$$

Combining Eqs. 1, 2 and 3 we get

$$T_{shaft} = \frac{\rho Q^2 r_{out} \cos\theta}{3\ A_{\substack{nozzle\\exit}}} \qquad (4)$$

To determine the rotor angular velocity associated with zero shaft torque we again use the moment-of-momentum torque equation (Eq. 5.50) to obtain, this time with rotation,

Allow rotor to spin

i.e. T=0

$$T_{shaft} = \dot{m}\, r_{out}\, (\underline{W_{out} \cos\theta} - U_{out}) \qquad (5)$$

Relative flow velocity in direction of circumference.

We note that

$$U_{out} = r_{out}\,\omega \qquad (6)$$

and

$$W_{out} = \frac{Q}{3A_{\substack{nozzle\\exit}}} \qquad (7)$$

Solve for $\underline{\omega}$

(con't)

5.67 (con't)

Combining Eqs. 2, 5, 6 and 7 we get

$$T_{shaft} = \rho Q r_{out} \left(\frac{Q}{3 A_{\substack{nozzle\\exit}}} \cos\theta - r_{out}\,\omega \right) \qquad (8)$$

(a) For $\theta = 0°$ we use Eq. 4 to get

$$T_{shaft} = \frac{\left(999\,\frac{kg}{m^3}\right)\left(5\,\frac{liters}{s}\right)^2 (0.5\,m)(\cos 0°)\left(1000\,\frac{mm}{m}\right)^2 \left(\frac{1\,N}{kg \cdot \frac{m}{s^2}}\right)}{\left(1000\,\frac{liters}{m^3}\right)^2 (3)(18\,mm^2)}$$

or

$$T_{shaft} = \underline{\underline{231\ N \cdot m}}$$

From Eq. 8 we obtain for $T_{shaft} = 0$

$$\omega = \frac{Q \cos\theta}{3 A_{\substack{nozzle\\exit}}\, r_{out}} = \frac{\left(5\,\frac{liters}{s}\right)(\cos 0°)\left(1000\,\frac{mm}{m}\right)^2}{3(18\,mm^2)\left(1000\,\frac{liters}{m^3}\right)(0.5\,m)} = \underline{\underline{185}}\,\frac{rad}{s}$$

(b) For $\theta = 30°$ we use Eq. 4 to get

$$T_{shaft} = \frac{\left(999\,\frac{kg}{m^3}\right)\left(5\,\frac{liters}{s}\right)^2 (0.5\,m)(\cos 30°)\left(1000\,\frac{mm}{m}\right)^2 \left(\frac{1\,N}{kg \cdot \frac{m}{s^2}}\right)}{\left(1000\,\frac{liters}{m^3}\right)^2 (3)(18\,mm^2)}$$

or

$$T_{shaft} = 200\ N \cdot m$$

From Eq. 8 we obtain for $T_{shaft} = 0$

$$\omega = \frac{\left(5\,\frac{liters}{s}\right)(\cos 30°)\left(1000\,\frac{mm}{m}\right)^2}{3(18\,mm^2)\left(1000\,\frac{liters}{m^3}\right)(0.5\,m)} = \underline{\underline{160}}\,\frac{rad}{s}$$

(c) For $\theta = 60°$ we use Eq. 4 to get

$$T_{shaft} = \frac{\left(999\,\frac{kg}{m^3}\right)\left(5\,\frac{liters}{s}\right)^2 (0.5\,m)(\cos 60°)\left(1000\,\frac{mm}{m}\right)^2 \left(\frac{1\,N}{kg \cdot \frac{m}{s^2}}\right)}{\left(1000\,\frac{liters}{m^3}\right)^2 (3)(18\,mm^2)}$$

or

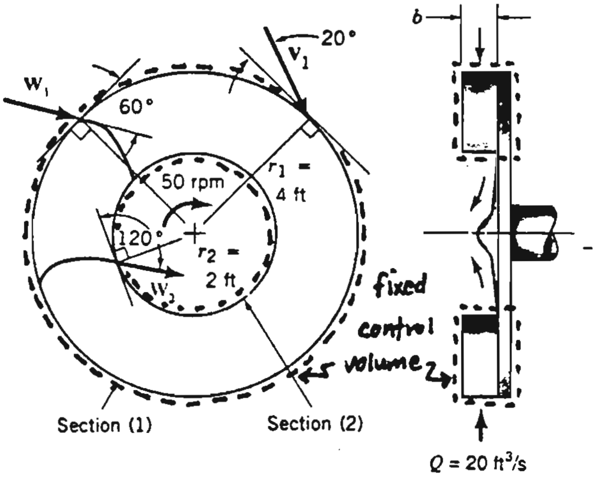
$$T_{shaft} = \underline{\underline{116\ N \cdot m}}$$

From Eq. 8 we obtain for $T_{shaft} = 0$

$$\omega = \frac{\left(5\,\frac{liters}{s}\right)(\cos 60°)\left(1000\,\frac{mm}{m}\right)^2}{(3)(18\,mm^2)\left(1000\,\frac{liters}{m^3}\right)(0.5\,m)} = \underline{\underline{92.5}}\,\frac{rad}{s}$$

5.70

5.70 A water turbine wheel rotates at the rate of 50 rpm in the direction shown in Fig. P5.70. The inner radius, r_2, of the blade row is 2 ft, and the outer radius, r_1, is 4 ft. The absolute velocity vector at the turbine rotor entrance makes an angle of 20° with the tangential direction. The inlet blade angle is 60° relative to the tangential direction. The blade outlet angle is 120°. The flowrate is 20 ft^3/s. For the flow tangent to the rotor blade surface at inlet and outlet, determine an appropriate constant blade height, b, and the corresponding power available at the rotor shaft.



■ FIGURE P5.70

since

$$Q = 2\pi r_1 b V_{R,1}$$

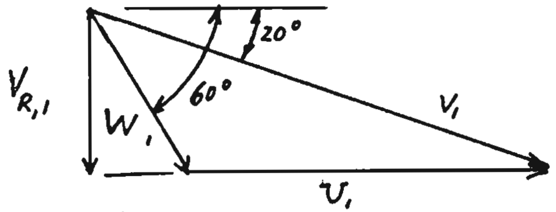
then the blade height, b, is determined with

$$b = \frac{Q}{2\pi r_1 V_{R,1}} \qquad (1)$$

The shaft power, $\dot{W}_{\substack{\text{shaft}\\ \text{net out}}}$, is obtained with the moment-of-momentum power equation (Eq. 5.53). Thus,

$$\dot{W}_{\substack{\text{shaft}\\ \text{net out}}} = \dot{m}\left(U_1 V_{\theta,1} \pm U_2 V_{\theta,2}\right) = \rho Q\left(U_1 V_{\theta,1} \pm U_2 V_{\theta,2}\right) \qquad (2)$$

and the use of "+" or "−" with $U_2 V_{\theta,2}$ depends on whether $V_{\theta,2}$ is opposite to or in the same direction as U_2 respectively. To determine the value of $V_{R,1}$ we use the velocity triangle at section (1). Thus, we have



With the velocity triangle we have

$$\frac{V_{R,1}}{\tan 20°} = \frac{V_{R,1}}{\tan 60°} + U_1 \qquad (3)$$

However

$$U_1 = r_1 \omega$$

(con't)

5.70 (con't)

thus Eq. 3 leads to

$$V_{R,1} = \frac{r_1 \omega}{\left(\frac{1}{\tan 20^\circ} - \frac{1}{\tan 60^\circ}\right)} = \frac{(4 ft)(50 rpm)(2\pi \frac{rad}{rev})}{\left(\frac{1}{\tan 20^\circ} - \frac{1}{\tan 60^\circ}\right)(60 \frac{s}{min})} = 9.651 \frac{ft}{s}$$

With Eq. 1 we obtain

$$b = \frac{(20 \frac{ft^3}{s})}{2\pi (4 ft)(9.651 \frac{ft}{s})} = 0.0825 ft$$

For the blade velocities in Eq. 2 we get

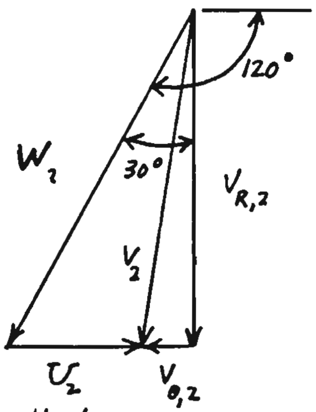
$$U_1 = r_1 \omega = \frac{(4 ft)(50 rpm)(2\pi \frac{rad}{rev})}{(60 \frac{s}{min})} = 20.94 \frac{ft}{s}$$

$$U_2 = r_2 \omega = \frac{(2 ft)(50 rpm)(2\pi \frac{rad}{rev})}{60 \frac{s}{min}} = 10.47 \frac{ft}{s}$$

For $V_{\theta,1}$ we use the velocity triangle at section (1) to obtain

$$V_{\theta,1} = \frac{V_{R,1}}{\tan 20^\circ} = \frac{9.651 \frac{m}{s}}{\tan 20^\circ} = 26.52 \frac{ft}{s}$$

For $V_{\theta,2}$ we construct the section (2) velocity triangle sketched below ($V_{\theta,2}$ not to scale)



and we realize that

$$V_{\theta,2} = V_{R,2} \tan 30^\circ - U_2 \qquad (4)$$

From conservation of mass

$$V_{R,2} = V_{R,1} \frac{A_1}{A_2} = V_{R,1}\left(\frac{r_1}{r_2}\right) = \left(9.651 \frac{ft}{s}\right)\left(\frac{4 ft}{2 ft}\right) = 19.3 \frac{ft}{s}$$

(con't)

5.70 (Con't)

so with Eq. 4 we obtain

$$V_{\theta,2} = \left(19.3 \ \frac{ft}{s}\right) \tan 30^\circ - 10.47 \ \frac{ft}{s} = 0.673 \ \frac{ft}{s}$$

Finally with Eq. 2 we obtain

$$\dot{W}_{\substack{shaft \\ net\ out}} = \left(1.94 \ \frac{slugs}{ft^3}\right)\left(20 \ \frac{ft^3}{s}\right)\left[\left(20.94 \ \frac{ft}{s}\right)\left(26.52 \ \frac{ft}{s}\right) + \left(10.47 \ \frac{ft}{s}\right)\left(0.673 \ \frac{ft}{s}\right)\right]\left(1 \ \frac{lb}{slug \cdot \frac{ft}{s^2}}\right)$$

or

$$\dot{W}_{\substack{shaft \\ net\ out}} = 2.18 \times 10^4 \ \frac{ft.lb}{s}$$

and

$$\dot{W}_{\substack{shaft \\ net\ out}} = \frac{2.18 \times 10^4 \ \frac{ft.lb}{s}}{550 \ \frac{ft.lb}{s.hp}} = \underline{\underline{39.6 \ hp}}$$

Summary: MASS $\frac{d}{dt}\int_{CV} \rho \, d\forall \quad + \quad \int_{CS} \rho \, \underline{W}\cdot\underline{n} \, dA \quad = 0$

Includes $\underline{V} = \underline{W} + \underline{V}_{cs}$

$$\rho_0 \frac{d\forall}{dt} + \forall \frac{d\rho}{dt} + \dot{m} = 0$$

LM $\int_{CS} \underline{W} \rho (\underline{W}\cdot\underline{n}) \, dA = \Sigma \underline{F}$

$$\underline{W} \dot{m} = \Sigma \underline{F}$$

Ang Momentum $\int (\underline{r} \times \underline{V}) \rho \, \underline{V}\cdot\underline{n} \, dA = \Sigma (\underline{r} \times \underline{F})$

$$\pm r V_\theta \dot{m} = T_{shaft}$$

[8]

Thermodynamics and Energy

Conservation Laws [7,8]

Relative velocities: $\mathbf{V} = \mathbf{W} + \mathbf{V_{cs}}$

Mass (continuity): $b = 1$ and $\dfrac{D}{Dt}M_{sys} = \dfrac{D}{Dt}\int_{sys} \rho d\cancel{V} = \dfrac{\partial}{\partial t}\int_{cv} \rho d\cancel{V} + \int_{cs} \rho \mathbf{W} \cdot \hat{n} dA = 0$

Linear Momentum:

Static: $b = \mathbf{V}$ and $\dfrac{D}{Dt}F_{sys} = \dfrac{D}{Dt}\int_{sys} \mathbf{V}\rho d\cancel{V} = \dfrac{\partial}{\partial t}\int_{cv} \mathbf{V}\rho d\cancel{V} + \int_{cs} \mathbf{V}\rho \mathbf{V} \cdot \hat{n} dA = \sum \mathbf{F}$

Moving and steady: $\int_{cs} (\mathbf{W} + \mathbf{V_{cs}})\rho \mathbf{W} \cdot \hat{n} dA = \sum \mathbf{F}$

Moment-of-Momentum:

Steady: $b = (\mathbf{r} \times \mathbf{V})$ and $\int_{cs} (\mathbf{r} \times \mathbf{V})\rho \mathbf{V} \cdot \hat{n} dA = \sum (\mathbf{r} \times \mathbf{F})$

$$T_{shaft} = \pm r V_{\theta} \dot{m}; \quad \dot{W}_{shaft} = T_{shaft}\omega; \quad w_{shaft} = \frac{\dot{W}_{shaft}}{\dot{m}}$$

First Law of Thermodynamics: $b = e$ and

$$\frac{\partial}{\partial t}\int_{cv} e\rho d\cancel{V} + \int_{cs} e\rho \mathbf{V} \cdot \hat{n} dA = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$\frac{p_{out}}{\gamma} + \frac{\alpha_o V_{out}^2}{2g} + z_{out} + h_L = \frac{p_{in}}{\gamma} + \frac{\alpha_i V_{in}^2}{2g} + z_{in} + h_P$$

$$\dot{m}[(\breve{h}_{out} - \breve{h}_{in}) + \tfrac{1}{2}(v_{out}^2 - v_{in}^2) + g(z_{out} - z_{in})] = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$h_P = \frac{w_{shaftin}}{g}; \quad w_{shaftin} = \frac{\dot{W}_{shaftin}}{\dot{m}}$$

$\alpha = 1$ for uniform flow.

Differential Analysis of Fluid Flow [7,8]

Euler′s Equation: $\rho \mathbf{g} - \nabla p = [\dfrac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}]$

Parallel plate flow: $q = -\dfrac{(2b)^3}{12\mu}\dfrac{\partial p}{\partial x}; \quad \hat{U} = -\dfrac{(2b)^2}{12\mu}\dfrac{\partial p}{\partial x}$

Circular pipe flow: $q = -\pi\dfrac{(2R)^4}{128\mu}\dfrac{\partial p}{\partial x}; \quad \hat{U} = -\dfrac{(2R)^2}{32\mu}\dfrac{\partial p}{\partial x}$

[8:1] Thermodynamics and Energy

Recap

Transport Theorem: $\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{cv} \rho b d\forall + \int_{cs} \rho b \mathbf{V} \cdot \hat{n} dA$ for $b = \frac{B}{m}$

Outline

First Law of Thermodynamics: $b = e$ and

$$\frac{\partial}{\partial t}\int_{cv} e\rho d\forall + \int_{cs} e\rho \mathbf{V} \cdot \hat{n} dA = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$\frac{p_{out}}{\gamma} + \frac{\alpha_o V_{out}^2}{2g} + z_{out} + h_L = \frac{p_{in}}{\gamma} + \frac{\alpha_i V_{in}^2}{2g} + z_{in} + h_P$$

$$\dot{m}[(\breve{h}_{out} - \breve{h}_{in}) + \tfrac{1}{2}(v_{out}^2 - v_{in}^2) + g(z_{out} - z_{in})] = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$h_P = \frac{w_{shaftin}}{g}; \quad w_{shaftin} = \frac{\dot{W}_{shaftin}}{\dot{m}}$$

FIRST LAW OF THERMODYNAMICS – ENERGY EQUATION

First law of Thermodynamics:

Rate of increase of total stored energy in the system = Net rate of addition by **heat** transfer + Net rate of addition by **work** transfer

---

$$\frac{D}{Dt}\int_{sys} e\rho \, d\forall = \left(\Sigma \dot{Q}_{in} - \Sigma \dot{Q}_{out}\right)_{sys} + \left(\Sigma \dot{W}_{in} - \Sigma \dot{W}_{out}\right)_{sys}$$

$$\frac{D}{Dt}\int_{sys} e\rho \, d\forall = \left(\dot{Q}_{\underset{in}{net}} + \dot{W}_{\underset{in}{net}}\right)_{sys}$$

What is total stored energy (per unit mass), e:

$$e = \check{u} + \frac{V^2}{2} + gz$$

Internal energy per unit mass

Kinetic energy per unit mass

Potential energy per unit mass

* N.B: $\dot{Q}$ and $\dot{W}$ defined (+ve) going into system
(−ve) exiting system
i.e. Opposite to $(\underline{V}\cdot\hat{n})$ concept.

REYNOLD'S TRANSPORT THEOREM

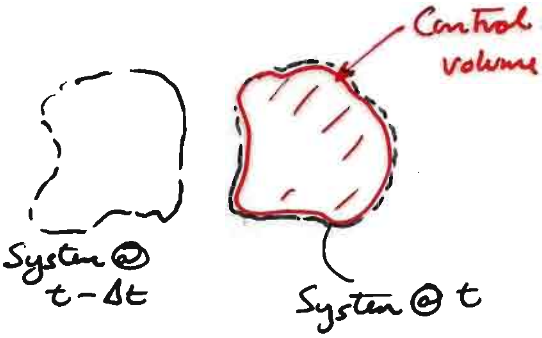
$$\boxed{\frac{D}{Dt}\int_{sys} b\rho \, d\forall = \frac{\partial}{\partial t}\int_{cv} b\rho \, d\forall + \int_{cs} b\rho(\underline{V}\cdot\hat{n})\, dA}$$

Extensive quantity : $B = em$

Intensive quantity : $b = \frac{B}{m} = e$

$$\frac{D}{Dt}\int_{sys} e\rho \, d\forall = \frac{\partial}{\partial t}\int_{cv} e\rho \, d\forall + \int_{cs} e\rho \, \underline{V}\cdot\hat{n} \, dA$$

Where : ① System and control volume are coincident



② Therefore $\frac{D}{Dt}\int_{sys} e\rho \, d\forall = \left(\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}}\right)_{\text{control volume}}$

$$\boxed{\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}} = \frac{\partial}{\partial t}\int_{cv} e\rho \, d\forall + \int_{cs} e\rho(\underline{V}\cdot\hat{n})\, dA}$$

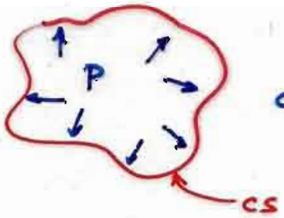
Heat transfer ($\dot{Q}$) :
- Radiation
- Conduction
- Convection

Adiabatic → $\dot{Q} = 0$

Work transfer ($\dot{W}$) : Power (+ve) when work done by surroundings on the control volume.

i.e. $\boxed{\dot{W}_{shaft} = T_{shaft}\,\omega}$

"WORK DONE" BY THE FLUID PRESSURE



P acts on control surface, cs

By definition: $\sigma = -p$.

Work done by normal stress, σ; or normal force $F_{\text{normal stress}}$.

$$\delta\dot{W} = \boxed{\delta\underline{F}_{\substack{\text{normal}\\\text{stress}}}}\cdot\underline{V}$$

i.e. Work = force × displacement

Work rate = force × displacement rate

$$\delta\dot{W}_{\substack{\text{normal}\\\text{stress}}} = \sigma\hat{n}\,\delta A\cdot\underline{V} = -p\hat{n}\,\delta A\cdot\underline{V} = -p\,\underline{V}\cdot\hat{n}\,\delta A$$

Total work due to normal stress; $\dot{W}_{\substack{\text{normal}\\\text{stress}}}$

$$\dot{W}_{\substack{\text{normal}\\\text{stress}}} = \int_{cs}\delta\dot{W} = -\int_{cs} p\,\underline{V}\cdot\hat{n}\,dA$$

The amount of power in the system is modified by work done by fluid pressures, p, acting on the "flowing" portion of the control volume.

L.H.S is modified as:

$$\dot{Q}_{\substack{\text{net}\\\text{in}}} + \dot{W}_{\substack{\text{net}\\\text{in}}}$$

$$\Downarrow$$

$$\dot{Q}_{\substack{\text{net}\\\text{in}}} + \dot{W}_{\substack{\text{net}\\\text{in}}} - \int_{cs} p\,\underline{V}\cdot\hat{n}\,dA$$

Mechanical work

Fluid work

SIMPLIFICATIONS TO THE GENERAL SYSTEM

$$\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}} = \frac{\partial}{\partial t}\int_{CV} e\rho \, d\forall + \rho\oint_{CS}\left(e+\frac{P}{\rho}\right)\underbrace{(\underline{V}\cdot\hat{n})\, dA}_{\uparrow \; \dot{m}}$$

Steady system : $\frac{\partial}{\partial t} \rightarrow 0$

Adiabatic : $\dot{Q}_{\substack{net\\in}} = 0$

'Stagnant' system : $\underline{V} = 0 \quad \therefore \quad \underline{V}\cdot\hat{n} = 0$

No transfer of power : $\dot{W}_{\substack{net\\in}} = 0.$

For "1-D" system: (Steady)

$$\dot{m}\left[(e_{out} - e_{in}) + \frac{1}{\rho}(p_{out} - p_{in})\right] = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}}$$

OR Denoting enthalpy as $\check{h} = \check{u} + \frac{P}{\rho}$ $\quad e = \check{u} + \frac{V^2}{2} + gz$

$$\boxed{\dot{m}\left[\check{h}_{out} - \check{h}_{in} + \frac{V_{out}^2 - V_{in}^2}{2} + g(z_{out} - z_{in})\right] = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}}}$$

$\check{u}$ = Internal energy per unit mass.

e = Total energy

COMPARISON WITH BERNOULLI EXPRESSION

Use "1-D" equation: but set $\dot{W} = 0$ (no abstraction from system).

$$\dot{m}\left[\check{u}_{out} - \check{u}_{in} + \frac{p_{out}}{\rho} - \frac{p_{in}}{\rho} + \frac{V_{out}^2 - V_{in}^2}{2} + g(z_{out} - z_{in})\right] = Q_{\substack{net\\in}}$$

Divide by mass flowrate, m, as

$$\frac{p_{out}}{\rho} + \frac{V_{out}^2}{2} + g z_{out} = \frac{p_{in}}{\rho} + \frac{V_{in}^2}{2} + g z_{in} - \underbrace{\left(\check{u}_{out} - \check{u}_{in} - q_{\substack{net\\in}}\right)}_{\text{loss}}$$

$$q_{\substack{net\\in}} = \frac{\dot{Q}_{net\,in}}{\dot{m}}$$

Energy loss due to "real" effects.

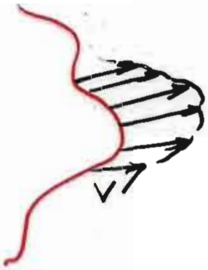
- ☐ Friction (viscous flow)
- ☐ Compressibility of flow

CORRECTION FOR NON-UNIFORM FLOWS

$$\int_{cs} e\rho\, \underline{V}\cdot\hat{n}\, dA$$

$$e = \check{u} + \frac{V^2}{2} + gz$$

Where:



Non-uniform velocity profile

$$\int_{cs} \frac{V^2}{2}\rho\, \underline{V}\cdot\hat{n}\, dA \neq \frac{V^2}{2}\int_{cs}\rho\, \underline{V}\cdot\hat{n}\, dA$$

Accommodated as:

$$\int \frac{V^2}{2}\rho\, \underline{V}\cdot\hat{n}\, dA = \dot{m}\left[\frac{\alpha_{out}\bar{V}_{out}^2}{2} - \frac{\alpha_{in}\bar{V}_{in}^2}{2}\right]$$

$\alpha = 1$ Uniform flow

$\alpha > 1$ Non-uniform flow.

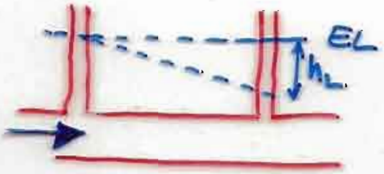
Modified Bernoulli expression:

$$\frac{P_{out}}{\gamma} + \frac{\alpha_{out}\bar{V}_{out}^2}{2g} + z_{out} = \frac{P_{in}}{\gamma} + \frac{\alpha_{in}\bar{V}^2}{2g} + z_{in} + \frac{W_{shaft\ net\ in}}{g} - h_L$$

head loss.

Important expression since it allows frictional losses to be accommodated.

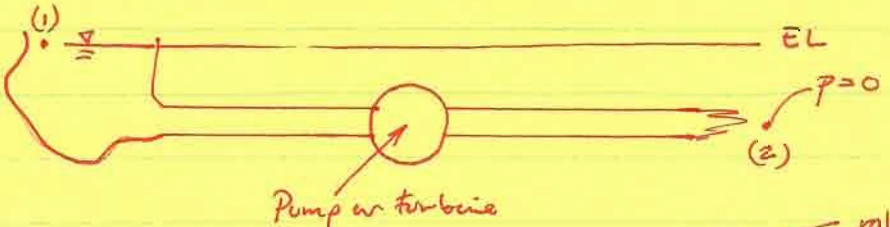
eg. Pipe flow / groundwater flow.



First Law of Thermodynamics - Energy Equation

Apply first law of thermodynamics to Reynold's transport theorem.
i.e. Total energy = Thermal energy + Mech work
Rearrange terms.

$$\frac{P_{in}}{\gamma} + \frac{\alpha_{in} V_{in}^2}{2g} + z_{in} + h_p = \frac{P_{out}}{\gamma} + \frac{\alpha_{out} V_{out}^2}{2g} + z_{out} + h_L$$



rotational velocity

$$\dot{W}_{shaft} = T_{shaft}\,\omega$$

no dot

$$h_p = \frac{w_{shaft\,in}}{g}$$

no dot

$$w_{shaft} = \frac{\dot{W}_{shaft}}{\dot{m}}$$

h_L = frictional losses in system

α = descriptor of uniformity of velocity $\int_{cs} \frac{V^2}{2}\rho\,\underline{V}\cdot\underline{n}\,dA \neq \frac{V^2}{2}\int_{cs}\rho\,\underline{V}\cdot\underline{n}\,dA$

$\therefore \alpha \frac{V^2}{2}\dot{m}$

$\alpha = 1$ uniform flow

$\alpha > 1$ non-uniform

Remember:

1. Always write equation with reference to upstream and downstream components.
   Important due to directionality of energy loss

2. H is no longer constant in system, as Bernoulli.
   due to h_L
   h_p → positive pump
   negative turbine

EXAMPLE 5.20

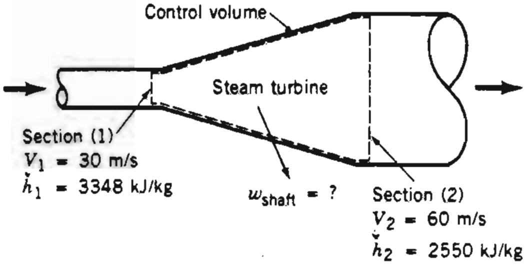
Steam enters a turbine with a velocity of 30 m/s and enthalpy, $\check{h}_1$, of 3348 kJ/kg (see Fig. E5.20). The steam leaves the turbine as a mixture of vapor and liquid having a velocity of 60 m/s and an enthalpy of 2550 kJ/kg. If the flow through the turbine is adiabatic and changes in elevation are negligible, determine the work output involved per unit mass of steam through-flow.

Note:

Work Joule, J ≡ Nm

Units: $\frac{P}{\rho} = \frac{N m^{-2}}{Kg/m^3} = \frac{J}{Kg}$

$\check{h} \equiv \check{u} + \frac{P}{\rho} \equiv \frac{ML^{-1}T^{-2}}{ML^{-3}}$



■ FIGURE E5.20

SOLUTION

We use a control volume that includes the steam in the turbine from the entrance to the exit as shown in Fig. E5.20. Applying Eq. 5.69 to the steam in this control volume we get

$$\dot{m}\left[\check{h}_2 - \check{h}_1 + \frac{V_2^2 - V_1^2}{2} + \overset{0 \text{ (elevation change is negligible)}}{g(\cancel{z_2 - z_1})}\right] = \overset{0 \text{ (adiabatic flow)}}{\cancel{\dot{Q}}_{\substack{\text{net}\\\text{in}}}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}} \qquad (1)$$

The work output per unit mass of steam through-flow, $w_{\substack{\text{shaft}\\\text{net in}}}$, can be obtained by dividing Eq. 1 by the mass flow rate, $\dot{m}$, to obtain

$$w_{\substack{\text{shaft}\\\text{net in}}} = \frac{\dot{W}_{\substack{\text{shaft}\\\text{net in}}}}{\dot{m}} = \check{h}_2 - \check{h}_1 + \frac{V_2^2 - V_1^2}{2} \qquad (2)$$

Since $w_{\text{shaft net out}} = -w_{\text{shaft net in}}$, we obtain

$$w_{\substack{\text{shaft}\\\text{net out}}} = \check{h}_1 - \check{h}_2 + \frac{V_1^2 - V_2^2}{2}$$

or

$$w_{\substack{\text{shaft}\\\text{net out}}} = 3348 \text{ kJ/kg} - 2550 \text{ kJ/kg} + \frac{[(30 \text{ m/s})^2 - (60 \text{ m/s})^2][1 \text{ J/(N·m)}]}{2[1 \text{ (kg·m)/(N·s}^2)](1000 \text{ J/kJ})}$$

Thus

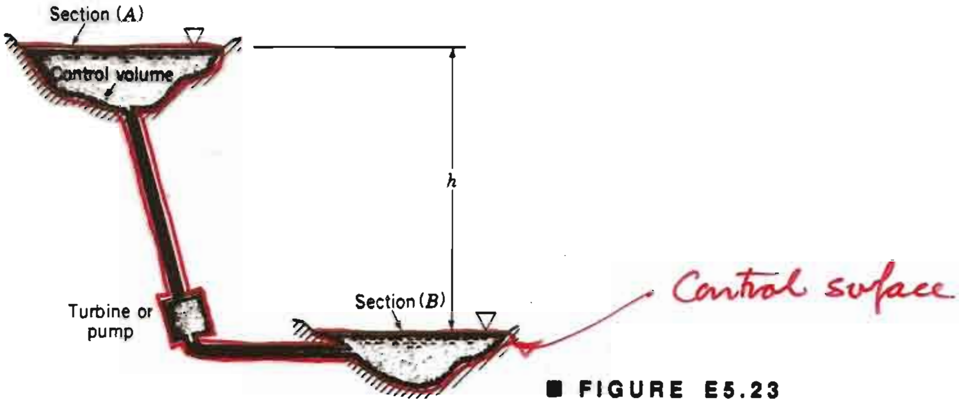
$$w_{\substack{\text{shaft}\\\text{net out}}} = 3348 \text{ kJ/kg} - 2550 \text{ kJ/kg} - 1.35 \text{ kJ/kg} = 797 \text{ kJ/kg} \qquad \text{(Ans)}$$

Assumes a perfect system with no losses!!

Note that in this particular example, the change in kinetic energy is small in comparison to the difference in enthalpy involved. This is often true in applications involving steam turbines. To determine the power output, $\dot{W}_{\text{shaft}}$, we must know the mass flowrate, $\dot{m}$.

EXAMPLE 5.23

Two large lakes exist as are shown in Fig. E5.23. The water level of one lake is appreciably higher than the other. During the day, water flows from the higher lake to the lower one through appropriate piping and a turbine to provide supplementary power to a nearby city. At night, water is pumped back from the lower lake to the higher one to "store" energy. Show that in each situation minimizing losses is beneficial.



■ FIGURE E5.23

SOLUTION

For turbine operation, application of Eq. 5.82 to the contents of the control volume shown in Fig. E5.23 leads to

0 (motionless water surface) 0 (motionless water surface) (large reservoir assumption)

$$\cancel{\frac{p_B}{\rho}} + \cancel{\frac{V_B^2}{2}} + gz_B = \cancel{\frac{p_A}{\rho}} + \cancel{\frac{V_A^2}{2}} + gz_A + w_{\substack{\text{shaft}\\ \text{net in}}} - {}_A\text{loss}_B \qquad (1)$$

(atmospheric pressures cancel)

or

Turbine loss A → B

$$-w_{\substack{\text{shaft}\\ \text{net in}}} = \boxed{g(z_A - z_B) - {}_A\text{loss}_B = w_{\substack{\text{shaft}\\ \text{net out}}}} \qquad (2)$$

Power gained from turbine ≡ $W_{\substack{\text{shaft}\\ \text{net out}}}$

We see clearly from Eq. 2 that the amount of work and, therefore, power obtained from the turbine is the maximum amount available diminished by the amount of loss due to frictional effects involved.

For pump operation (i.e., flow from *B* to *A*), application of Eq. 5.82 to the contents of the control volume shown in Fig. E5.23 leads to

$$gz_A = gz_B + w_{\substack{\text{shaft}\\ \text{net in}}} - {}_B\text{loss}_A$$

$$\boxed{w_{\substack{\text{shaft}\\ \text{net in}}} = g(z_A - z_B) + {}_B\text{loss}_A}$$

Power needed to pump uphill ≡ $W_{\substack{\text{shaft}\\ \text{net in}}}$

We note that the amount of work and, therefore, power required to move water from the lower lake to the higher one is increased above the minimum amount by the loss due to frictional effects involved. In Chapter 8 we discuss ways to calculate such losses based on the pipe diameter, length, material, and other parameters.

NO FREE LUNCH !!

LIKE FRICTION

5.100

5.100 For the 180° elbow and nozzle flow shown in Fig. P5.100, determine the loss in available energy from section (1) to section (2). How much additional available energy is lost from section (2) to where the water comes to rest?

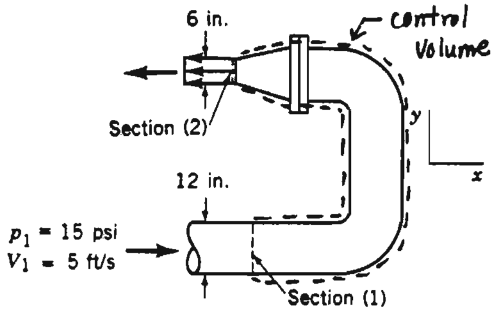


FIGURE P5.100

For solving the first part of this problem, the control volume shown in the sketch above is used. To determine the loss accompanying flow from section 1 to section 2 Eq. 5.79 can be used as follows.

$$ {}_1loss_2 = \frac{p_1 - p_2}{\rho} + \frac{V_1^2 - V_2^2}{2} + g(z_1 - z_2) $$

Since x-y coordinates are specified we assume that the flow is horizontal and $z_1 - z_2 = 0$. Also, $p_2 = p_{atm} = 0$ psi.

From the conservation of mass principle we conclude that

$$ V_2 = V_1 \frac{A_1}{A_2} = V_1 \left(\frac{D_1^2}{D_2^2}\right) $$

Thus

$$ {}_1loss_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2}\left[1 - \left(\frac{D_1^2}{D_2^2}\right)^2\right] = \frac{p_1}{\rho} + \frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] $$

or

$$ {}_1loss_2 = \frac{\left(15\ \frac{lb}{in.^2}\right)\left(144\ \frac{in.^2}{ft^2}\right)}{\left(1.94\ \frac{slugs}{ft^3}\right)} + \frac{\left(5\ \frac{ft}{s}\right)^2}{2}\left[1 - \left(\frac{12\ in.}{6\ in.}\right)^4\right]\left(\frac{1\ lb}{slug \cdot \frac{ft}{s^2}}\right) $$

$$ {}_1loss_2 = \underline{\underline{926}}\ \frac{ft \cdot lb}{slug} $$

For the second part of this problem we consider the flow of a fluid particle from section 2 to a state of rest, a. Eq. 5.79 leads to

$$ {}_2loss_a = \frac{V_2^2}{2} $$

Note that we have assumed that $p_2 = p_a = p_{atm}$ and $z_2 = z_a$.

Thus

$$ {}_2loss_A = \frac{V_2^2}{2} = \frac{V_1^2}{2}\left(\frac{D_1^2}{D_2^2}\right)^2 = \frac{V_1^2}{2}\left(\frac{D_1}{D_2}\right)^4 = \frac{\left(5\ \frac{ft}{s}\right)^2\left(\frac{12\ in.}{6\ in.}\right)^4\left(1\ \frac{lb}{slug \cdot \frac{ft}{s^2}}\right)}{2} $$

$$ {}_2loss_A = \underline{\underline{200}}\ \frac{ft \cdot lb}{slug} $$

[8:2] Thermodynamics and Energy

Recap

$$\dot{m}[(\breve{h}_{out} - \breve{h}_{in}) + \tfrac{1}{2}(v_{out}^2 - v_{in}^2) + g(z_{out} - z_{in})] = \dot{Q}_{netin} + \dot{W}_{netin}$$

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 + h_P = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_L$$

$$h_P = \frac{\dot{W}_{shaftin}}{\dot{m}g} = \frac{w_{shaftin}}{g}; \qquad h_L = K_L \frac{V^2}{2g}$$

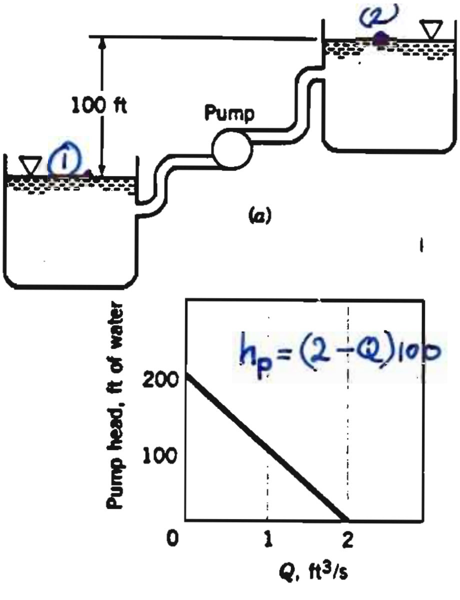
Outline

- Examples
- Differential analysis
  - Porous medium flow
  - Flow between plates
  - Flow in pipe

5.117

5.117 A pump transfers water from one large reservoir to another as shown in Fig. P5.117*a*. The difference in elevation between the two reservoirs is 100 ft. The friction head loss in the piping is given by $K_L\overline{V}^2/2g$, where $\overline{V}$ is the average fluid velocity in the pipe and K_L is the loss coefficient, which is considered constant. The relation between the total head rise H across the pump and the flowrate Q through the pump is given in Fig. 5.117*b*. If $K_L = 20$, and the pipe diameter is 4 in., what is the flowrate through the pump?

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$



(b)

FIGURE P5.117

For the flow from section (1) to section (2) Eq. 5.84 leads to

$$h_p = z_2 - z_1 + h_L \qquad (1)$$

From Fig. P5.117 b we conclude that

$$h_p = 200 - 100\,Q \qquad (2)$$

From the problem statement

$$h_L = K_L \frac{\overline{V}^2}{2g}$$

or since

$$\overline{V} = \frac{Q}{A} = \frac{Q}{\frac{\pi D^2}{4}}$$

we have

$$h_L = \frac{K_L Q^2}{2g\left(\frac{\pi D^2}{4}\right)^2} = \frac{(20)\left(Q\,\frac{ft^3}{s}\right)^2}{(2)\left(32.2\,\frac{ft}{s^2}\right)\left[\frac{\pi\,(4\text{ in.})^2}{\left(12\,\frac{in.}{ft}\right)^2(4)}\right]^2} = 40.78\,Q^2 \text{ ft} \qquad (3)$$

Combining Eqs. 1, 2 and 3 we obtain

$$40.78\,Q^2 + 100\,Q - 100 = 0 \qquad (4)$$

The root of Eq. 4 that makes physical sense is

$$Q = \underline{\underline{0.763\,\frac{ft^3}{s}}}$$

$V_2 = \frac{Q}{A_2}$

5.121

5.121 Gasoline ($SG = 0.68$) flows through a pump at 0.12 m³/s as indicated in Fig. P5.121. The head loss between sections (1) and (2) is equal to $0.3V_1^2/2$. What will the difference in pressures between sections (1) and (2) be if $\underset{\nwarrow\ \dot{W} = \text{power}}{\underline{20\text{ kW}}}$ is delivered by the pump to the fluid?

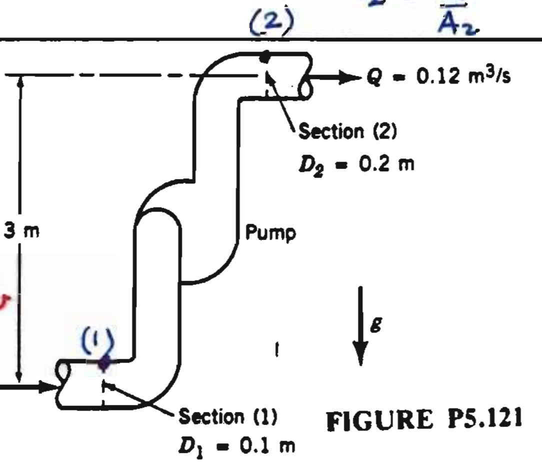


FIGURE P5.121

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + \frac{W_{in}}{g} = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

From Eq. 5.82 we get for the flow from section (1) to section (2)

$$p_1 - p_2 = \rho\left[\frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) - w_{\substack{shaft\\ net\ in}} + loss\right] \qquad (1)$$

From the volume flowrate we obtain

$$V_2 = \frac{Q}{A_2} = \frac{Q}{\frac{\pi D_2^2}{4}} = \frac{\left(0.12\ \frac{m^3}{s}\right)}{\frac{\pi\ (0.2\ m)^2}{4}} = 3.82\ \frac{m}{s}$$

and from conservation of mass (Eq. 5.13) it follows that

$$V_1 = V_2\ \frac{A_2}{A_1} = V_2\ \frac{D_2^2}{D_1^2} = \left(3.82\ \frac{m}{s}\right)\frac{(0.2\ m)^2}{(0.1\ m)^2} = 15.28\ \frac{m}{s}$$

Also

$$w_{\substack{shaft\\ net\ in}} = \frac{\dot{W}_{\substack{shaft\\ net\ in}}}{\underset{kg/s \nearrow}{\boxed{\rho Q}}} = \frac{\left(20{,}000\ \frac{N\cdot m}{s}\right)}{(0.68)\left(999\ \frac{kg}{m^3}\right)\left(0.12\ \frac{m^3}{s}\right)} = 245.3\ \frac{N\cdot m}{kg}$$

and

$$loss = 0.3\ \frac{V_1^2}{2} = (0.3)\frac{\left(15.28\ \frac{m}{s}\right)^2}{2}\left(1\ \frac{N}{kg\cdot\frac{m}{s^2}}\right) = 35.02\ \frac{N\cdot m}{kg}$$

From Eq. 1 then

$$p_1 - p_2 = (0.68)\left(999\ \frac{kg}{m^3}\right)\left\{\left[\frac{\left(3.82\ \frac{m}{s}\right)^2 - \left(15.28\ \frac{m}{s}\right)^2}{2} + \left(9.81\ \frac{m}{s^2}\right)(3\ m)\right]\left(\frac{1\ N}{kg\cdot\frac{m}{s^2}}\right) - 245.3\ \frac{N\cdot m}{kg} + 35.02\ \frac{N\cdot m}{kg}\right\}$$

or

$$p_1 - p_2 = \underline{\underline{-197{,}000\ \frac{N}{m^2}}} = \underline{\underline{-197\ kPa}}$$

Power (SI) = Watt   1 Watt = 1 Nm/s.

5.107

5.107 What is the maximum possible power output of the hydroelectric turbine shown in Fig. P5.107?

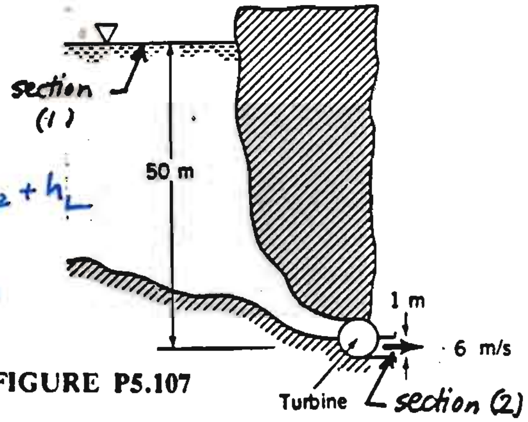


FIGURE P5.107

Relations: $\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+z_1+h_p=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+z_2+h_L$

No dot

Also: $h_p=\frac{w_{shaft\,in}}{g}$ $w_{shaft}=\frac{\dot{W}_{shaft\,in}}{\dot{m}}$

For flow from section (1) to section (2), Eq. 5.82 yields

$$\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2=\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1+w_{\substack{shaft\\ net\ in}}-loss \qquad (1)$$

Since $p_1=p_2=p_{atm}$, $w_{\substack{shaft\\ net\ in}}=-w_{\substack{shaft\\ net\ out}}$ Eq. 1 can be expressed as

$$w_{\substack{shaft\\ net\ out}}=g(z_1-z_2)-\frac{V_2^2}{2}-loss$$

The maximum work or power output is achieved when loss = 0.

(con't)

5.107 (con't)

Thus

$$\dot{W}_{\substack{shaft\\ net\ out\\ maximum}}=\dot{m}\,w_{\substack{shaft\\ net\ out\\ maximum}}=\dot{m}\left[g(z_1-z_2)-\frac{V_2^2}{2}\right]$$

Now

$$\dot{m}=\rho V_2 A_2=\rho V_2\frac{\pi D_2^2}{4}=\left(999\frac{kg}{m^3}\right)\left(6\frac{m}{s}\right)\frac{\pi(1\,m)^2}{4}=4710\frac{kg}{s}$$

and

$$\dot{W}_{\substack{shaft\\ net\ out\\ maximum}}=\left(4710\frac{kg}{s}\right)\left[\left(9.81\frac{m}{s^2}\right)(50m)-\frac{\left(6\frac{m}{s}\right)^2}{2}\right]\left(\frac{1\,N}{kg\cdot\frac{m}{s^2}}\right)$$

$$\dot{W}_{\substack{shaft\\ net\ out\\ maximum}}=\underline{\underline{2.23\times10^6}}\,\frac{N.m}{s}=\underline{\underline{2.23\times10^6\,W}}=\underline{\underline{2.23}}\,MW$$

1 Watt = 1 Nm/s

Evaluation of Power ($\dot{W}$) in BGS.

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s}$$

5.108

5.108 A hydroelectric turbine passes 4 million gal/min across a head of 100 ft of water. What is the maximum amount of power output possible? Why will the actual amount be less?

From the analysis of problem 5.107 we conclude that

$$\dot{W}_{\substack{\text{shaft}\\ \text{net out}\\ \text{maximum}}} = \dot{m}g(\text{head}) = \rho Q g(\text{head})$$

$$\dot{W}_{\substack{\text{shaft}\\ \text{net out}\\ \text{maximum}}} = \left(1.94\,\frac{\text{slugs}}{\text{ft}^3}\right)\left(4\times10^6\,\frac{\text{gal}}{\text{min}}\right)\frac{(100\text{ ft})\left(32.2\,\frac{\text{ft}}{\text{s}^2}\right)\left(1\,\frac{\text{lb}}{\text{slug}\cdot\frac{\text{ft}}{\text{s}^2}}\right)}{\left(60\,\frac{\text{s}}{\text{min}}\right)\left(7.48\,\frac{\text{gal}}{\text{ft}^3}\right)\left(550\,\frac{\text{ft}\cdot\text{lb}}{\text{s}\cdot\text{hp}}\right)}$$

$$\dot{W}_{\substack{\text{shaft}\\ \text{net out}\\ \text{maximum}}} = \underline{\underline{1.01\times10^5}}\text{ hp}$$

FLUID KINEMATICS

Where are we:

1. Equations developed for: (BERNOULLI)
   a) Inviscid $\mu = 0$ ← Neglect shear stresses in derivation along streamline
   b) Incompressible
   c) Isothermal

$h_L = 0$

2. Thermodynamic concerns:
   - ☐ Reynolds' Transport Theorem
   - ☐ Gives "Bernoulli like" expression

$h_L \neq 0$

If $h_L \neq 0$ then how do we evaluate h_L. ?

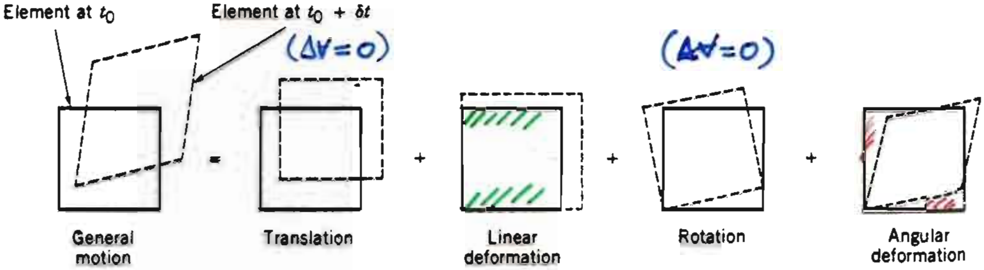
Need a more comprehensive analysis that:

1. Incorporates viscous effects as $\mu \neq 0$
   ∴ includes $-p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$
   and τ_{xy} etc.

2. Incorporates compressibility of fluid.
   (Constitutive equn. or equn. of state)

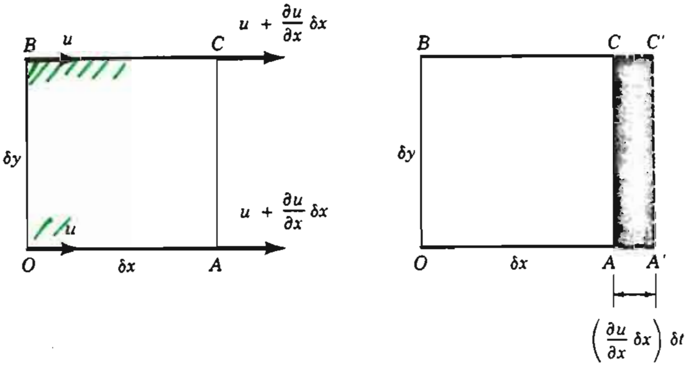
3. Uses "differential" control volumes. (Define local behavior).

FLUID KINEMATICS (INCORPORATE COMPRESSIBILITY)



■ **FIGURE 6.1** Types of motion and deformation for a fluid element.

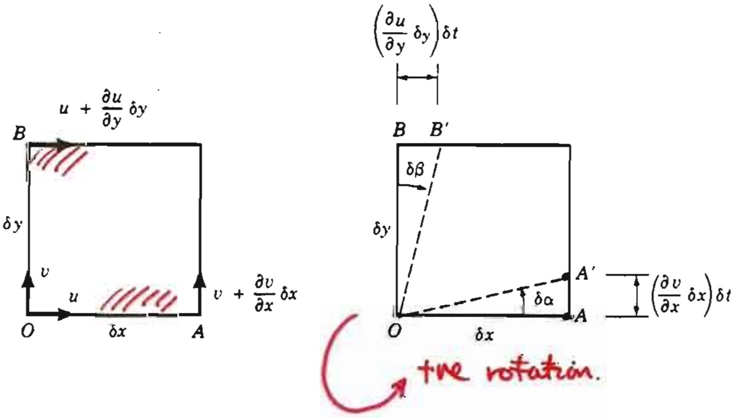
LINEAR MOTION & DEFORMATION



$$\frac{1}{\delta \forall}\frac{d\forall}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \underline{\nabla}\cdot\underline{V}$$

(6.9)

ANGULAR MOTION & DEFORMATION

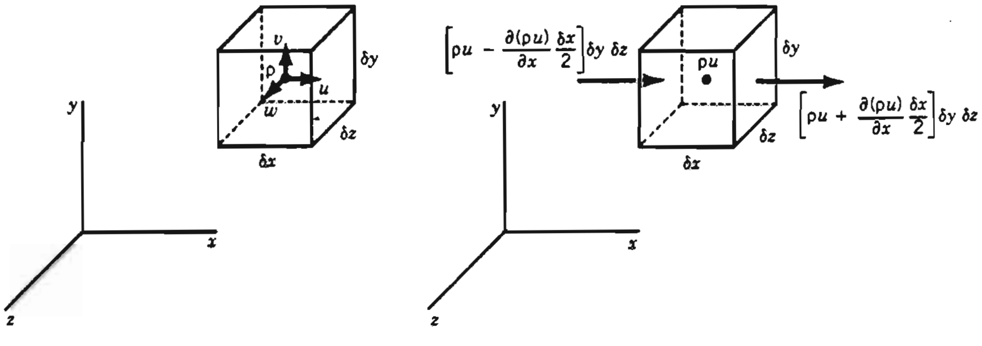


$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (6.12)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad (6.13)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad (6.14)$$

CONSERVATION OF MASS



(a) (b)

■ **FIGURE 6.5** A differential element for the development of conservation of mass equation.

Integral Form :

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \overset{\underline{V} \text{ if static}}{\cancel{\underline{W}}} \cdot \hat{n} \, dA = 0$$

Differential Form:

x-direction:

$$\underbrace{\left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]}_{\rho(\underline{V}\cdot\hat{n})}\underbrace{\delta y\,\delta z}_{dA} - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z = \frac{\partial(\rho u)}{\partial x}\delta x\,\delta y\,\delta z$$

y-direction: Mass outflow $= \frac{\partial}{\partial y}(\rho v)\,\delta x\,\delta y\,\delta z$ z-: $\frac{\partial}{\partial z}(\rho w)\,\delta x\,\delta y\,\delta z$

Adding components:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad (6.27)$$

Simplifying:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\rho\,\underline{V} = 0$$ Nonsteady, compressible

$$\underline{\nabla}\cdot\rho\,\underline{V} = 0$$ Steady, compressible

$$\underline{\nabla}\cdot\underline{V} = 0$$ Steady, incompressible

[8:3] Thermodynamics and Energy

Recap

Conservation of mass

$$\cancel{V}\dot{\rho}+\rho\dot{\cancel{V}}+\dot{M}=0$$

$$\cancel{V}\frac{\partial \rho}{\partial t}+\cancel{V}\rho[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}]=0$$

Outline

Differential analysis

- Conservation of mass – kinematics
- Stream potentials
- Flow between plates
- Flow in pipe

STREAM FUNCTION

2-D; steady; incompressible flow: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

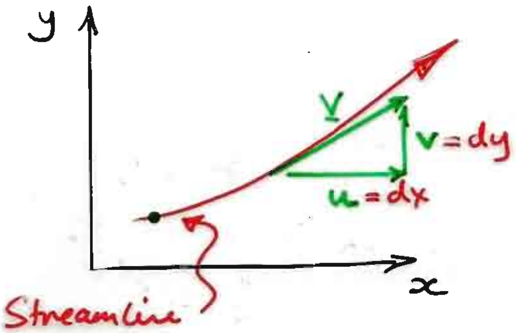
Arbitrarily define a function $\psi(x,y) \equiv$ "stream function"

$$\boxed{u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x}} \qquad (6.37)$$

Then we note that this function satisfies "continuity" or "conservation of mass" equation as:

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Usefully: ψ = constant defines streamlines (that are everywhere tangent to velocities).

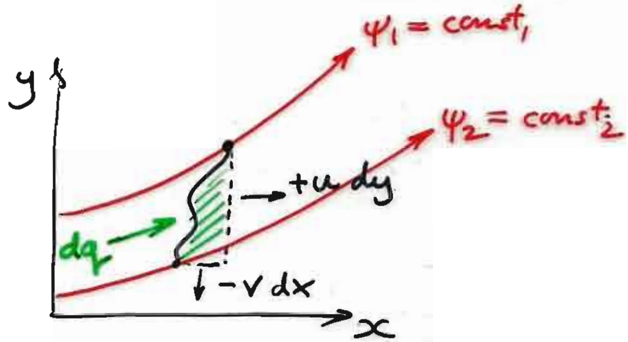


$$\frac{v}{u} = \frac{dy}{dx} \Rightarrow v\,dx - u\,dy = 0$$

Gives: $$-\frac{\partial \psi}{\partial x}dx - \frac{\partial \psi}{\partial y}dy = 0$$

$$\therefore \quad d\psi = 0 \quad \text{and} \quad \psi = \text{constant}$$

STREAM FUNCTIONS (CONT'D)



Note from figure that:

$$dq = u\,dy - v\,dx$$

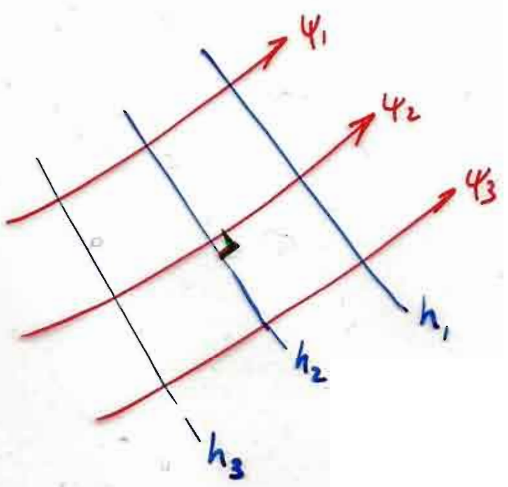
We know that: $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

∴ substituting gives $dq = d\psi$

$$q = \int_{\psi_2}^{\psi_1} d\psi = \psi_1 - \psi_2$$

i.e. q is constant between two streamlines, within a streamtube. !!

STREAMLINES AND EQUIPOTENTIALS



Streamlines and equipotentials are perpendicular.

Flow nets may be drawn.

Darcy's Law

The flow of fluids in porous and fractured media are governed by Darcy's Law that states flow rate, v, is directly proportional to the driving gradient of total head, h. This is described as,

$$v=k\frac{\partial h}{\partial x}$$

where x is the longitudinal direction of flow and k is the hydraulic conductivity of the porous medium. The parameter, v, is often referred to as the Darcy velocity. The mean discharge, Q, across a plane of area A, oriented perpendicular to the direction of flow ($x-axis$) is defined as,

$$Q=Ak\frac{\partial h}{\partial x}$$

or when the partial derivatives are written as finite derivatives,

$$Q=Ak\frac{\Delta h}{\Delta x}.$$

In this the total head, h, is the sum of elevation and pressure head, and velocity head is assumed negligible. The total head at any point is given by the elevation of that point above an arbitrary datum, plus the pressure head that is experienced at that point.

Flow Nets

Despite the increasing use of computer methods, graphical methods remain an important, rapid and robust method of computing pressure distributions and flow rates in nominally homogeneous bodies. Flow net methods apply to flow in two-dimensional sections under steady state conditions. The material may be porous or porous-fractured providing it may be represented by an equivalent isotropic ($k_x=k_y$) or anisotropic hydraulic conductivity($k_x\neq k_y$). The method requires that a net of orthogonal trajectories is drawn to cover the saturated flow domain representing, respectively, streamlines and equipotentials.

Streamlines trace the path of a individual particle of fluid (in an average sense) as it transits the system.

Equipotentials locate a locus of constant total head, h. It can further be demonstrated that the streamlines represent boundaries that no fluid may cross, and therefore the groundwater surface is a streamline. .PP Any orthogonal grid of streamlines and equipotentials that simultaneously satisfy the boundary conditions of the flow system and the requirements for orthogonality also satisfy the conditions for groundwater flow. A simple example is illustrated in the Figure 1 where a net of curvilinear quadrilaterals is drawn that satisfies the constant head boundary conditions of heads h_0 and h_9. The phreatic surface (groundwater surface) represents the topmost streamline dropping uniformly between equipotentials such that $\Delta_{23}=\Delta_{45}$, etc. The lowermost streamline corresponds to a prescribed no-flow boundary beneath the system. It may be noted that the equipotentials remain orthogonal to the upper (phreatic) and lower bounding streamlines, and also to all intermediate streamlines.

From Darcy's law, the unidirectional flow confined between streamlines (characterized by the inset of Figure 1) may be represented as

$$Q_{12}=dw\ k\frac{(h_1-h_2)}{dl}$$

where Q is the volumetric flow rate of a single streamtube. Noting the equidimensionality of the system as $dw=dl$ then it follows that

$$Q_{12}=k(h_1-h_2)$$

and further realizing that fluid cannot leave the streamtube, then Q_{12}=Q_{34} and the head drops between streamtubes must be uniform, as h_2-h_1=h_4-h_3, etc. Consequently, the total head along any equipotential may be evaluated. For equipotential h_4 the corresponding head is given as

$$h_4=(h_{in}-h_{out})\frac{5}{N_d}+h_{out}$$

where N_D is the number of potential drops in the system. This is 9 for this particular example. Total flow rate may also be determined by summing the contribution of each of the streamtubes. Due to the orthogonality of the flow net the flux contribution of each streamtube is identical. Consequently, for a total of N_S streamtubes, the total flow, Q_{total}, is given as

$$Q_{total}=\frac{N_S}{N_D}k(h_{in}-h_{out}).$$

This extremely simple technique is powerful and versatile, and gives surprisingly good estimates of pressure distributions and flow rates.

Flow Nets in Anisotropic Media

Where the system is described by different hydraulic conductivities in the x- and y- directions, the method may be extended. The following procedure must be adopted.

1 Redraw the flow section with the original (x,y) coordinates scaled to $(\bar{x},y)$ where $\bar{x}=x\sqrt{k_y/k_x}$.
2. Draw a flow net in the distorted geometry described in 1., above.
3. The head distribution is determined by applying the reverse distortion of 1. and 2. to return the geometry to its real form.
4. The flow rate may be evaluated from

$$Q_{total}=\frac{N_S}{N_D}\sqrt{k_x k_y}(h_{in}-h_{out})$$

In groundwater environments, where conductivity magnitudes and distributions are commonly poorly defined, or potentially indeterminate, approximate analysis by flow net sketching is of eminent use.

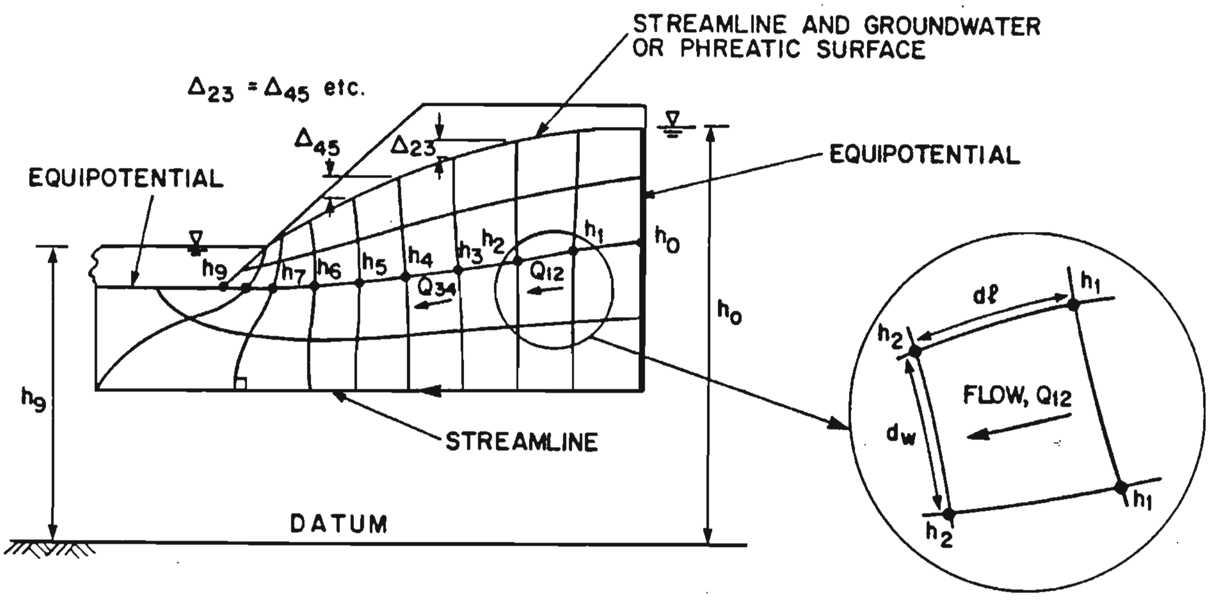


FIGURE 1.

EQUATIONS OF FLOW — INCLUDING MOMENTUM & VISCOUS LOSSES

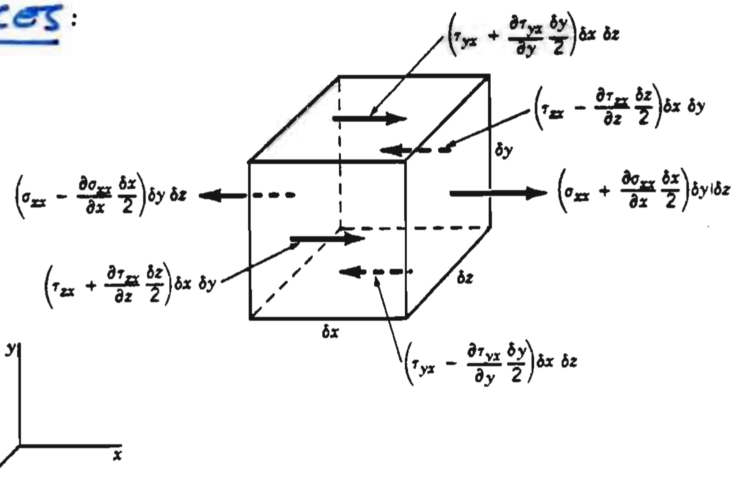
BODY FORCES:

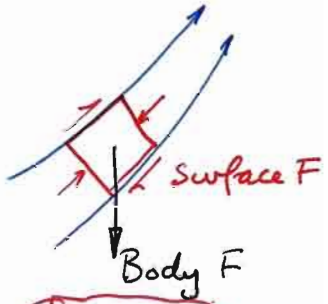
$$\delta \mathbf{F}_b = \delta m\, \mathbf{g} \qquad (6.46)$$

$$\begin{cases} \delta F_{bx} = \delta m\, g_x \\ \delta F_{by} = \delta m\, g_y \\ \delta F_{bz} = \delta m\, g_z \end{cases}$$

SURFACE FORCES:

$$\tau = \mu \frac{du}{dy}$$





$$+\ \underline{F} = m\underline{a}$$

EQUATION OF MOTION:

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\delta \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \qquad \textbf{(6.50a)}$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad \textbf{(6.50b)}$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\delta \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \qquad \textbf{(6.50c)}$$

Body $\underline{F}$ | Surface $\underline{F}$ | Inertial $\underline{F}$ $(\underline{F} = m\underline{a}) = \rho\underline{a}$

EULER'S EQUATIONS:

$\mu = 0 \therefore \tau = 0 \quad ; \quad -p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \qquad \textbf{(6.51a)}$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad \textbf{(6.51b)}$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \qquad \textbf{(6.51c)}$$

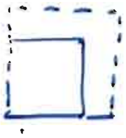
$$\boxed{\rho \mathbf{g} - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]} \qquad (6.52)$$

Integrate along streamline → Bernoulli. Eqn.

VISCOUS FLOW

If we desire to solve full equations (6.50) for $\mu \neq 0$ then we need a constitutive law, relating σ and μ and p.

STRESS-DEFORMATION RELATIONS:

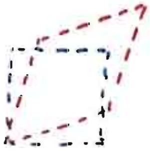


$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \quad (6.125a)$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \quad (6.125b)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \quad (6.125c)$$

$$-p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$



$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (6.125d)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad (6.125e)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (6.125f)$$

NAVIER-STOKES EQUATIONS

Substitute (6.125) into (6.50) and note that $\underline{\nabla} \cdot \underline{V} = 0$ (Continuity)

(x direction)

① $$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (6.127a)$$

(y direction)

② $$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (6.127b)$$

(z direction)

③ $$\rho\left(\frac{\partial w}{\partial t} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad (6.127c)$$

Convective accn. is nonlinear

INERTIAL TERMS
$\underline{F} = m\underline{a}$

BODY FORCE

VISCOUS LOSSES

FLUID PRESSURE LOSS

continuity

④ $$\underline{\nabla} \cdot \rho \underline{V} = 0$$

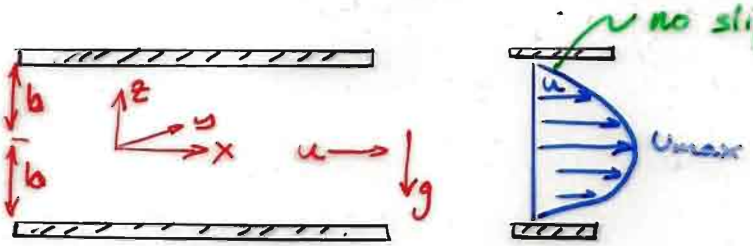
☐ 4 equations and 4 unknowns as (p, u, v, w)

∴ Well posed. but non-linear 2^{nd} order pde.

☐ Solve to determine h_L due to μ.

APPLICATION OF N-S EQUATIONS TO DETERMINE h_L

STEADY LAMINAR FLOW BETWEEN PARALLEL PLATES



no slip condition $u = 0$ @ $z = \pm b$

KNOWN CONDITIONS:

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

Plane flow: $v = w = 0$

Continuity: $\frac{du}{dx} = 0$

Steady $\frac{\partial(\,)}{\partial t} = 0$

NAVIER-STOKES EQUATIONS:

① $$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$$

② $$0 = -\frac{\partial p}{\partial y}$$

③ $$0 = -\frac{\partial p}{\partial z} - \rho g$$

Pressure variation as hydrostatic in a static fluid!!

Only equation ① remains:

$$\frac{d^2u}{dz^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Integrate (once): $$\frac{du}{dz} = \frac{1}{\mu}\left(\frac{dp}{dx}\right) z + c_1$$

Integrate (again): $$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) z^2 + \underline{c_1} z + \underline{c_2}$$

Apply boundary conditions: i.e. $u = 0$ for $z = \pm b$

$$\therefore \quad c_1 = 0$$

$$c_2 = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) b^2$$

Resubstitute for u, as:

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)(z^2 - b^2)$$ (parabolic $\underline{u}$)

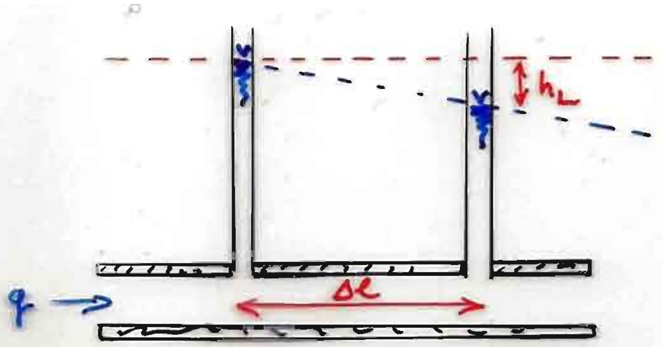
Flow rate, q given by:

$$q = \int_{-b}^{+b} u\, dz = \int_{-b}^{+b} \frac{1}{2\mu}\left(\frac{dp}{dx}\right)(z^2 - b^2)\, dz$$

$$\boxed{q = -\frac{(2b)^3}{12\mu}\left(\frac{dp}{dx}\right) \equiv -\underbrace{\frac{(2b)^3}{12\mu}\frac{1}{\Delta l}}_{\text{Viscous loss (Inverse/reciprocal loss)}} \Delta p}$$

Mean flow velocity, $\bar{u}$

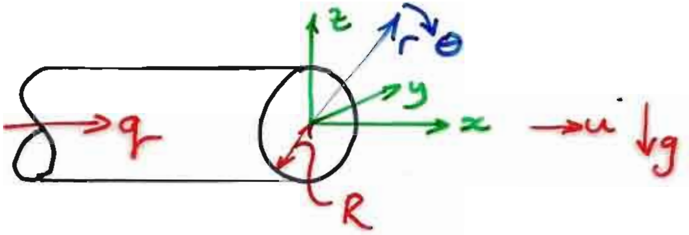
$$\boxed{\bar{u} = \frac{q}{A} = \frac{q}{2b} = -\frac{(2b)^2}{12\mu}\left(\frac{dp}{dx}\right)}$$

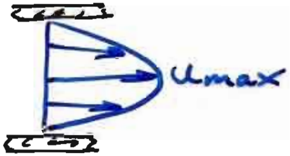


EL = HGL (for V small)
Bernoulli tells us this !!

Reality with viscous loss, h_L

STEADY LAMINAR FLOW IN CIRCULAR (SECTION) TUBES





Velocity distribution; u $\quad u = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$

Flow rate; q $\quad q = \frac{\pi(2R)^4}{128\mu}\left(\frac{dp}{dx}\right)$

Mean flow velocity; $\bar{u}$: $\quad \bar{u} = \frac{q}{A} = \frac{q}{\frac{1}{4}\pi(2R)^2} = \frac{(2R)^2}{32\mu}\left(\frac{dp}{dx}\right)$

This term will reappear as a friction factor for pipe flow as $f = \frac{64}{Re}$

[9]

Dimensional Analysis

Dimensional Analysis [9]

$$\mathbf{Re} = \frac{\rho V l}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{g l}}; \quad \mathbf{Eu} = \frac{p}{\rho V^2}$$

[9:1] Dimensional Analysis

Outline

Buckingham Pi $\mathbf{Re} = \frac{\rho V l}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{gl}}; \quad \mathbf{Eu} = \frac{p}{\rho V^2}$

SIMILITUDE, DIMENSIONAL ANALYSIS & (PHYSICAL) MODELING

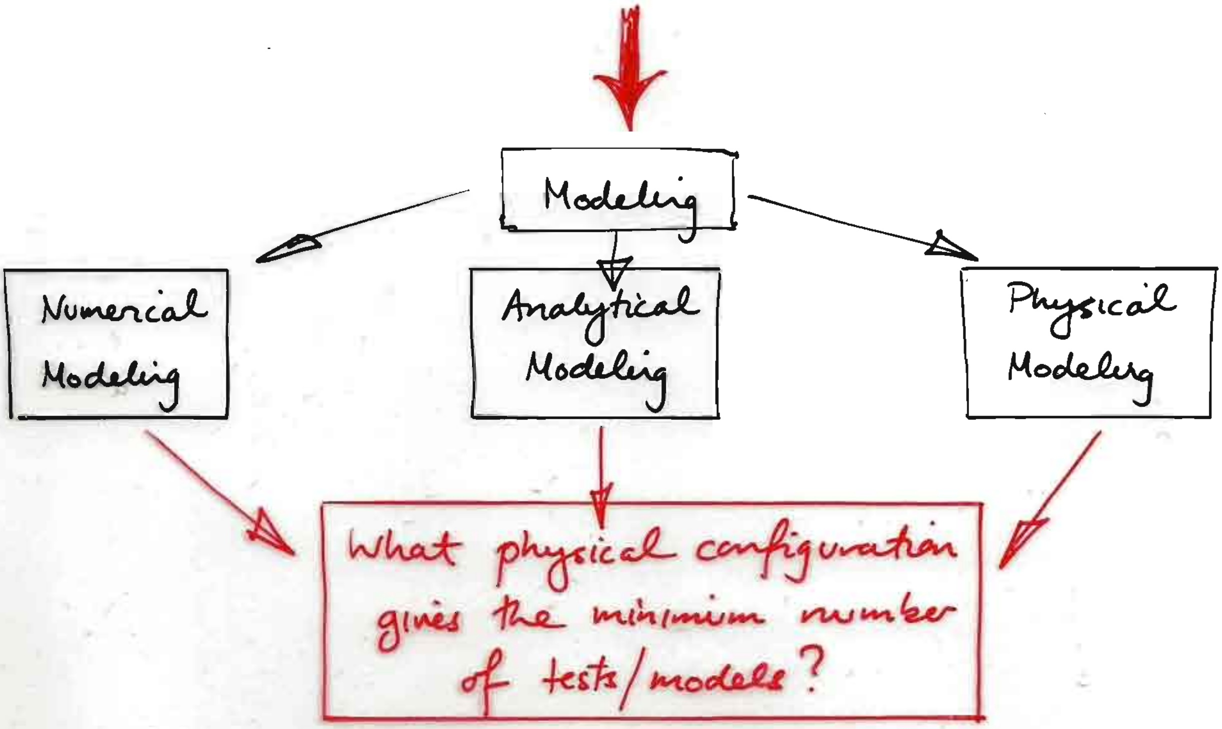
What solutions do we have so far?

Invicid flow: Euler equations → Bernoulli equation
- Free jets
- Weir → $V \propto \sqrt{2gH}$
- Sluice

Viscous flow: Navier - Stokes equations: → Simple geometry solns.
- Parallel plate flow
- Circular section pipe (Non circular?)

How to obtain other solutions for more complex problems?

eg.
- Circulation of water in North Sea
- Flow over an airfoil
- Flow over a weir


Modeling

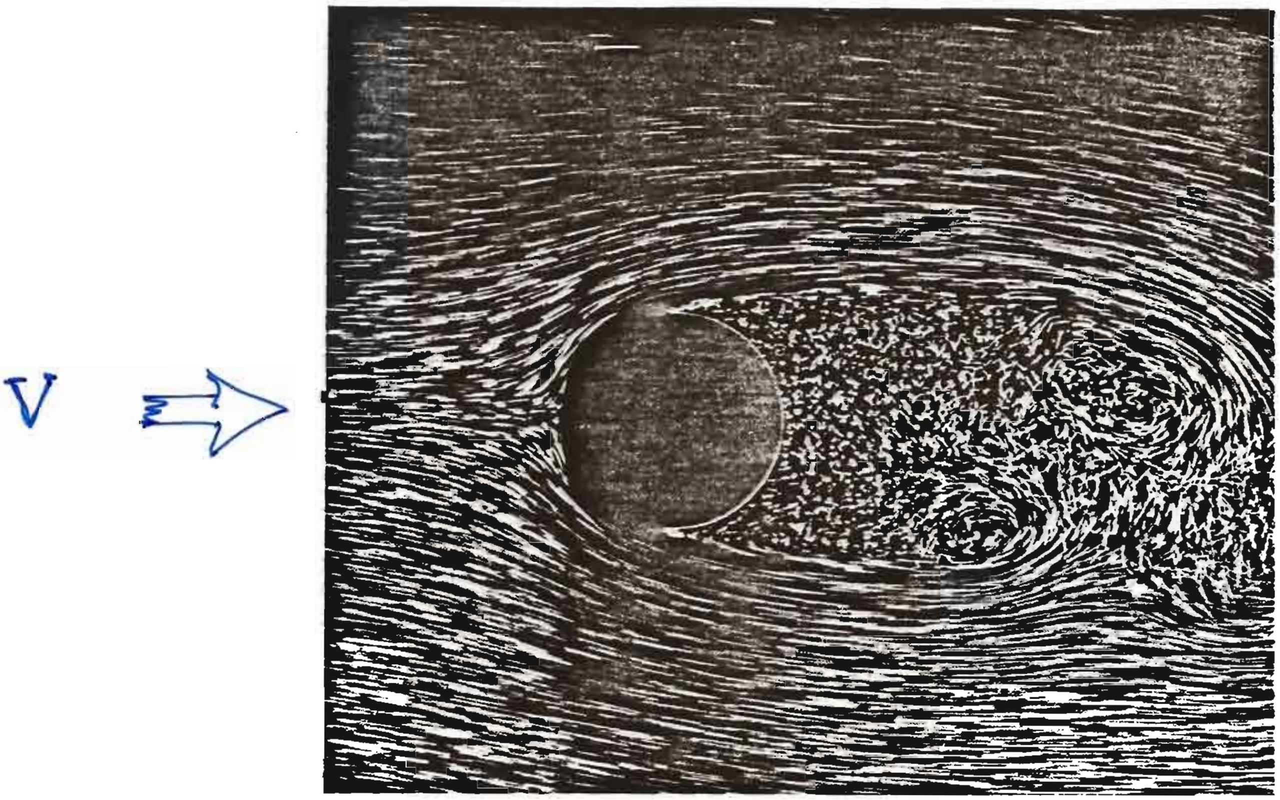
Numerical Modeling

Analytical Modeling

Physical Modeling

What physical configuration gives the minimum number of tests/models?

FLOW PAST A CYLINDER



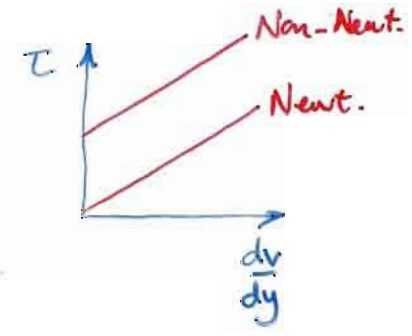
Reynolds No., $Re = \frac{\rho V l}{\mu} = \frac{V l}{\nu}$

Photo is for Re = 2000

Streaklines identical for all flows at Re = 2000

- Water
- Gasoline
- Molasses ?

Any Newtonian fluid



HOW TO DETERMINE THE CONTROLLING NONDIMENSIONAL GROUPS

1. Inspection – (guessing).

2. Fundamental equations (eg. N-S equns)
   (a) Solved for simple cases (eg Weir) (// plate duct)
   (b) Unsolved for exact case

3. Buckingham Pi Theorem
   – Formalism of (1.).

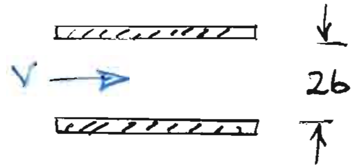
Define: How many parameters control the system – eg. μ, P, ρ, g ..

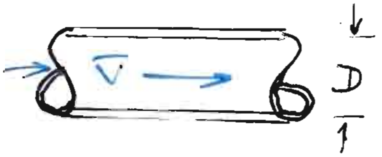
Determine: 1. How many "unique" nondimensional groups

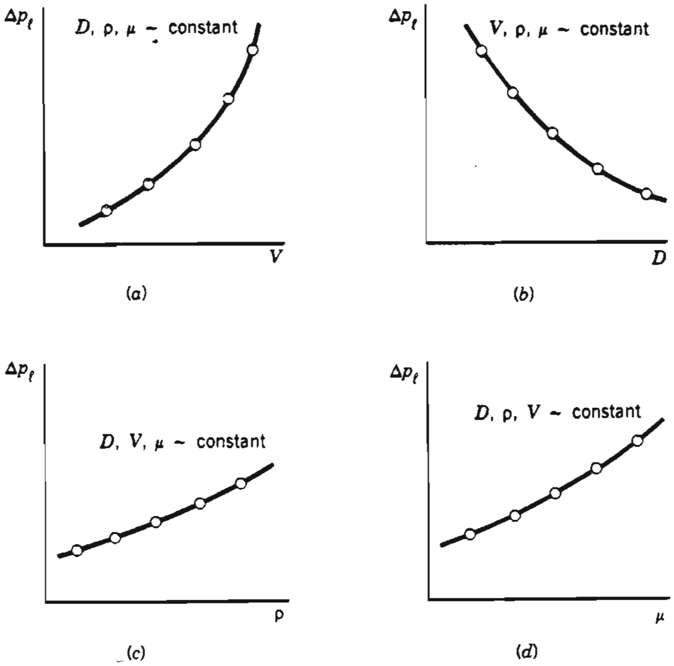
2. What are the groups?

FLOW IN PIPE (D) OR PARALLEL PLATE DUCT (2b)

5 variables Δp; V; D; μ; ρ







EXPERIMENTAL VARIABLES

$$V = f\left[D, \frac{\overset{\Delta p_L}{\Delta p}}{\Delta l}, \mu, g\right] \Rightarrow$$ 5 variables

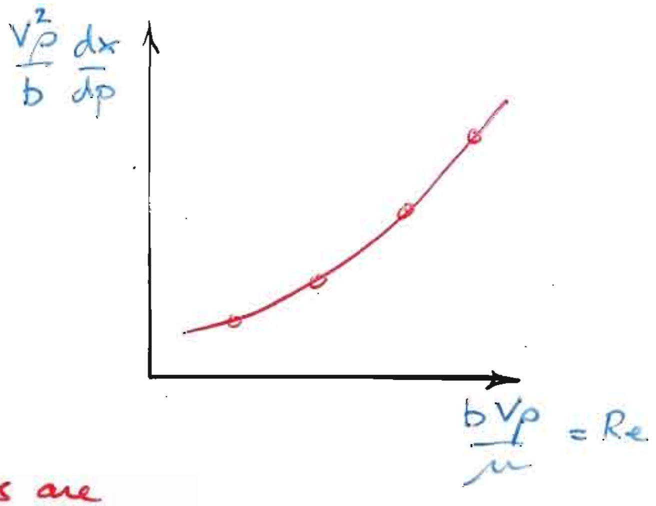
---

Do we need to run all these tests? No!!

Two "unique" groupings of variables as:

$$\frac{V^2\rho}{b}\cdot\frac{dx}{dp} = f\left[\underbrace{\frac{bV\rho}{\mu}}_{Re}\right]$$

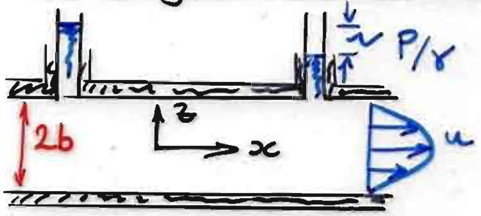
or D



Note: The two groups are both non-dimensional.

FLOW IN // SIDED DUCT

Governing (N-S) Equations



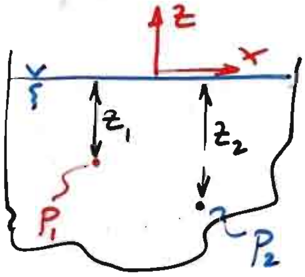
$$\frac{dp}{dx} = \mu \frac{d^2u}{dz^2} \Big\}$$ Along stream

Static

$$\left\{ \begin{array}{l} \dfrac{dp}{dz} = -\gamma \\[2ex] \dfrac{dp}{dy} = 0 \end{array} \right.$$

Behavior of a static fluid:

$$\frac{dp}{dz} = -\rho(g+a_z) \Rightarrow \frac{P}{z} \propto -\rho(g+a_z)$$



$$\frac{P}{z(g+a_z)\rho} = \underline{\text{Constant}}$$

Calibrate, say:

Measure p_1 @ z_1 for ρ, g

Determine Constant

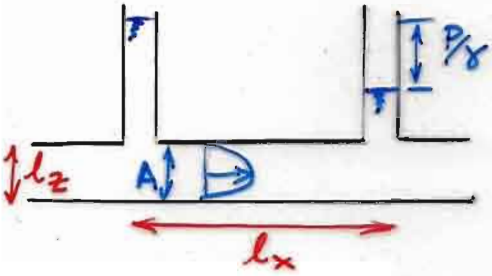
Use, say:

Evaluate p_2 @ z_2 for ρ, g, a_z

or ρ_2 ... etc.

FLOW IN X-DIRECTION

$$\frac{dp}{dx} = \mu \frac{d^2 u}{dz^2} = \mu \frac{d^2}{dz^2}(u)$$



Ignore the differential operator (units).

$$\frac{\Delta p}{\Delta x} \propto \mu \frac{\Delta u}{(\Delta z)^2}$$

Replace with characteristic lengths,

$\Delta x \rightarrow l_x$

$\Delta z \rightarrow l_z$

$\Delta u \rightarrow u$

$$u = c_1 \frac{(l_z)^2}{\mu} \frac{\Delta p}{\Delta x}$$

$$q = \int_{-b}^{+b} u \, dz = \int_{-b}^{+b} c_1 \frac{(l_z)^2}{\mu} \frac{\Delta p}{\Delta x} dz = \frac{c_1}{\mu} \frac{1}{3} (l_z)^3 c_2 \frac{\Delta p}{\Delta x}$$

$$q = \frac{c_3 (l_z)^3}{\mu} \frac{\Delta p}{\Delta x} \longrightarrow \boxed{q = \frac{-(2b)^3}{12\mu} \left(\frac{\Delta p}{\Delta l} \right)}$$

$$\bar{V} = \frac{q}{A} = \frac{c_4 \overset{2b}{(l_z)}^2}{\mu} \frac{\Delta p}{\Delta x} \longrightarrow \boxed{\bar{V} = \frac{-(2b)^2}{12\mu} \left(\frac{\Delta p}{\Delta l} \right)}$$

Actual Solns!!

↑

EXAMINING NON-DIMENSIONAL GROUPINGS (Cont'd)

$$\bar{V} = C_4 \frac{b^2}{\mu}\frac{dp}{dx} = \frac{L^2}{ML^{-1}T^{-1}} \quad \frac{ML^{-1}T^{-2}}{L} = \frac{L}{T} \quad \text{OK} \checkmark$$

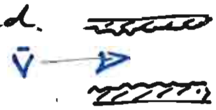
Note C_4 is a dimensionless constant. (v. imp).

Rearrange:

$$\frac{\bar{V}\mu}{b^2}\frac{dx}{dp} = C_4$$

This is ok for
- ☐ Laminar flow
- ☐ Perfectly smooth sided conduit

but not so if Turbulent or rough-sided.



Split into two-non-dimensional groups:

i.e. Multiply by: ρ, b, V

Divide by: μ

$$\frac{\rho \bar{V}^2}{b}\left(\frac{dx}{dp}\right) = f\Big[\underbrace{\frac{\rho b \bar{V}}{\mu}}_{Re}\Big]$$

Rearrange:

$$\underbrace{\frac{b}{\rho V^2}}_{\text{Friction factor}}\left(\frac{dp}{dx}\right) = f\Big[\underbrace{\frac{\rho b V}{\mu}}_{Re}\Big]$$

1. Applies to pipe flow $b \rightarrow D$ (Characteristic ℓ)
2. Applies for various ε/b ; $\frac{\varepsilon}{D}$ & Turbulent.

[9:2] Dimensional Analysis

Outline

Buckingham Pi $\mathbf{Re} = \frac{\rho V l}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{gl}}; \quad \mathbf{Eu} = \frac{p}{\rho V^2}$

BUCKINGHAM PI THEOREM - Formalism for selecting groupings.

" If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k-r dimensionless products (groups), where r is the minimum number of reference dimensions used to describe the variables."

Pipe flow: Variables: $[V; \rho; \mu; b; \Delta p_L]$ $k = 5$

Ref. dimensions: M L T $r = 3$

$k - r \rightarrow 2$

Re and friction f.

Solution Steps:

1. List all variables - i.e. Non dimensional & dimensional.

eg. geometry - b or D
fluid properties μ, σ - - - ρ, γ
driving forces $g, \Delta p_L$

Be careful not to oversupply eg. $\gamma = \rho g$ $\longrightarrow k$

2. Express each variable in terms of its dimensions

eg. $\rho \doteq ML^{-3}$ etc $\longrightarrow r$

3. Determine the number of Π terms $\Pi = k - r$

4. Select the number of repeating variables.

Remove from the list of variables some that may be combined to give a Π (dimensionless) term. Must include all ref. dimensions (MLT) in each group.

Do <u>not</u> choose the <u>dependent</u> variable as one of the repeating variables.

(we want to isolate behavior of dependent variable)

5. Form a Π term by multiplying the nonrepeating variables by the product of the repeating variables.

6. Repeat <u>step 5</u> for all the non-repeating variables.

7. Check the resulting Π terms to ensure non-dimensionality.

8. Express final relationship as

$$\Pi_1 = \phi\,[\Pi_2 , \Pi_3 \; \ldots \; , \Pi_{k-r}]$$

$\nearrow$

Contains dependent variable in numerator

⇩

Run experiment to determine the form of ϕ relating the non-dimensional terms.

PIPE FLOW

Dependent variable

Step 1: $$\Delta p_e = f[D ; \rho ; \mu ; V]$$ $k = 5$

Step 2:

$\Delta p_e \doteq ML^{-1}T^{-2} \; (L^{-1}) \leftarrow$ * Note Δp_e is $\frac{dp}{dx}$ not p.

$D \doteq L$

$\rho \doteq ML^{-3}$

$\mu \doteq ML^{-1}T^{-1}$

$V \doteq LT^{-1}$

$r = 3$ [ie $M L \,\&\, T$]

Step 3: No. of π terms $\equiv k - r = 2$

Step 4: Select repeating varables fm $[D, \rho, \mu, V]$

$r = 3$ ∴ need three repeating variables

– pick the dimensionally simplest:

$$\boxed{D ; V ; \rho}$$

L $\quad LT^{-1}$ $\quad ML^{-3}$

Check that they are dimensionally independent – i.e.

none of D, V and ρ has same dimensions (units).

Step 5 → over.

Step 5: Form The Π terms.

Variable 1:

$$\Pi_1 = \overbrace{\Delta p_\ell}^{\text{dependent variable}}\ \overbrace{D^a V^b \rho^c}^{\text{repeating variables}}$$

*note Δp_ℓ is $\frac{dp}{dx}$ not p

To be dimensionless: $$\underbrace{(ML^{-2}T^{-2})}_{\Delta p_\ell}(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

Determine exponents:

$$\begin{array}{rcrcrcrcll} 1 & & & & & + & c & = 0 & (\text{for } M) \\ -2 & + & a & + & b & - & 3c & = 0 & (\text{for } L) \\ -2 & & & - & b & & & = 0 & (\text{for } T) \end{array}$$

Solve system for a, b, c. →

$$c = -1$$
$$b = -2$$
$$a = 1$$

Resubstitute into Π_1 as: $$\Pi_1 = \frac{\Delta p_\ell D}{V^2 \rho}$$

$\boxed{\text{Step 6:}}$

Variable 2: Add remaining variable (μ) to Π_1 term

$$\Pi_2 = \mu\ D^a V^b \rho^c$$

To be dimensionless: $$(ML^{-1}T^{-1})\ (L)^a\ (LT^{-1})^b\ (ML^{-3})^c = M^0L^0T^0$$

$$\begin{array}{rcrcrcrcll} 1 & & & & & + & c & = 0 & (\text{for } M) \\ -1 & + & a & + & b & - & 3c & = 0 & (\text{for } L) \\ -1 & & & - & b & & & = 0 & (\text{for } T) \end{array}$$

$c = -1$; $b = -1$; $a = -1$ $\therefore$ $$\Pi_2 = \frac{\mu}{DV\rho}$$

Step 6 ....

If there were any more variables, say σ (surface tension) then the next Π group is:

$$\Pi_3 = \sigma \; D^a V^b \rho^c$$

etc .....

Step 7: Check non-dimensionality of Π terms

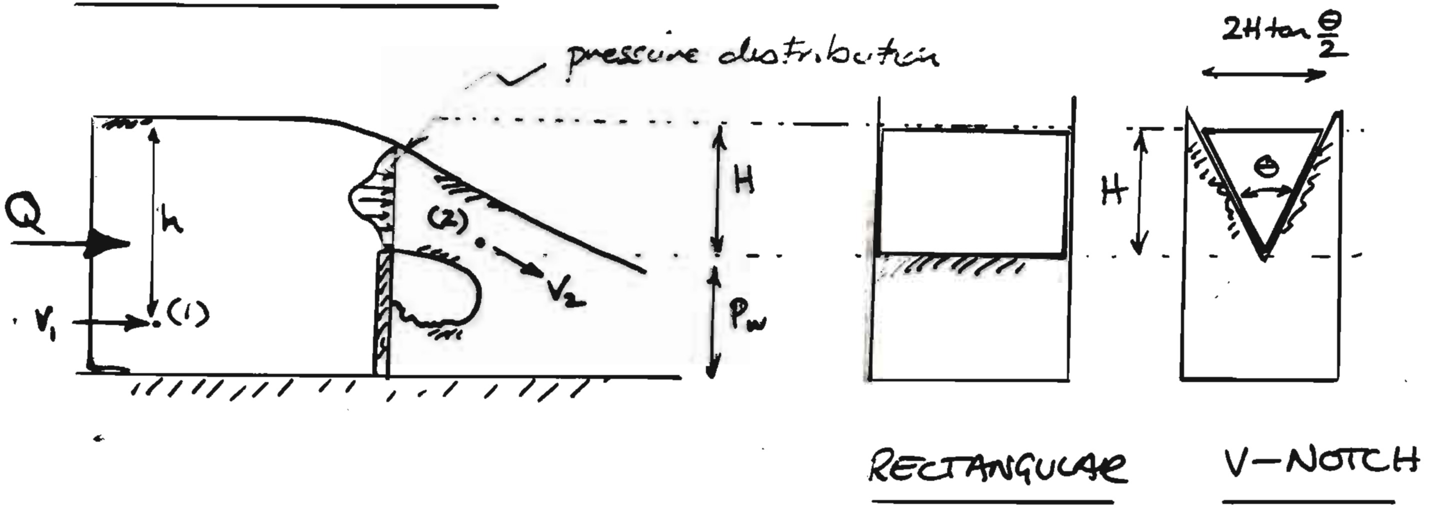
Step 8: Final relationship:

$$\frac{\Delta p_L D}{V^2 \rho} = \phi \left[\underbrace{\frac{\rho V D}{\mu}}_{Re} \right]$$

Note, since the form of ϕ is not defined, we can use Re or $\frac{1}{Re}$

since nondimensional!!

SHARP CRESTED WEIR



ANALOG TO FREE JET

$$V \propto \sqrt{2gh}$$

$$V = C_1\sqrt{2gH}$$

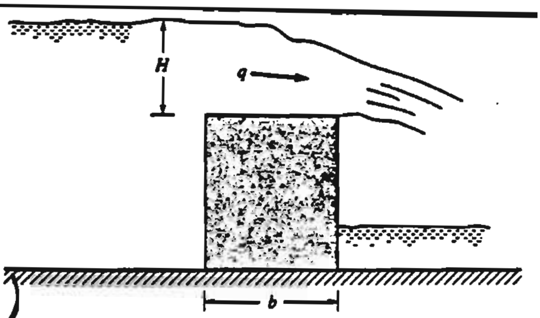
$$Q \propto V b H = C_1 b H \sqrt{2gH}$$

$$Q = C_1 b \sqrt{2g}\, H^{3/2}$$

Determine C_1 experimentally. $C_1 = f\left(H_w/p_w ;\ b\,?\ldots\right)$

7.10

7.10 Water flows over a dam as illustrated in Fig. P7.10. Assume the flowrate, q, per unit length along the dam depends on the head, H, width, b, acceleration of gravity, g, fluid density, ρ, and fluid viscosity, μ. Develop a suitable set of dimensionless parameters for this problem using b, g, and ρ as repeating variables.



■ FIGURE P7.10

$$\underline{q} = f(H, \underbrace{b, g, \rho}_{\text{Repeat}}, \mu)$$

$q \doteq L^2T^{-1}$ $H \doteq L$ $b \doteq L$ $g \doteq LT^{-2}$ $\rho \doteq FL^{-4}T^2$ $\mu \doteq FL^{-2}T$

From the pi theorem $6 - 3 = 3$ pi terms required. Use b, g, and ρ as repeating variables. Thus,

$$\Pi_1 = q\, b^a g^b \rho^c$$

and

$$(L^2T^{-1})(L)^a(LT^{-2})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

so that

$$c = 0 \qquad (\text{for } F)$$
$$2 + a + b - 4c = 0 \qquad (\text{for } L)$$
$$-1 - 2b + 2c = 0 \qquad (\text{for } T)$$

It follows that $a = -3/2$, $b = -1/2$, $c = 0$, and therefore

$$\Pi_1 = \frac{q}{b^{3/2} g^{1/2}}$$

Check dimensions using MLT system:

$$\frac{q}{b^{3/2} g^{1/2}} \doteq \frac{L^2T^{-1}}{L^{3/2}(LT^{-2})^{1/2}} \doteq M^0L^0T^0 \quad \therefore \text{ok}$$

For Π_2:

$$\Pi_2 = H\, b^a g^b \rho^c$$

$$(L)(L)^a(LT^{-2})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$$c = 0 \qquad (\text{for } F)$$
$$1 + a + b - 4c = 0 \qquad (\text{for } L)$$
$$-2b + 2c = 0 \qquad (\text{for } T)$$

It follows that $a = -1$, $b = 0$, $c = 0$, and therefore

$$\Pi_2 = \frac{H}{b}$$

which is obviously dimensionless.

(con't)

7.10 (con't)

For Π_3:

$$\Pi_3 = \mu\, b^a g^b \rho^c$$

$$(FL^{-2}T)(L)^a(LT^{-2})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$1 + c = 0$ (for F)

$-2 + a + b - 4c = 0$ (for L)

$1 - 2b + 2c = 0$ (for T)

It follows that $a = -3/2$, $b = -1/2$, $c = -1$, and therefore

$$\Pi_3 = \frac{\mu}{b^{3/2} g^{1/2} \rho}$$

Check dimensions using MLT system:

$$\frac{\mu}{b^{3/2} g^{1/2} \rho} \doteq \frac{(ML^{-1}T^{-1})}{(L)^{3/2}(LT^{-2})^{1/2}(ML^{-3})} \doteq M^0L^0T^0 \quad \therefore \text{OK}$$

Thus,

$$\underline{\underline{\boxed{\frac{q}{b^{3/2}\sqrt{g}}} = \phi\left(\frac{H}{b}, \frac{\mu}{b^{3/2}\sqrt{g}\,\rho}\right)}}$$

Froude number — ratio of $\frac{\text{Inertia forces}}{\text{Gravity forces}}$

controls behaviour of free surface flows

v. important non-dimensional number.

RELEVANCE OF TERMS

$$\frac{q}{b^{3/2}\sqrt{g}} \quad \left\{ \begin{array}{l} q \doteq VH \\ b^{3/2} = \dfrac{H^{3/2}\cdot b^{3/2}}{H^{3/2}} = H^{3/2}\left(\dfrac{b}{H}\right)^{3/2} \end{array} \right.$$

Substituting:

$$\underbrace{\frac{VH}{H^{3/2}\sqrt{g}}}\ \underbrace{\left(\frac{H}{b}\right)^{3/2}}$$

Both terms are dimensionless.

① $\dfrac{V}{\sqrt{gH}} \doteq$ Froude No.

Ratio of $\dfrac{\text{inertia force}}{\text{gravitational force}}$.

New groups: $$\frac{V}{\sqrt{gH}}\underbrace{\left(\frac{H}{b}\right)^{3/2}} = \phi\left[\underbrace{\frac{H}{b}}\ ;\ \frac{\mu}{b^{3/2}\sqrt{g}\,\rho}\right]$$

Third dimensionless group

$$\frac{\mu}{b^{3/2}\sqrt{g}\,\rho} \Rightarrow \underbrace{\frac{\mu}{H^{3/2}\sqrt{g}\,\rho}}\ \underbrace{\left(\frac{H}{b}\right)^{3/2}}$$

②

Both dimensionless

Dividing ① by ②

$$\frac{V}{\sqrt{g\cancel{H}}} \times \frac{H^{\cancel{3/2}}\cancel{\sqrt{g}}\,\rho}{\mu} = \frac{VH\rho}{\mu}$$

= Reynolds' No., Re.

$$\underset{Fr}{\frac{V}{\sqrt{gH}}} = \phi\left[\frac{H}{b}\ ;\ \underset{Re}{\frac{VH\rho}{\mu}}\right]$$

∴ Π groups useful but not unique and may mask important variables.

7.32

7.32 The flowrate, Q, in an open canal or channel can be measured by placing a plate with a V-notch across the channel as illustrated in Fig. P7.32. This type of device is called a V-notch *weir*. The height, H, of the liquid above the crest can be used to determine Q. Assume that

$$Q = f(H, g, \theta)$$

where g is the acceleration of gravity. What are the significant dimensionless parameters for this problem?

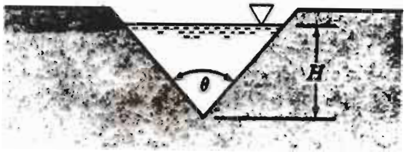


FIGURE P7.32

$Q \doteq L^3T^{-1}$ $\quad H \doteq L \quad g \doteq LT^{-2} \quad \Theta = F^0L^0T^0$ $\left\{ \right.$ Only L & T represented.

From the pi theorem, 4-2 = 2 pi terms required.

By inspection, for Π_1 (containing Q):

$$\Pi_1 = \frac{Q}{g^{1/2}\,H^{5/2}} \doteq \frac{L^3T^{-1}}{(LT^{-2})^{1/2}\,(L)^{5/2}} \doteq L^0T^0$$

Since the angle, θ, is dimensionless

$$\Pi_2 = \Theta$$

So that

$$\underline{\underline{\frac{Q}{\sqrt{gH^5}} = \phi(\Theta)}}$$

For rectangular weir (or channel) $Q \propto \sqrt{gh^3}\ b$

For Δ channel, width, b, is proportional to H.

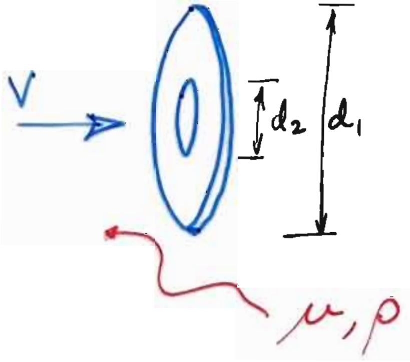
$$\therefore \quad Q \propto \sqrt{gH^3}\cdot H$$

7.12

7.12 The drag, $\mathcal{D}$, on a washer shaped plate placed normal to a stream of fluid can be expressed as

$$\mathcal{D} = f(\underline{d_1}, d_2, \underline{V}, \mu, \underline{\rho})$$

where d_1 is the outer diameter, d_2 the inner diameter, V the fluid velocity, μ the fluid viscosity, and ρ the fluid density. Some experiments are to be performed in a wind tunnel to determine the drag. What dimensionless parameters would you use to organize these data?



$\mathcal{D} \doteq F$ $\quad d_1 \doteq L$ $\quad d_2 \doteq L$ $\quad V \doteq LT^{-1}$ $\quad \mu \doteq FL^{-2}T$ $\quad \rho \doteq FL^{-4}T^2$

From the pi theorem, 6−3=3 pi terms required. Use d_1, V, and ρ as repeating variables. Thus,

$$\Pi_1 = \mathcal{D}\, \underbrace{d_1^a V^b \rho^c}$$

and

$$(F)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c = F^0L^0T^0$$

so that

$$1 + c = 0 \qquad \text{(for } F)$$
$$a + b - 4c = 0 \qquad \text{(for } L)$$
$$-b + 2c = 0 \qquad \text{(for } T)$$

It follows that $a = -2$, $b = -2$, $c = -1$, and therefore

$$\Pi_1 = \frac{\mathcal{D}}{d_1^2 V^2 \rho}$$

Check dimensions using MLT system:

$$\boxed{\Pi_1 = \frac{\mathcal{D}}{d_1^2 V^2 \rho}} \doteq \frac{MLT^{-2}}{(L)^2(LT^{-1})^2(ML^{-3})} \doteq M^0L^0T^0 \qquad \therefore \text{OK}$$

For Π_2:

$$\Pi_2 = d_2\, \underbrace{d_1^a V^b \rho^c}$$

$$(L)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c = F^0L^0T^0$$

$$c = 0 \qquad \text{(for } F)$$
$$1 + a + b - 4c = 0 \qquad \text{(for } L)$$
$$-b + 2c = 0 \qquad \text{(for } T)$$

(con't)

7.12 (con't)

It follows that $a=-1$, $b=0$, $c=0$, and therefore

$$\boxed{\Pi_2 = \frac{d_2}{d_1}}$$

which is obviously dimensionless.

For Π_3:

$$\Pi_3 = \mu \underbrace{d_1^{\,a} V^b \rho^c}$$

$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$$1+c=0 \qquad \text{(for } F\text{)}$$

$$-2+a+b-4c=0 \qquad \text{(for } L\text{)}$$

$$1-b+2c=0 \qquad \text{(for } T\text{)}$$

It follows that $a=-1$, $b=-1$, $c=-1$, and therefore

$$\boxed{\Pi_3 = \frac{\mu}{d_1 V \rho}}$$

Check dimensions using MLT system:

$$\frac{\mu}{d_1 V\rho} \doteq \frac{ML^{-1}T^{-1}}{(L)(LT^{-1})(ML^{-3})} \doteq M^0L^0T^0 \quad \therefore \text{OK}$$

Thus,

$$\frac{\mathcal{D}}{d_1^{\,2}V^2\rho} = \phi\left(\frac{d_2}{d_1}, \frac{\mu}{d_1 V\rho}\right) \qquad (1)$$

Since $\frac{\rho V d_1}{\mu}$ is a standard dimensionless parameter (Reynolds number), Eq. (1) would more commonly be expressed as

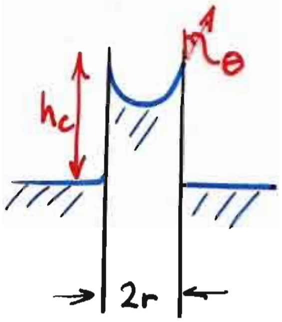
Equivalent "pressure" $\rightarrow$

$$\underline{\underline{\frac{\mathcal{D}}{d_1^{\,2}V^2\rho} = \phi\left(\overset{\text{Geometry}}{\frac{d_2}{d_1}}, \overset{\text{Re.}}{\frac{\rho V d_1}{\mu}}\right)}} \qquad (2)$$

As far as dimensional analysis is concerned, Eqs. (1) and (2) are equivalent.

Euler No., $Eu = \dfrac{P}{\rho V^2}$

BUCKINGHAM Π THEOREM — SOMETIMES INDETERMINATE



$$2\pi r\,\sigma\cos\theta = \pi r^2 h_c\,\gamma$$

or:

$$1 = \frac{\pi r^2 h_c \gamma}{2\pi r\,\sigma\cos\theta} = \frac{r h_c \gamma}{2\sigma\cos\theta} \qquad (1)$$

Non dimensional

<u>Using Π theorem:</u>

$$\begin{array}{ccccc} h_c & \gamma & r & \sigma & \theta \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ L & ML^{-2}T^{-2} & L & MT^{-2} & \text{none} \end{array}$$

$$h_c \quad \gamma^a \quad r^b \quad \sigma^c$$

$$\begin{array}{rrrrcl} & a & & +c & = 0 & (M) \\ 1 & -2a & +b & & = 0 & (L) \\ & -2a & & -2c & = 0 & (T) \end{array}$$

Note: Indeterminate since equations for (M) and (T) are not independent.

Note: From equation (1) that $\left\{\begin{array}{l} a = 1 \\ b = 1 \\ c = -1 \end{array}\right\}$ and these satisfy the 3 Buckingham Π equations.

Note: θ is dimensionless so keep as separate variable
Of the remainder there are $k-r$ groups, or $4-3 = 1$ group

$$(r h_c \gamma / 2\sigma) = \phi(\theta).$$

RESOLVE Π INDETERMINACY BY REORDERING VARIABLES

Change to:

$$\begin{array}{cccc} \gamma & h_c & r & \sigma \\ \downarrow & \downarrow & \downarrow & \downarrow \\ (ML^{-2}T^{-2}) & L & L & (MT^{-2}) \end{array}$$

$$\gamma \quad h_c^{a} \quad r^{b} \quad \sigma^{c}$$

$$\begin{array}{lllll} 1 & & + c & = 0 & (M) \\ -2 & + a + b & & = 0 & (L) \\ -2 & & - 2c & = 0 & (T) \end{array}$$

And solution is $c = -1$

but a and b remain indeterminate

Potential solutions: $a = 1$; $b = 1$ $\Rightarrow$ $\dfrac{\gamma h_c r}{\sigma} = \Pi_1$

or $a = 3$; $b = -1$ $\Rightarrow$ $\dfrac{\gamma h_c^3}{r\sigma} \equiv \dfrac{\gamma h_c^2}{\sigma} \cdot \dfrac{h_c}{r}$

∴ Non unique!!

UNIQUENESS OF Π TERMS

Variables chosen as primary and dependent control the resulting form of the Π terms.
Therefore the Π terms are not unique.

Pipeflow:

$$\Delta pe = f[\underbrace{D; V; \rho}_{\text{Repeating variables}}; \mu]$$

$$\frac{\Delta pe\, D}{V^2 \rho} = \phi_1\left[\frac{\rho V D}{\mu}\right]$$

Not identical
∴ non-unique

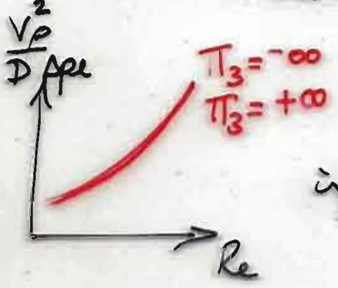
$$\Delta pe = f[\underbrace{D; V; \mu}_{\text{Repeating variables}}; \rho]$$

$$\frac{\Delta pe\, D^2}{V \mu} = \phi_2\left[\frac{\rho V D}{\mu}\right]$$

Reynold's no. is identical since D is the only available characteristic dimension.

- Number of Π terms is however fixed as $k - r$.

- Could choose extra variable, eg. σ (surface tension) that would result in Π_3, a 3rd term. For this example, since behaviour is not influenced by σ, this term, Π_3, would have no influence.



[9:3] Dimensional Analysis

Recap

Buckingham Pi & $\mathbf{Re} = \frac{\rho V l}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{gl}}; \quad \mathbf{Eu} = \frac{p}{\rho V^2}$

Outline

Relevance of dimensionless terms

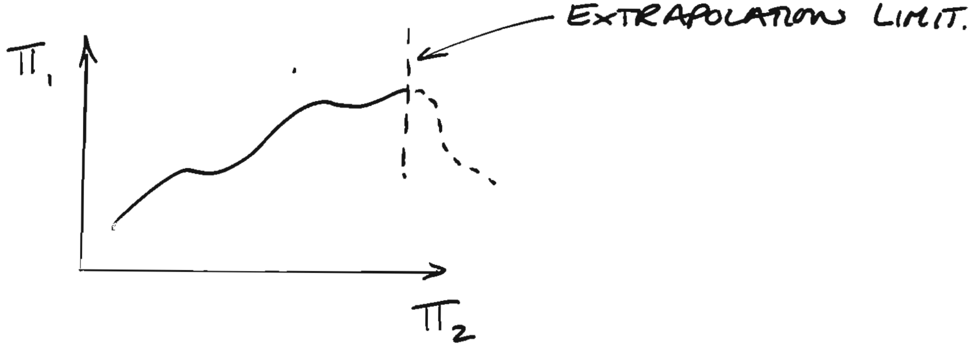
Use of models

Similitude

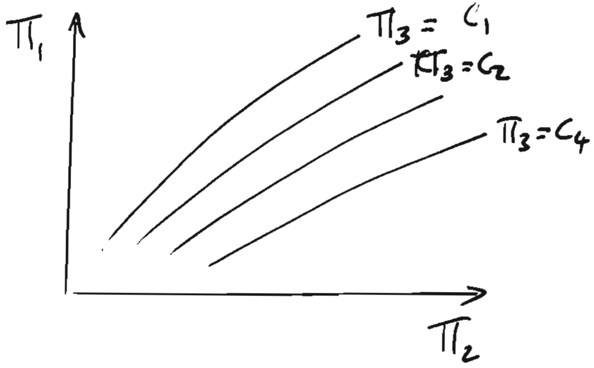
- Geometric
- Kinematic
- Dynamic

RELEVANCE OF THE NUMBER OF Π TERMS

TWO TERMS:



THREE TERMS:



Four terms needs more imaginative representation.

DEPARTMENT OF MINERAL ENGINEERING
CE 261 FLUID MECHANICS

Exam 3 - November 11th 1994
Exam duration 50 minutes. Open book & open notes.
Answer all questions. Underline or box all answers.
Assume: $\gamma_w = 62.4\ pcf\ = 9.8\ kN/m^3$ *and* $g = 9.8\ m/s^2$.
$Energy:\ 1J = 1N.m\ \ Force:\ 1N = 1\ Kg.m/s^2\ \ Power: 1W = 1\ J/s$
$ax^2 + bx + c = 0;\ \ \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

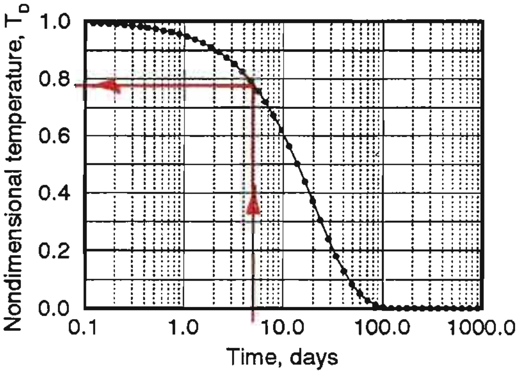
Name: MASTER COPY.

SSN: ______________________

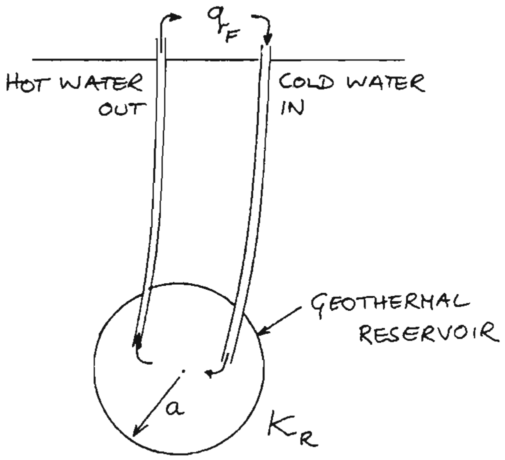
Question	Points	Score
1	30	
2	30	
3	30	
4	10	
Total	100	

Question 1 (30 points)

Thermal recovery from a geothermal reservoir may be described as an analog to a geological heat exchanger where fluid (water) is circulated at a volumetric flow rate, q_F $[L^3T^{-1}]$, through a reservoir of nominal radius, a $[L]$. Thermal recovery from the system is indexed by the four dimensional parameters, q_F, a, thermal diffusivity of the rock, κ_R $[L^2T^{-1}]$, and time, t, and the single nondimensional parameter of nondimensional temperature, T_D.



1. Determine the number of dimensionless group(s) that completely define the system.
2. Using the repeating variables κ_R and a, determine the dimensionless group(s) that define the system.
3. A field test of a of $20m$ radius reservoir produced at $0.1m^3/s$ yields the thermal drawdown shown. Determine the nondimensional temperature, T_D, of a larger $40m$ radius reservoir at the same site after 20 days if it is produced at double the volumetric flowrate.



1. 4 dimensional parameters; q_F, a, K_R, t

2 dimensions; $\underline{L, T}$

2 nondimensional groups plus T_D. $= 3$

---

2. $$\begin{array}{cccc} q_F & t & K_R & a \\ L^3T^{-1} & T & \underbrace{L^2T^{-1} \quad L}_{\text{Repeating}} & \end{array}$$

Π_1

$$q_F \quad K_R \quad a$$

$$(L^3T^{-1})(L^2T^{-1})^a(L)^b = L^0T^0$$

$$3 + 2a + b = 0 \text{ (for } L)$$

$$-1 - a = 0 \text{ (for } T)$$

$$a = -1 \qquad b = -1$$

$$\boxed{\Pi_1 = \frac{q_F}{K_R a}}$$

Π_2

$$t \quad K_R \quad a$$

$$(T)(L^2T^{-1})^a(L)^b = L^0T^0$$

$$2a + b = 0 \text{ (for } L)$$

$$1 - a = 0 \text{ (for } T)$$

$$a = 1 \qquad b = -2$$

$$\boxed{\Pi_2 = \frac{tK_R}{a^2}}$$

Groups $$T_D = \phi\left[\frac{q_F}{K_R a} ; \frac{tK_R}{a^2}\right]$$

---

3. Reservior I $$\left(\frac{q_F}{K_R a}\right)_I = \left(\frac{0.1}{K_R 20}\right)$$

Reservoir II $$\left(\frac{q_F}{K_R a}\right)_{II} = \left(\frac{0.2}{K_R 40}\right)$$

} equivalent. ✓ OK.

System I : $t_I = ?$; K_R ; $a_I = 20\,m$

System II : $t_{II} = 20d$; K_R ; $a_{II} = 40m$

$$\frac{t_I K_R}{a_I^2} = \frac{t_{II} K_R}{a_{II}^2} \Rightarrow t_I = \frac{K_R}{K_R}\left(\frac{a_I}{a_{II}}\right)^2 t_{II} = \left(\frac{20}{40}\right)^2 20d = 5 \text{ days.}$$

$\boxed{@\ 5 \text{ days};\ T_D = 0.78}$ from Figure

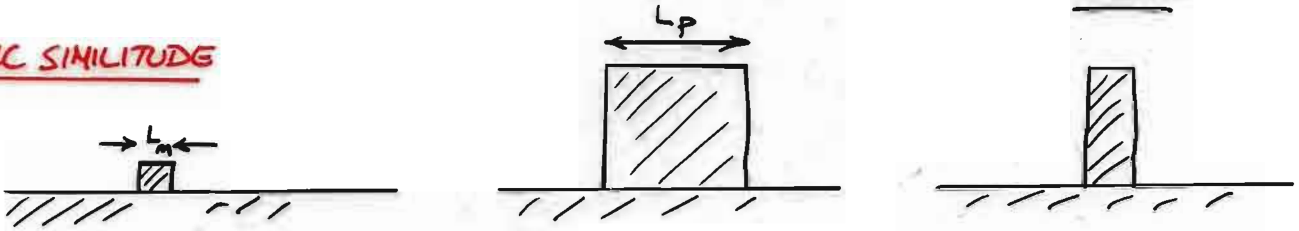
equivalent to 20 days for 40 m reservoir.

USE OF MODELS

Require to match similitude between model and prototype:

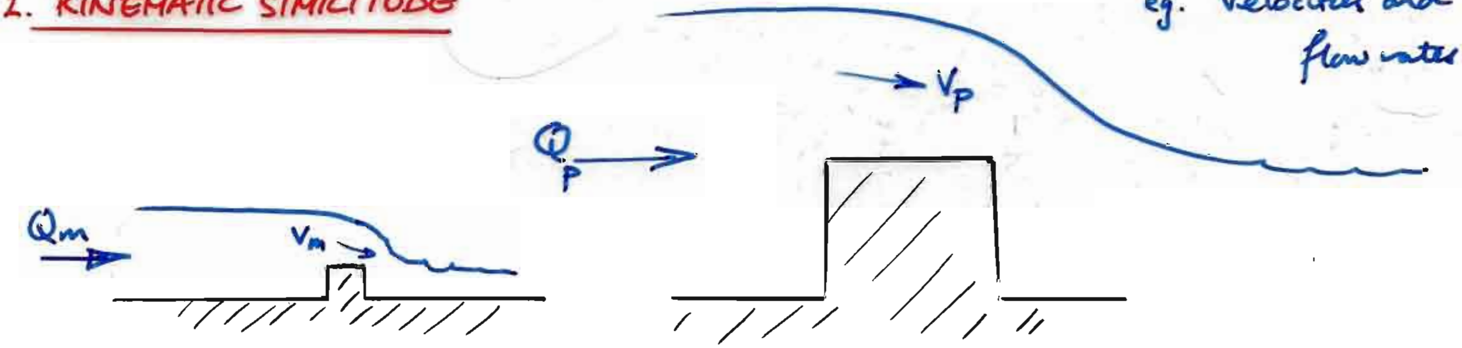
Model | Prototype | Distorted Model

1. GEOMETRIC SIMILITUDE



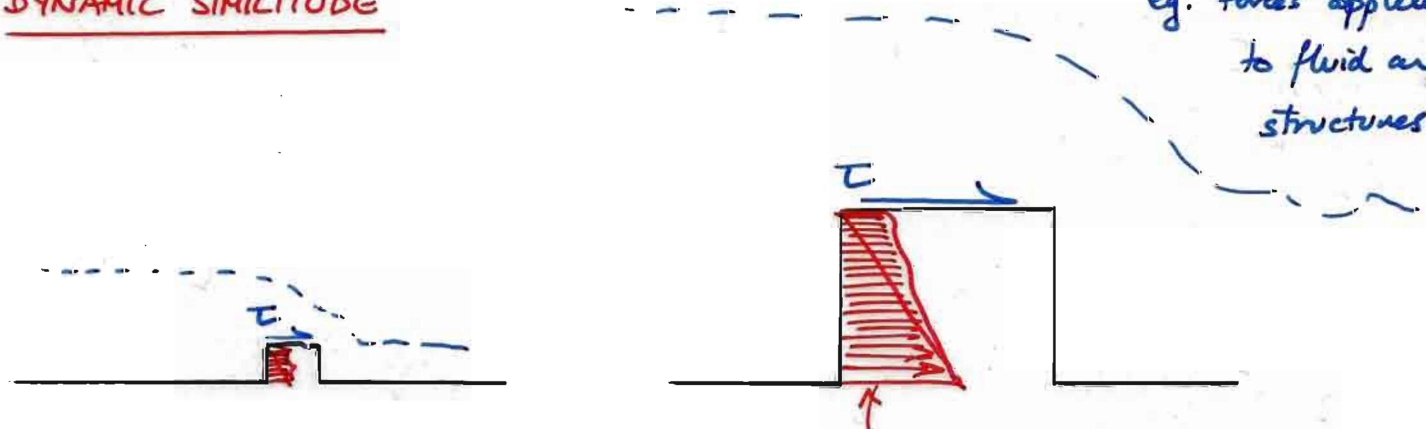
2. KINEMATIC SIMILITUDE

eg. Velocities and flow rates



3. DYNAMIC SIMILITUDE

eg. Forces applied to fluid and structures.



pressure distribution on flow obstruction

THREE TYPES OF SIMILITUDE

1. GEOMETRIC

   - Ratios of all corresponding dimensions on the model and prototype are equal. i.e $\boxed{L_m/L_p = l_r}$

     otherwise a distorted model. $(l_r)_H \neq (l_r)_V$

2. KINEMATIC

   - Ratios of all corresponding velocities and accelerations in the model and prototype are equal

1. Time

$$g_m = g_p$$

$$\frac{l_m}{t_m^2} = \frac{l_p}{t_p^2} \; ; \qquad \frac{t_m}{t_p} = \left(\frac{l_m}{l_p}\right)^{1/2} \qquad \boxed{t_r = (l_r)^{1/2}}$$

2. Velocity

$$V_r = \frac{l_r}{t_r} = \frac{l_r}{l_r^{1/2}} \qquad \boxed{V_r = l_r^{1/2}}$$

3. Flow

$$Q_r = \frac{l_r^3}{t_r} = \frac{l_r^3}{l_r^{1/2}} \qquad \boxed{Q_r = l_r^{5/2}}$$

4. Acceleration

$$a_r = \frac{l_r}{t_r^2} = \frac{l_r}{l_r} = 1 \qquad \boxed{a_r = 1}$$

3. DYNAMIC

- Ratios of all corresponding forces on model and prototype are equal.

Forces include:

1. Inertia: $F_I = ma = \rho \forall a = \rho L^4 T^{-2} = \rho L^2 V^2$
2. Gravity: $F_g = mg = \rho \forall g = \rho L^3 g$
3. Viscosity: $F_V = \mu \left(\frac{du}{dy}\right) A = \mu \left(\frac{V}{L}\right) L^2 = \mu V L$
4. Pressure: $F_P = (\Delta p) A = (\Delta p) L^2$
5. Surface tension: $F_T = \sigma L$
6. Elasticity: $F_E = E_v A = E L^2$

Evaluate ratios of these pairs, (1–6). eg.

Inertia / Viscous forces:

$$\frac{F_I}{F_V} = \frac{\rho L^2 V^2}{\mu V L} = \frac{\rho L V}{\mu} = Re$$

Inertia / Gravity forces:

$$\frac{F_I}{F_g} = \frac{\rho L^2 V^2}{\rho L^3 g} = \frac{V}{\sqrt{Lg}} = Fr$$

Pressure / Inertial forces:

$$\frac{F_P}{F_I} = \frac{(\Delta p) L^2}{\rho L^2 V^2} = \frac{\Delta P}{\rho V^2} = Eu$$

also Cavitation Number $\frac{(P_r - P_v)}{\frac{1}{2} \rho V^2}$

■ **TABLE 7.1**

Some Common Variables and Dimensionless Groups in Fluid Mechanics

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number,[a] Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{\omega \ell}{V}$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

[a] The Cauchy number and the Mach number are related and either can be used as an index of the relative effects of inertia and compressibility. See accompanying discussion.

All flows

Gravity drive →

Pressure drive →

Also:
Cavitation no:

$$\frac{(P_r - P_v)}{\frac{1}{2}\rho V^2}$$

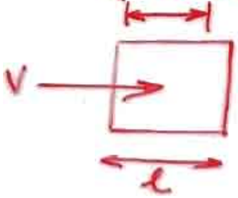
(Same form as Euler No.).

P_r = reference pressure.

Peclet No., $Pe = \frac{VL}{D} = \frac{\text{Advective flux}}{\text{Diffusive flux}}$

D= Dispersion coef. or Coef. of Molecular Diffusion L^2T^{-1}

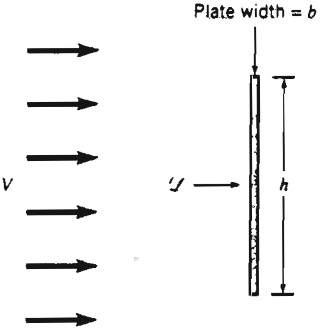
($V\Delta t$=length) Courant No., $Ce = \frac{V\Delta t}{\ell}$



Δt is time step (increment) or time

7.54

7.54 A very viscous fluid flows slowly past the submerged rectangular plate of Fig. P7.54. The drag, $\mathfrak{D}$, is known to be a function of the plate height, h, plate width, b, fluid velocity, V, and fluid viscosity, μ. A model is to be used to predict the drag and during a certain model test using glycerin ($\mu_m = 0.03$ lb·s/ft²), with $h_m = 1$ in. and $b_m = 3$ in., it was found that $\mathfrak{D}_m = 0.2$ lb when $V_m = 0.5$ ft/s. If possible, predict the drag on a geometrically similar larger plate with $h = 4$ in. and $b = 12$ in. immersed in the same glycerin moving with a velocity of 2 ft/s. If it is *not* possible, explain why.



■ FIGURE P7.54

$$\mathscr{D} = f(h, b, V, \mu)$$

$$\mathscr{D} \doteq F \quad h \doteq L \quad b \doteq L \quad V \doteq LT^{-1} \quad \mu \doteq FL^{-2}T$$

From the pi theorem, $5-3 = 2$ pi terms required, and a dimensional analysis yields

$$\boxed{\frac{\mathscr{D}}{hV\mu} = \phi\left(\frac{b}{h}\right)}$$

The required similarity condition is

$$\frac{b}{h} = \frac{b_m}{h_m}$$

and from the data given

$$\frac{12\text{ in.}}{4\text{ in.}} = \frac{3\text{ in}}{1\text{ in.}} = 3$$

so that this condition is satisfied. Thus, the prediction equation is

$$\frac{\mathscr{D}}{hV\mu} = \frac{\mathscr{D}_m}{h_m V_m \mu_m}$$

or

$$\mathscr{D} = \frac{h}{h_m}\frac{V}{V_m}\frac{\mu}{\mu_m}\mathscr{D}_m$$

and for the data given

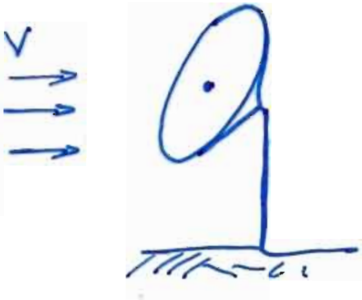
$$\mathscr{D} = \left(\frac{4\text{ in.}}{1\text{ in.}}\right)\left(\frac{2\frac{ft}{s}}{0.5\frac{ft}{s}}\right)(1)(0.2\text{ lb}) = \underline{\underline{3.20\text{ lb}}}$$

GEOMETRIC SIMILITUDE; Same geometric ratios

DYNAMIC SIMILITUDE; Re is the same model and full-scal

7.47

7.47 The drag on a 2-m-diameter satellite dish due to an 80 km/hr wind is to be determined through a wind tunnel test using a geometrically similar 0.4-m-diameter model dish. Assume standard air for both model and prototype. **(a)** At what air speed should the model test be run? **(b)** With all similarity conditions satisfied, the measured drag on the model was determined to be 170 N. What is the predicted drag on the prototype dish?



(a) From Eq. 7.19, Reynolds number similarity is required. Thus,

$$\frac{V_m D_m}{\nu_m} = \frac{VD}{\nu}$$

where D is the dish diameter. It follows that

$$V_m = \frac{\nu_m}{\nu}\frac{D}{D_m}V$$

and with $\nu_m/\nu = 1$

$$V_m = \left(\frac{2\,m}{0.4\,m}\right)\left(80\,\frac{km}{hr}\right) = \underline{\underline{400\,\frac{km}{hr}}}$$

(b) From Eq. 7.19,

$$\frac{\mathcal{D}_m}{\frac{1}{2}\rho_m V_m^2 D_m^2} = \frac{\mathcal{D}}{\frac{1}{2}\rho V^2 D^2}$$

so that (with $\rho_m = \rho$)

$$\mathcal{D} = \frac{V^2}{V_m^2}\frac{D^2}{D_m^2}\mathcal{D}_m$$

$$= \frac{\left(80\,\frac{km}{hr}\right)^2}{\left(400\,\frac{km}{hr}\right)^2}\frac{(2m)^2}{(0.4m)^2}(170\,N) = \underline{\underline{170\,N}}$$

(Note that $\mathcal{D} = \mathcal{D}_m$ in this problem, since from the condition of Reynolds number similarity, $V^2/V_m^2 = D_m^2/D^2$. This is not true in general.)

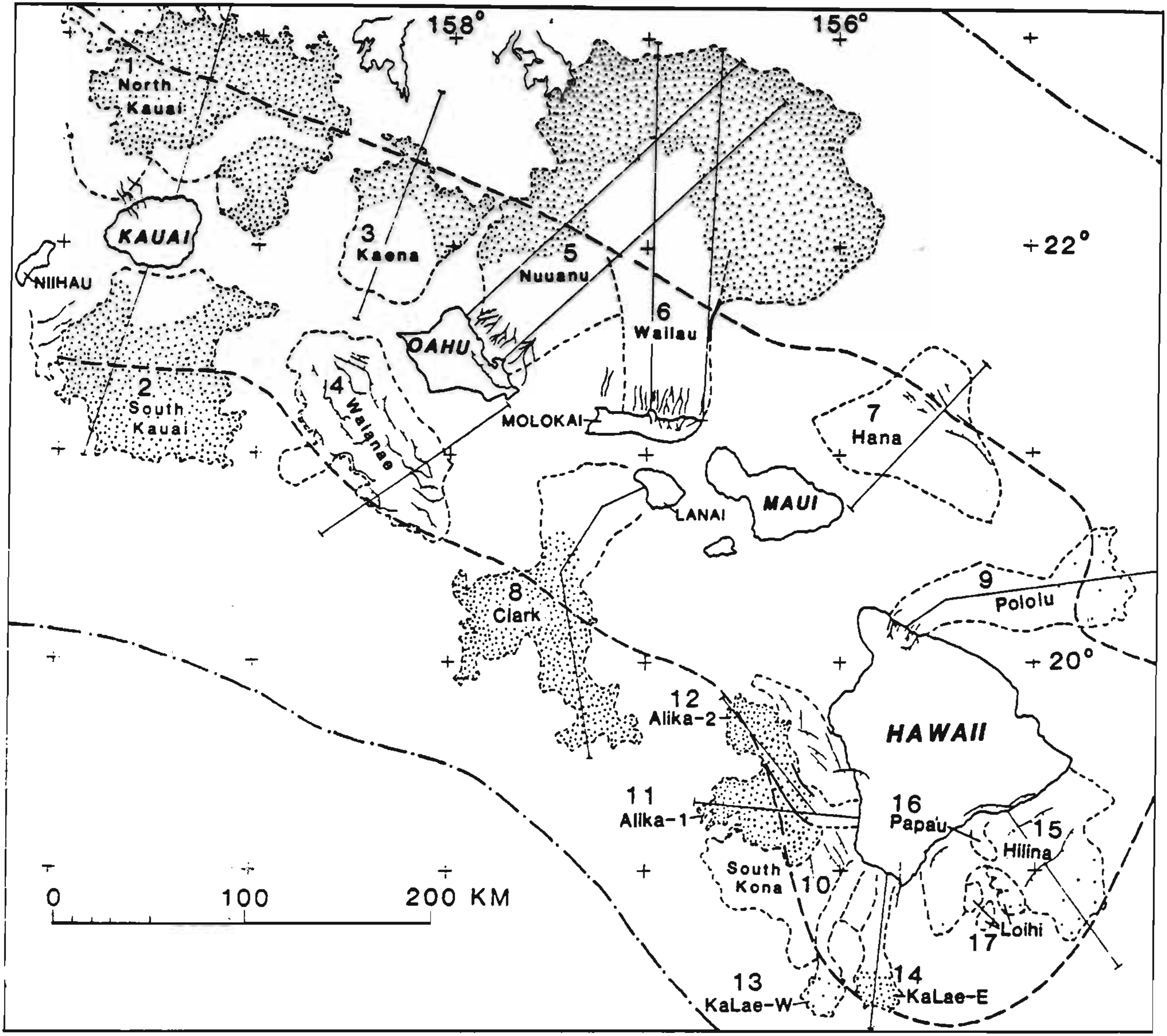
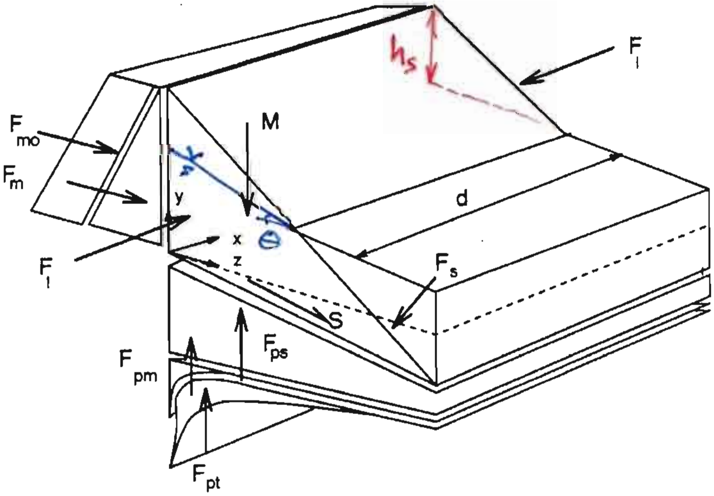
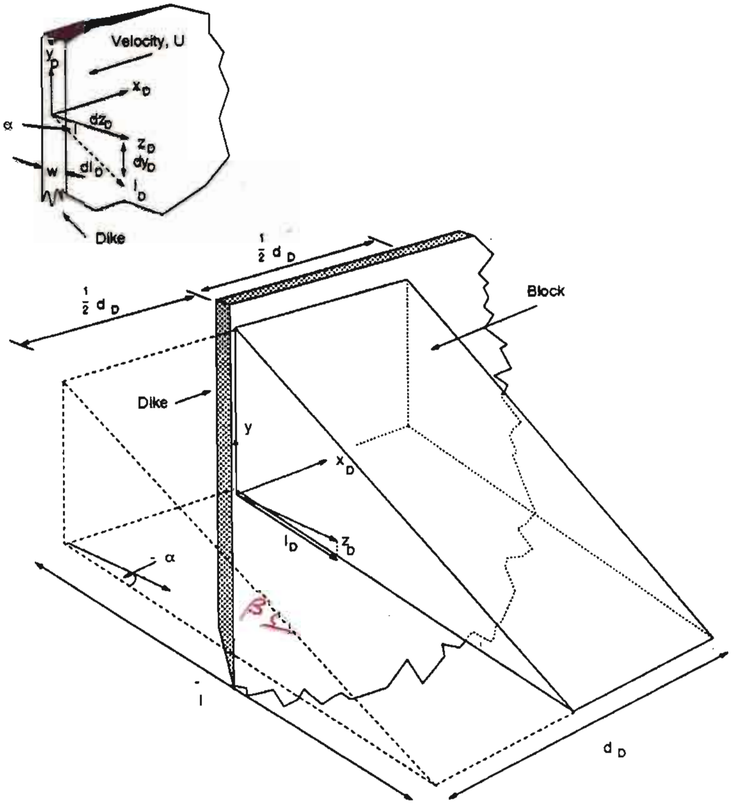


Fig. 2. Map of southeastern Hawaiian Ridge showing major slides bounded by dashed lines identified by number in text and Table 1; compare with Figure 1. Dotted area, hummocky ground (widely spaced where subdued); hachured lines, scarps; thin, downslope-directed lines, submarine canyons and their subaerial counterparts; heavy dashed line, axis of the Hawaiian Deep: dash-dotted line, crest of the Hawaiian Arch.

VOLCANO INSTABILITY — DIMENSIONAL ANALYSIS





Volcano Instability — Non-Dimensional Parameters

Isothermal

$$\frac{F}{\tan\phi} = f[\underbrace{\alpha,\ \beta,\ \theta,\ \overbrace{\frac{\gamma_m}{\gamma_w},\ \frac{\gamma_r}{\gamma_w}},\ \frac{h_m}{h_s},\ \frac{d}{h_s},\ \frac{h_o}{h_s}}_{\text{GEOMETRY}},\ k_0,\ \underbrace{\overset{W_D}{\frac{\mu}{k}\frac{wc}{\pi\gamma_w h_s^2}},\ \overset{U_D}{\frac{Uh_s}{2c}}}_{\text{MECHANICAL EFFECT}}]$$

Non-isothermal

$$\frac{F}{\tan\phi} = f[\cdots\ \underbrace{\frac{\kappa n}{c},\ \frac{4\kappa t}{h_s^2},\ \overset{A_D}{\frac{AK_bD}{\gamma_w h_s}}}_{\text{THERMAL EFFECT}}]$$

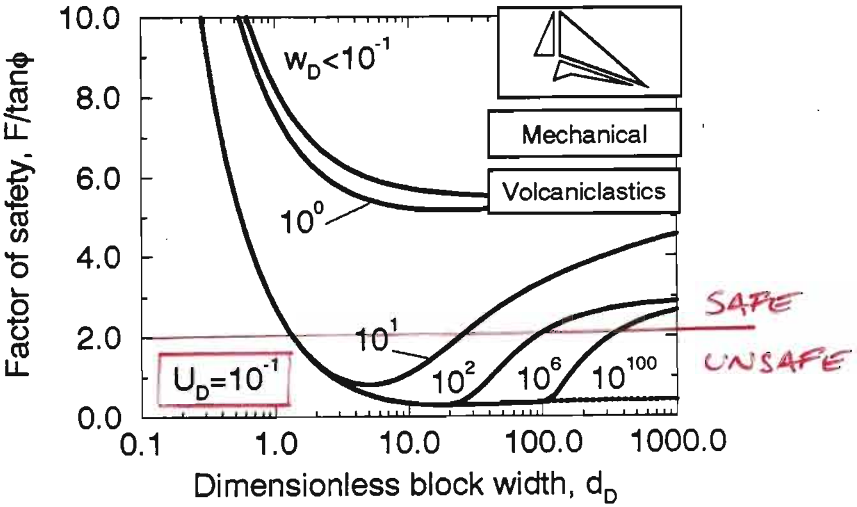
Volcano Instability — Non-Dimensional Parameters

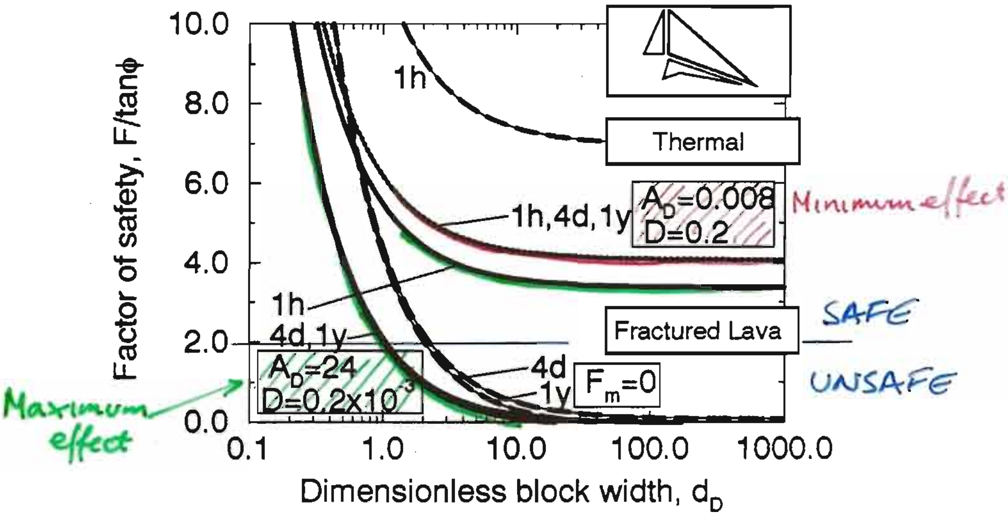
Isothermal

$$\frac{F}{\tan\phi} = f[\alpha,\ \beta,\ \theta,\ \frac{\gamma_m}{\gamma_w},\ \frac{\gamma_r}{\gamma_w},\ \frac{h_m}{h_s},\ \frac{d}{h_s},\ \frac{h_o}{h_s},\ k_0,\ \underbrace{\frac{\mu}{k}\frac{wc}{\pi\gamma_w h_s^2},\ \frac{Uh_s}{2c}}_{}]$$

Non-isothermal

$$\frac{F}{\tan\phi} = f[\cdots\ \ \underbrace{\frac{\kappa n}{c},\ \frac{4\kappa t}{h_s^2},\ \frac{AK_bD}{\gamma_w h_s}}_{}]$$





HOT DRY ROCK (HDR) GEOTHERMAL ENERGY

Solve coupled PDEs.

RESERVOIR:

$$k \frac{\partial T_R}{\partial r}\bigg|_{r=a} = q_F \rho_F c_F (T_{F_o} - T_{F_i}) + \frac{4}{3}\pi a^3 \rho_s c_s \frac{\partial T_{F_o}}{\partial t}$$

EXTERIOR:

$$K\left[\frac{\partial^2 T_R}{\partial r^2} + \frac{2}{r}\frac{\partial T_R}{\partial r}\right] = \frac{\partial T_R}{\partial t}$$

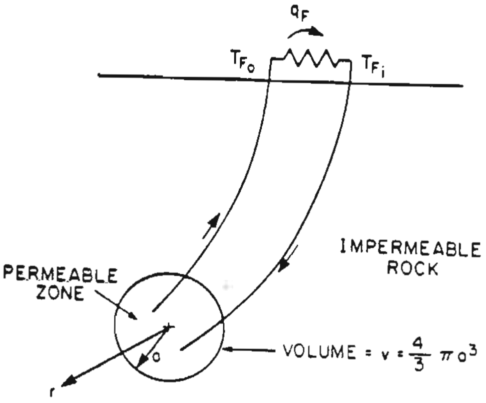


Fig. 2. Geometry of the Spherical Reservoir Model (SRM) for geothermal energy extraction.

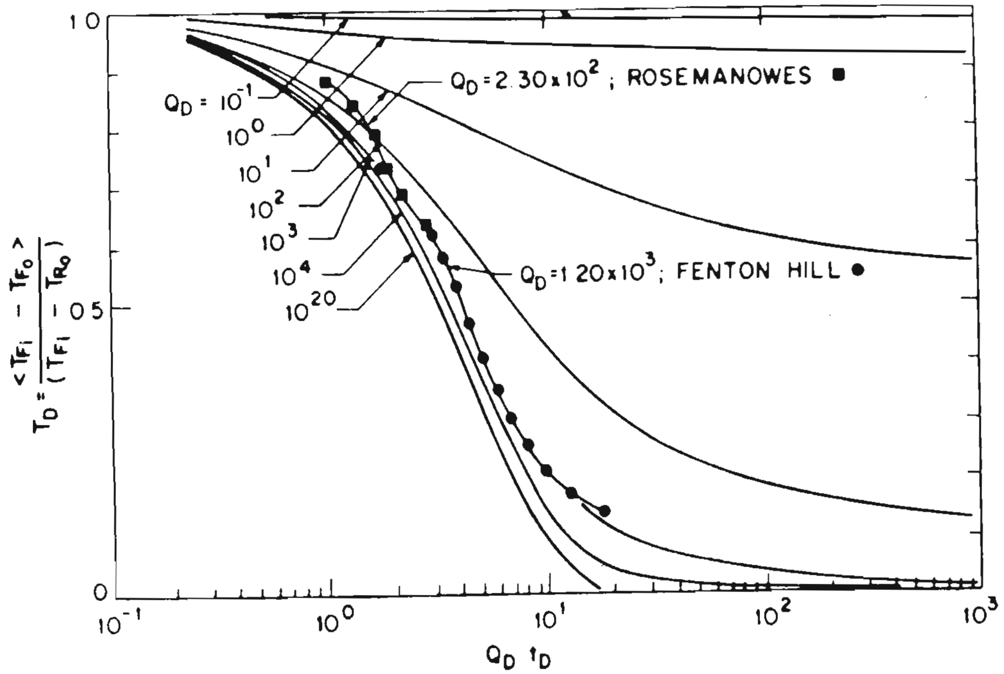
$$T_D = \frac{< T_{F_i} - T_{F_o} >}{(T_{F_i} - T_{R_0})} \tag{4}$$

$$Q_D = \frac{q_F \varrho_F c_F}{K_R a} = \frac{q_F}{\kappa a} \qquad \kappa = \frac{K}{\rho c} \tag{5}$$

$$\Phi_D = \frac{\varrho_S c_S}{\varrho_R c_R} \tag{6}$$

$$t_D = \frac{K_R t}{\varrho_R c_R a^2} = \frac{\kappa t}{a^2} \tag{7}$$

where $\varrho_S c_S = \varrho_R c_R (1 - \phi) + \varrho_F c_F \phi$



[10-11]

Flow in Pipes

Pipe Flow [10-11]

$$\tau_w = \frac{\rho V^2}{8} f; \quad h_L^{major} = f(\frac{l}{D})\frac{V^2}{2g}; \quad h_P = \frac{Power}{\gamma Q}$$

$$\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 + h_P = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 + \sum h_L^{major} + \sum h_L^{minor}$$

$$h_L^{minor} = K_L \frac{V^2}{2g}; \quad l_{eq}^{minor} = \frac{K_L D}{f}; \quad K_L = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

Non-circular: Laminar: $[f = \frac{C}{\mathbf{Re}_h}; D_h = \frac{4A}{P}]$ Turbulent: [Use Moody; $f = \varphi(\frac{\varepsilon}{D_h})$]

Series: $h_L = h_{L_1} + h_{L_2} + ... + h_{L_n}$; Parallel: $h_{L_1} = h_{L_2} = ... = h_{L_n}$

Flow meters: $Q = CA\sqrt{\frac{2(p_1 - p_2)}{\rho(1-\beta^4)}}$; $\beta = \frac{D_2}{D_1}$

[10:1] Pipe Flow

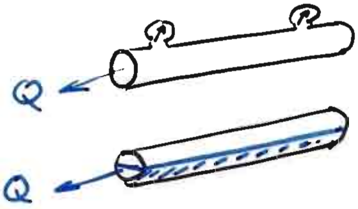
Outline

Open and closed flows

Bernoulli/Energy equation

Frictional losses

PIPE FLOW



Pipe flow – Pressure

Open-channel – Gravity

Important aspects neglected so far:

1. Viscous flow (Friction factor, f)
2. Roughness effects (ε/D)

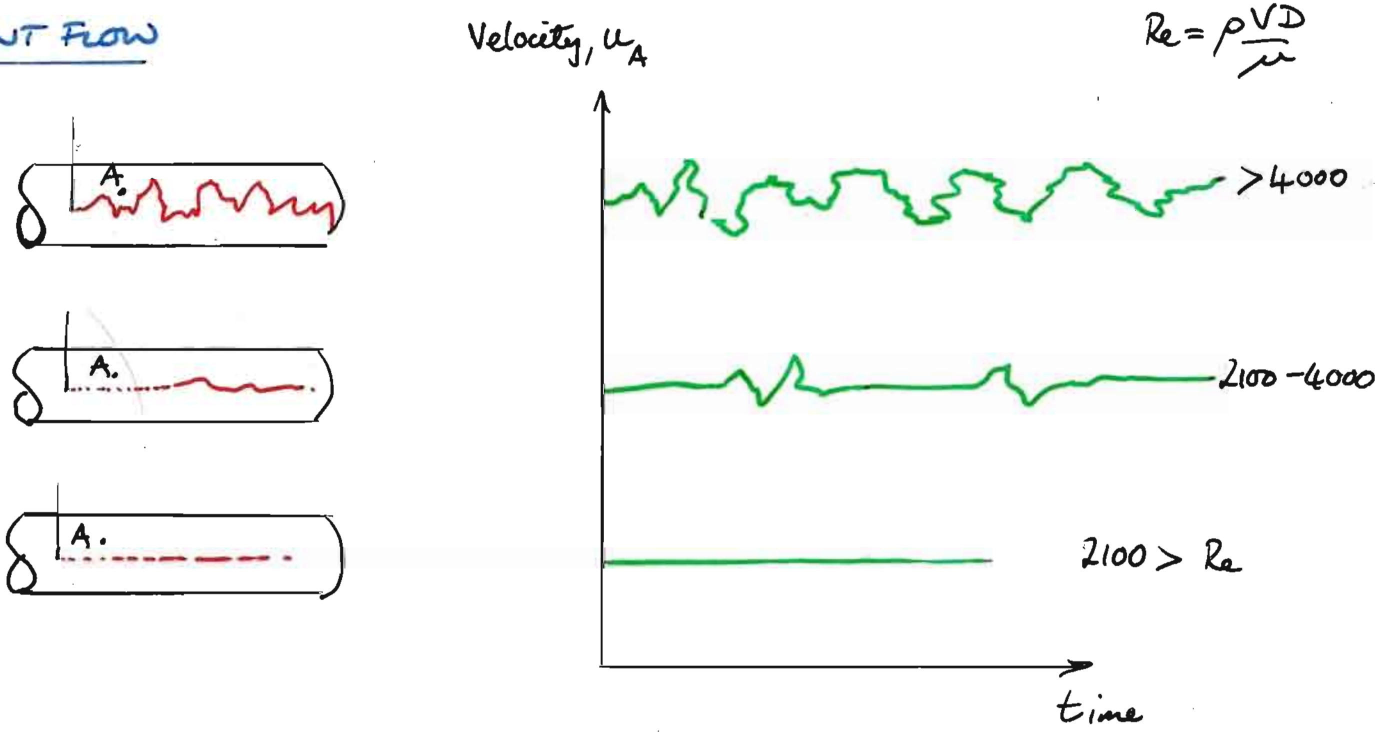
3. Turbulence (Re)

Laminar flow: $f = 64/Re$

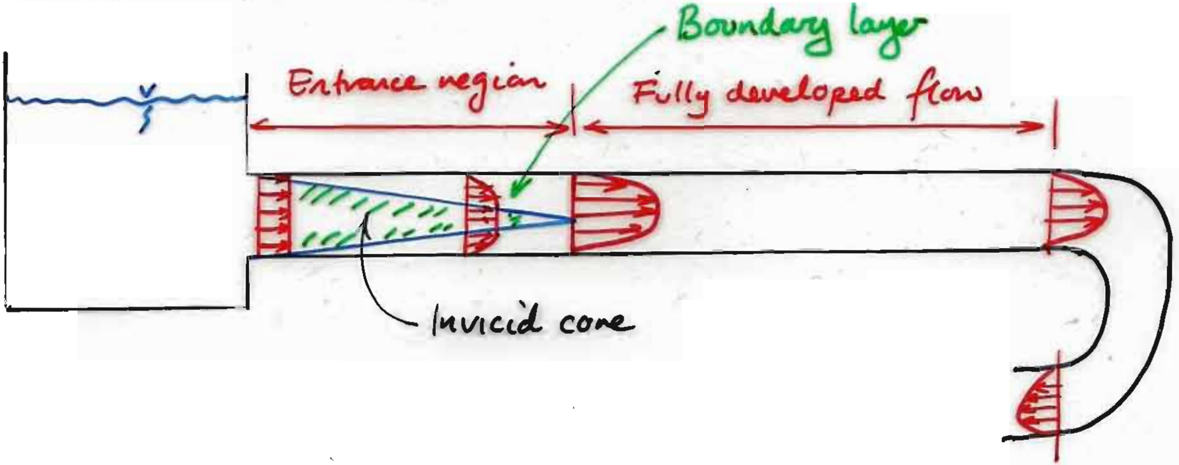
Turbulent flow: $f = \phi[Re; \frac{\varepsilon}{D}]$

TURBULENT FLOW

$$Re = \rho \frac{VD}{\mu}$$



ENTRANCE REGIONS



ENTRANCE LENGTH: l_e

$$\frac{l_e}{D} = 0.06\ Re \qquad \text{Laminar}$$

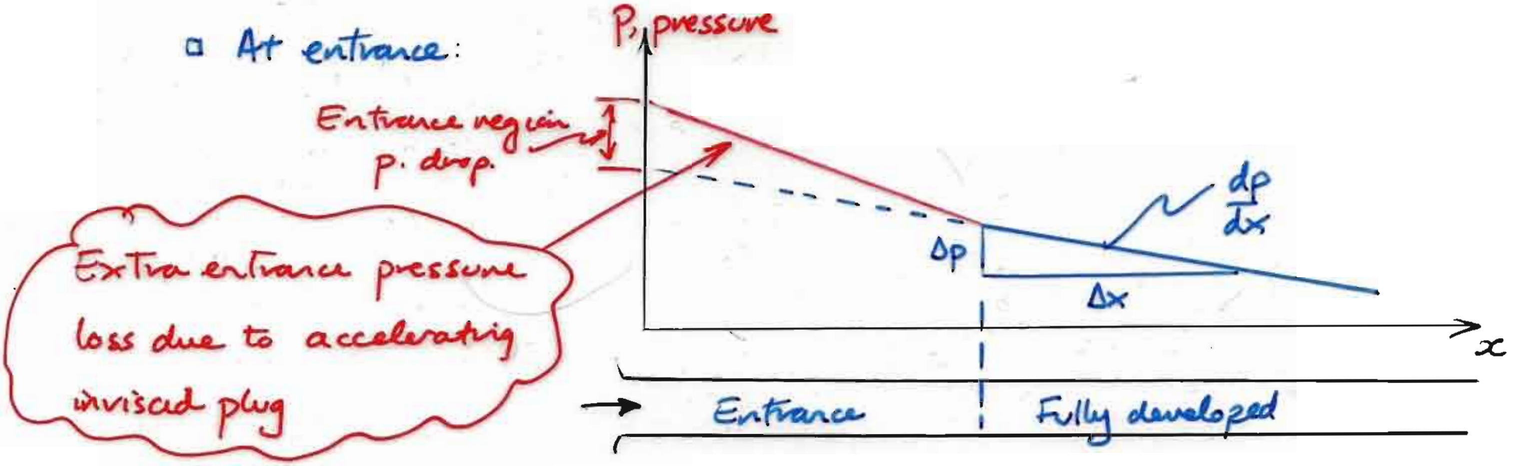
$$\frac{l_e}{D} = 4.4\,(Re)^{1/6} \qquad \text{Turbulent}$$

Low Re ⟶ small entrance region.
length "stretches" with increased Re.

PRESSURE & SHEAR STRESS

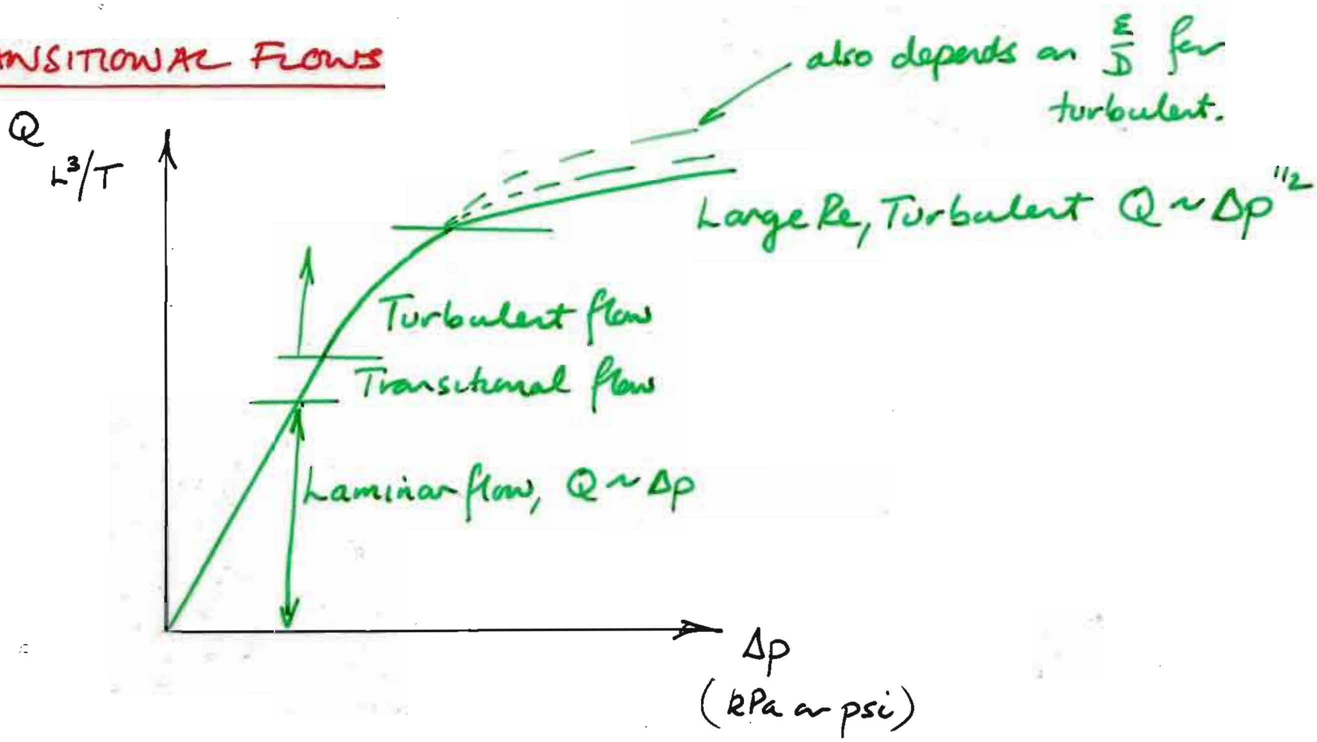
- Flow driven by – pressure gradient (typically large)
  gravitational forces (typically negligible)

- At entrance:



Extra entrance pressure loss due to accelerating inviscid plug

TRANSITIONAL FLOWS

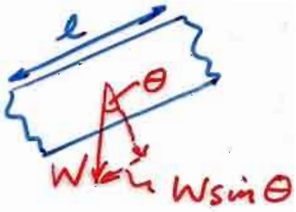


also depends on $\frac{\varepsilon}{D}$ for turbulent.

LAMINAR FLOW IN PIPES

Develop pipe flow equation:

1. Apply $F = ma$ to fluid element and note that $\tau = \mu \, du/dy$.
2. Solve Navier - Stokes equations.



$$Q = \frac{\pi D^4}{128\mu} \left(\frac{\Delta p - \gamma \ell \sin\theta}{\ell} \right)$$

pressure gradient

gravitational 'gradient' (along pipe).

3. Dimensional analysis.

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} \doteq \underbrace{\frac{64}{Re}}_{f = \text{friction factor}} \left(\frac{\ell}{D} \right)$$

"horizontal pipe" !!

wall shear stress

$$f = \frac{64}{Re} = \frac{8\tau_w}{\rho V^2}$$

Laminar only

Laminar or turbulent

ENERGY CONSIDERATIONS

General energy equation:

$$\frac{P_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

$\alpha \geqslant 1$ and h_L = viscous losses.

For constant section pipe flow $V_1 = V_2$ and

$$\left(\frac{P_1}{\gamma} + z_1 \right) = \left(\frac{P_2}{\gamma} + z_2 \right) + h_L$$

$$h_L = \frac{2 \tau_w l}{\gamma r}$$

$$h_L = \frac{4 l \tau_w}{\gamma D}$$

Applies equally well to laminar & turbulent.

Substituting $f = \frac{8 \tau_w}{\rho V^2}$ then $\tau_w = \frac{\rho V^2}{8} f$

$$h_L = f \left(\frac{l}{D} \right) \frac{V^2}{2g}$$

Equation ok for laminar and turbulent if (f) is determined correctly. (Moody).

DIMENSIONAL ANALYSIS OF PIPE FLOW

Desire to accommodate :
1. Viscosity, μ
2. Laminar → turbulence   function of Re
3. Pipe wall roughness.

MOODY CHARTS

$$\Delta p = F(V, D, \ell, \varepsilon, \rho, \mu)$$

Dimensional analysis gives:
$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \bar{\phi}\left[\frac{\rho V D}{\mu} ; \frac{\ell}{D} ; \frac{\varepsilon}{D}\right]$$

Alternatively:
$$\frac{\Delta p}{\frac{1}{2}\rho V^2}\frac{D}{\ell} = \underbrace{\phi\left[Re ; \frac{\varepsilon}{D}\right]}$$

or:
$$\Delta p = f \frac{\ell}{D} \frac{\rho V^2}{2}$$

Friction factor, $f = \phi\left[Re, \frac{\varepsilon}{D}\right]$

Energy Arguments

$$\frac{P_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

$D_1 = D_2$; $z_1 = z_2$ Then $\alpha_1 = \alpha_2$ This reduces to

$$\Delta p = p_1 - p_2 = \gamma h_L$$

$$\boxed{h_L = f \frac{\ell}{D} \frac{V^2}{2g}}$$

[10:2] Pipe Flow

Recap

Bernoulli/Energy equation

Frictional losses

Outline

Moody chart

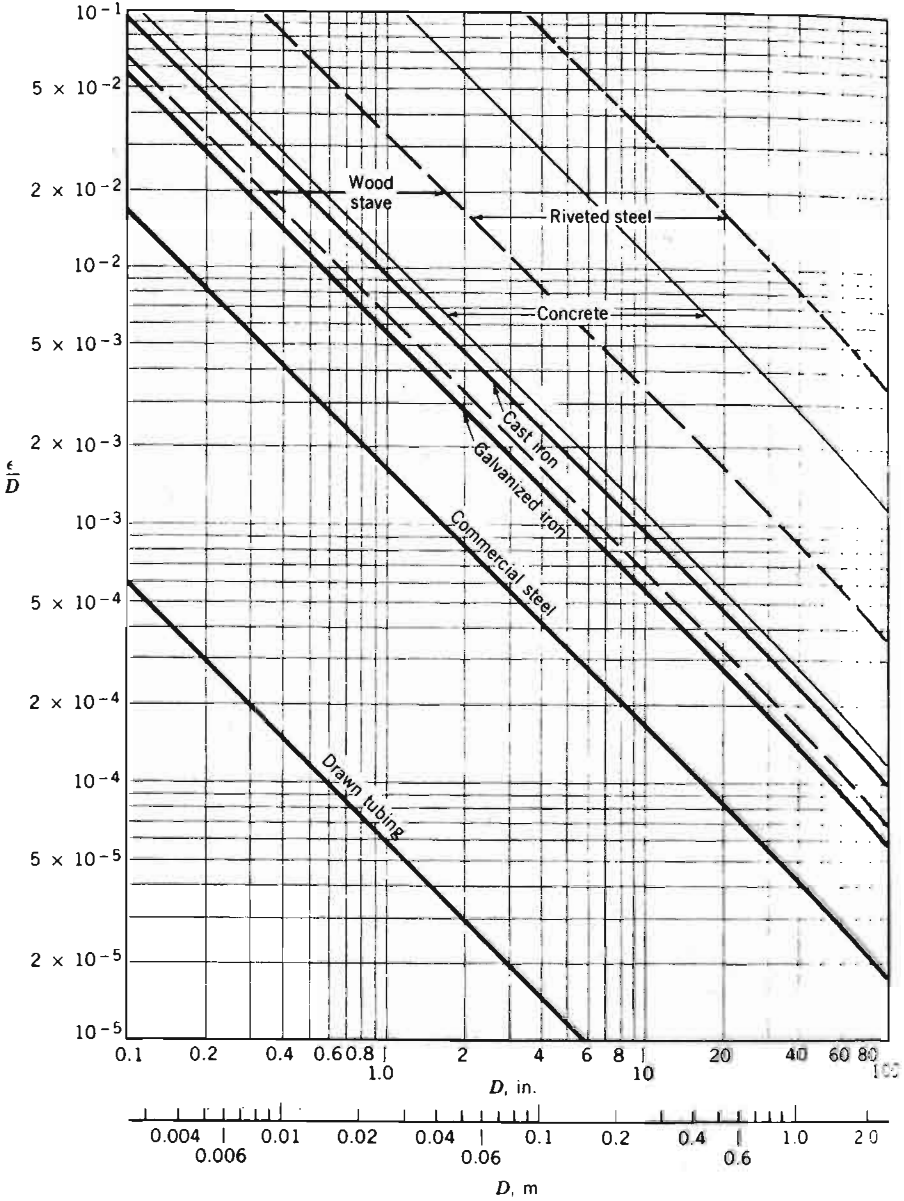
Minor Losses

Exit/Entry losses

■ **TABLE 8.1**

Equivalent Roughness for New Pipes [from Moody (Ref. 7) and Colebrook (Ref. 8)].

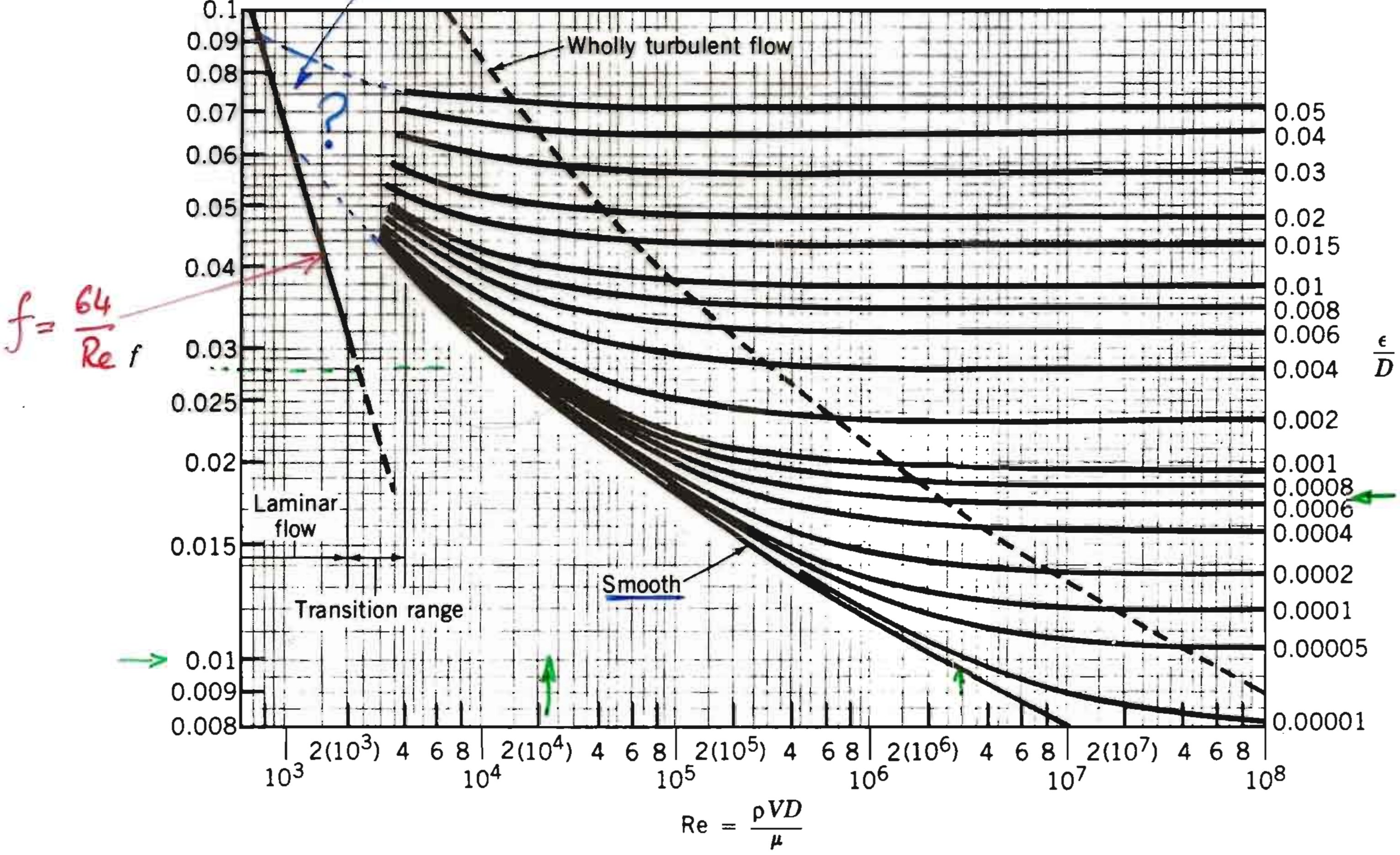
Pipe	Equivalent Roughness, ε	
	Feet	Millimeters
Riveted steel	0.003–0.03	0.9–9.0
Concrete	0.001–0.01	0.3–3.0
Wood stave	0.0006–0.003	0.18–0.9
Cast iron	0.00085	0.26
Galvanized iron	0.0005	0.15
Commercial steel or wrought iron	0.00015	0.045
Drawn tubing	0.000005	0.0015
Plastic, glass	0.0 (smooth)	0.0 (smooth)



■ **FIGURE 8.22** Relative roughness of new pipes (adapted from Ref. 7).

Transitional range

Laminar flow : $f = \frac{64}{Re}$; Complete turbulent flow : $f = \phi(\frac{\varepsilon}{D})$



■ FIGURE 8.23 Friction factor as a function of Reynolds number and relative roughness for round pipes—the Moody chart (Data from Ref. 7 with permission).

Colebrook Formula (Non-laminar range, only)

$$\frac{1}{\sqrt{f}} = -2.0 \log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right)$$

$$\left(\text{Laminar} \quad f = \frac{64}{Re}\right)$$

$$\boxed{h_L = f \frac{l}{D} \frac{V^2}{2g}}$$

[10:3] Pipe Flow

Recap

Bernoulli/Energy equation

Moody chart

Outline

Minor Losses

Exit/Entry losses

MINOR LOSSES

Major losses : h_L due to friction (f) in pipe sections

Minor losses : h_L due to
- valves
- bends
- contractions/expansions

Losses due to 'minor' causes are usually less than the linear losses in pipes, but may be significant.

Nondimensionalize the loss coefficient, K_L:

$$K_L = \frac{h_L}{(V^2/2g)} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

For $K_L \equiv 1$; the pressure drop across the "component" (Δp) is equal to the dynamic pressure $\frac{1}{2}\rho V^2$.

$$\boxed{h_L = K_L \frac{V^2}{2g}}$$

$$K_L = \phi(\text{geometry}, Re)$$

Re for pipe (not component)

$$Re = \frac{\rho V D}{\mu}$$

If flow is turbulent (often the case) then $K_L = \phi(\text{geometry})$ (not controlled by Re). ← All practical applications.

May also classify losses (h_L) as the "equivalent" length of pipe needed to give the same pressure/head loss, (l_{eq}).

$$h_L = K_L \frac{V^2}{2g} = \underbrace{f \frac{l_{eq}}{D}} \frac{V^2}{2g}$$

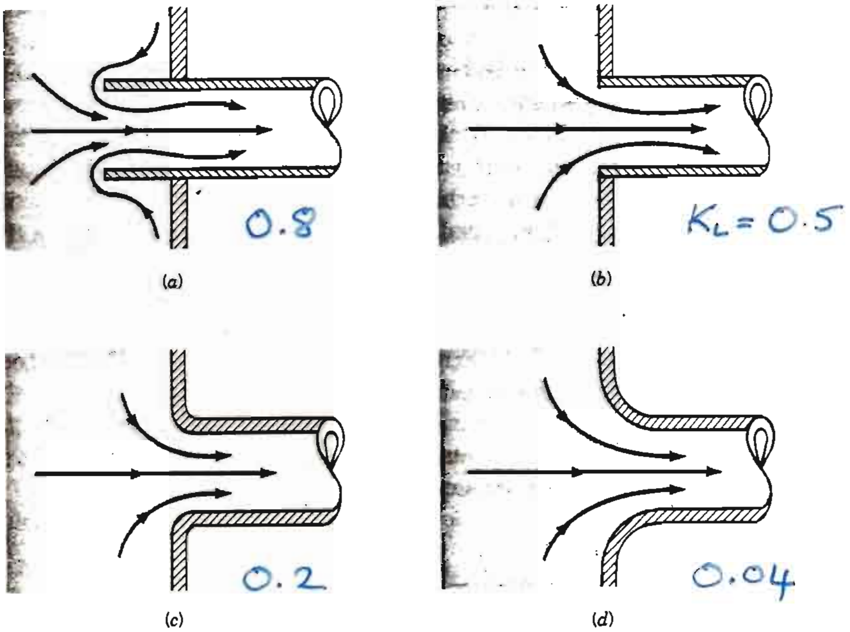
$$l_{eq} = \frac{K_L D}{f}$$

Again, f and D are based on the pipe, not the component.

ENTRY LOSSES

Two effects:

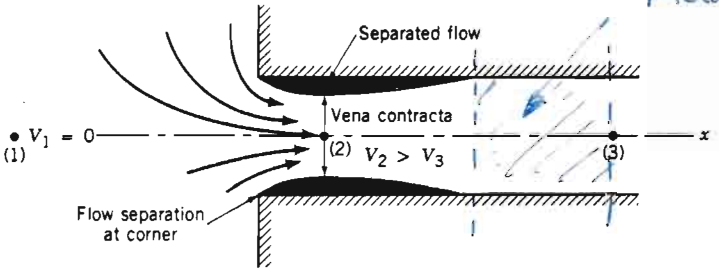
1. Acceleration due to change in direction



ACCELERATION

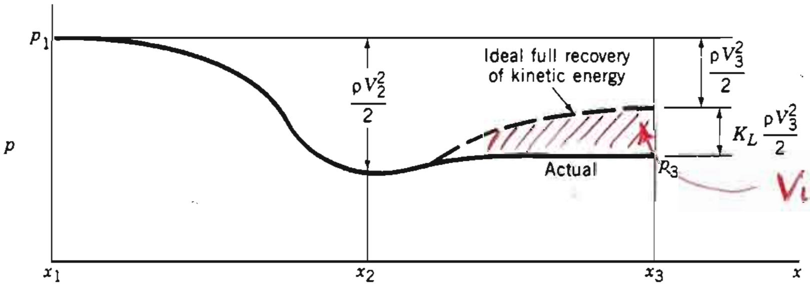
■ **FIGURE 8.25** **Entrance flow conditions and loss coefficient (Refs. 28, 29). (*a*)** Reen-**trant, $K_L = 0.8$, (*b*) sharp-edged, $K_L = 0.5$, (*c*) slightly rounded, $K_L = 0.2$ (see Fig. 8.27) (*d*) well-rounded, $K_L = 0.04$ (see Fig. 8.27).**

2. Separation of flow (Vena contracta)



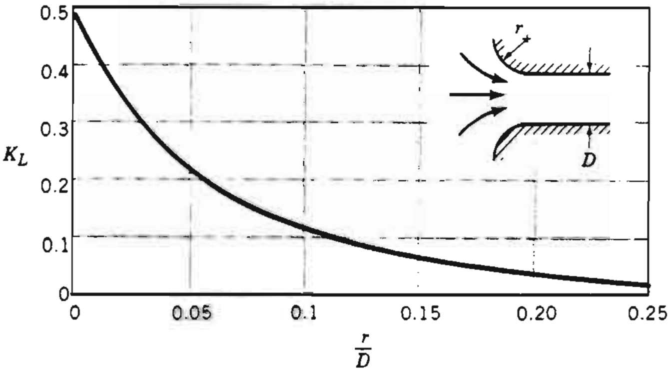
Fluid decelerates and dissipates energy by viscous losses

(*a*)



Viscous loss

Experimental Characterization



■ **FIGURE 8.27** **Entrance loss coefficient as a function of rounding of the inlet edge (Ref. 9).**

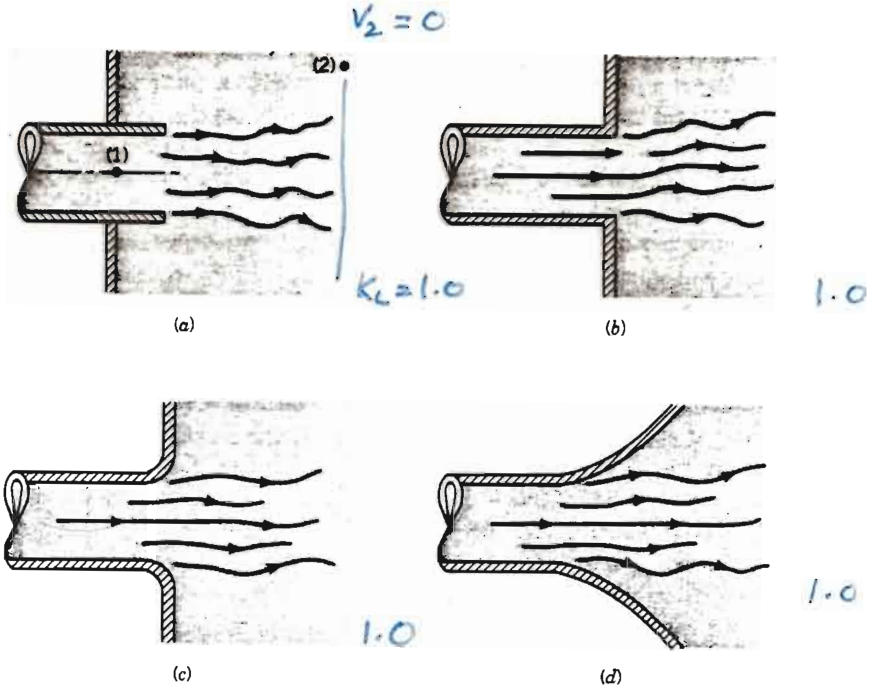
EXIT LOSSES

Exit into 'stagnant' tank.

$V_2 = 0$ ∴ flow is arrested.

∴ K_L is uniformly equivalent to 1.0

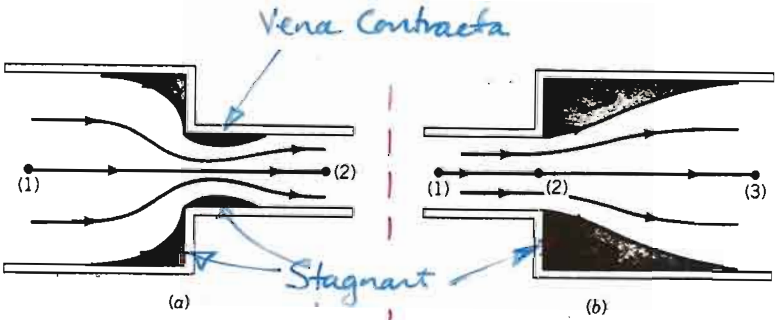
Not shape dependent.



■ FIGURE 8.28 Exit flow conditions and loss coefficient. (*a*) Reentrant, $K_L = 1.0$, (*b*) sharp-edged, $K_L = 1.0$, (*c*) slightly rounded, $K_L = 1.0$, (*d*) well rounded, $K_L = 1.0$.

EXIT & ENTRY LOSSES ARE NOT THE SAME. (EXPANSION / CONTRACTION)

Fluid accelerates



Fluid decelerates

■ FIGURE 8.29 Flow conditions near a sudden change in diameter. (*a*) Contraction, (*b*) expansion.

ENTRY

- Vena contracta forms

$\frac{A_1}{A_2} = 1$ $K_L = 0$?

$\frac{A_1}{A_2} \to \infty$ $K_L = ?$

Experimental results available.

EXIT

Limiting $K_L \to 1.0$ as $\frac{A_1}{A_3} \to 0$

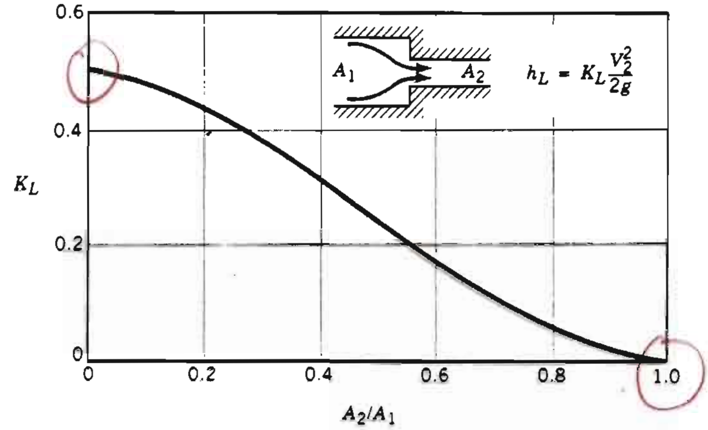
i.e exit into stagnant tank

$V_1 = V_2 \neq 0$

$V_3 = 0$

Analytical expression available

CONTRACTION



■ **FIGURE 8.30** Loss coefficient for a sudden contraction (Ref. 10).

EXPANSION (ABRUPT)

Analytical soln for conservation equations:

Mass: $A_1V_1 = A_3V_3$ ①

Momentum (x):

$$p_1A_3 - p_3A_3 = \rho A_3V_3(V_3 - V_1) \quad ②$$

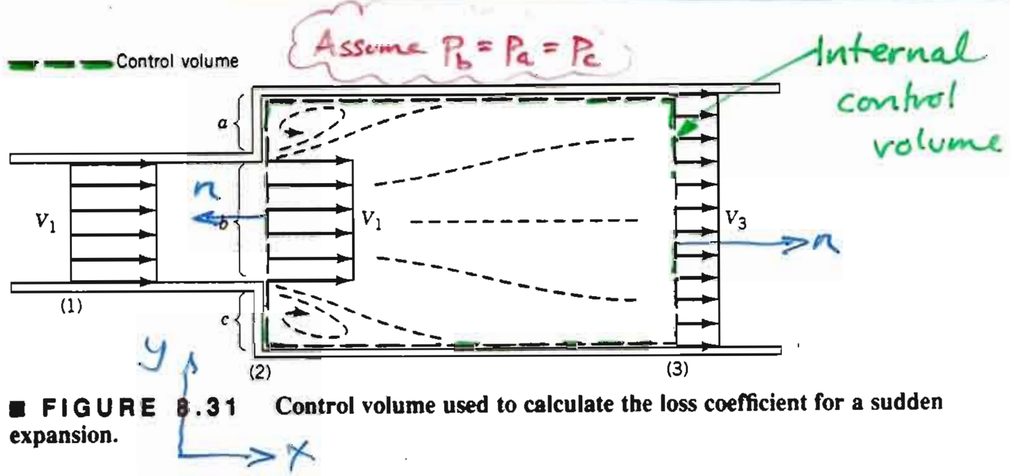
Energy:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + h_L \quad ③$$

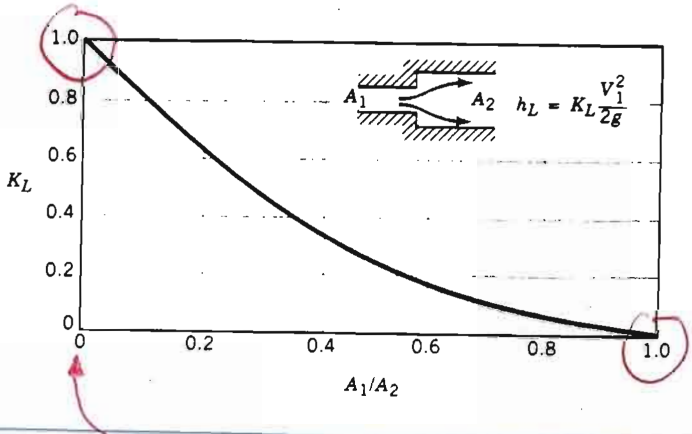
Combining ①, ② & ③

$$K_L = \left(1 - \frac{A_1}{A_3}\right)^2$$

2 in figure

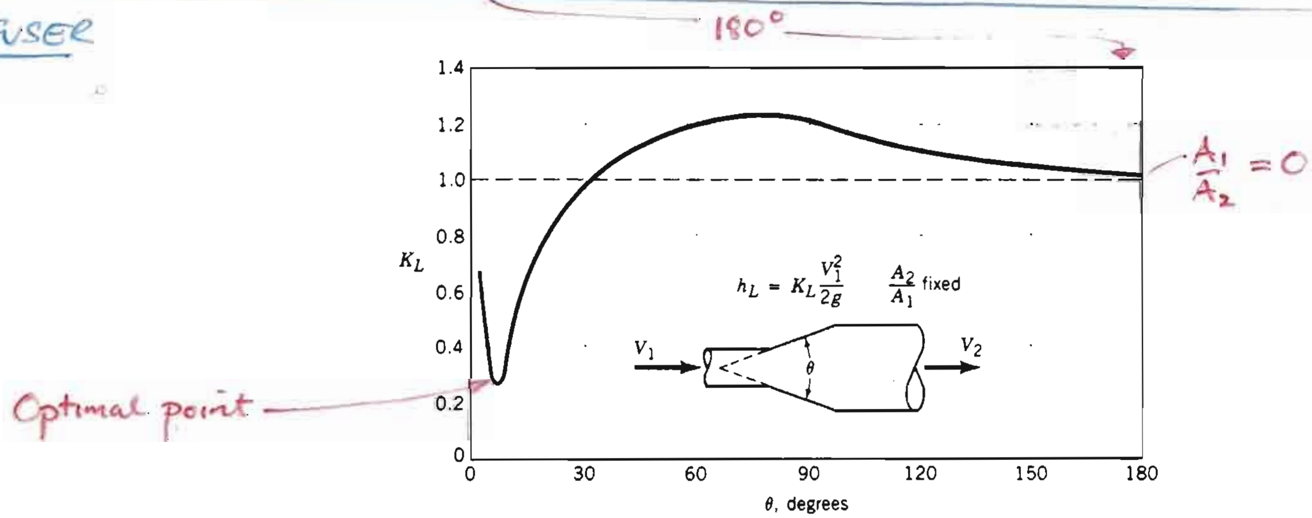


■ **FIGURE 8.31** Control volume used to calculate the loss coefficient for a sudden expansion.



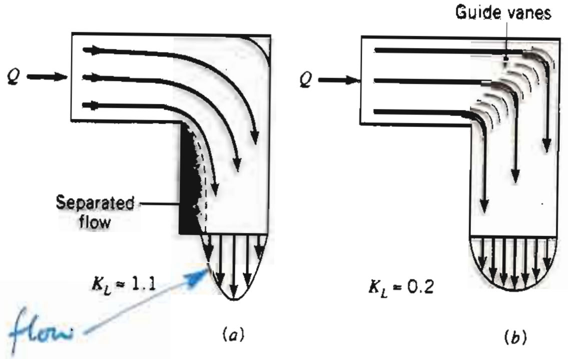
■ **FIGURE 8.32** Loss coefficient for a sudden expansion (Ref. 10).

DIFFUSER



■ **FIGURE 8.33** Loss coefficient for a typical conical diffuser (Ref. 5).

FLOW ARCUND BENDS



Accelerated flow

■ **FIGURE 8.35** **Character of the flow in a 90° miter bend and the associated loss coefficient: (*a*) without guide vanes. (*b*) with guide vanes.**

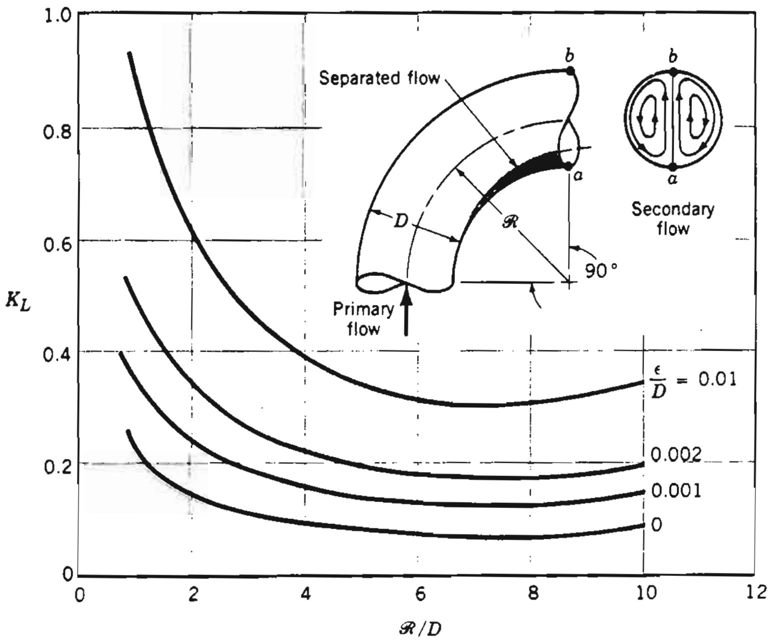
Remember: Add loss due to axial portion of pipe, h_L.

$$h_L^{maj} = f \frac{\ell}{D} \frac{v^2}{2g}$$

$$+$$

$$h_L^{min} = K_L \frac{v^2}{2g}$$

Sharp bend ⟶ Gentle bend

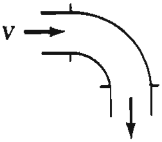
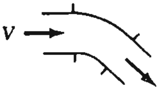
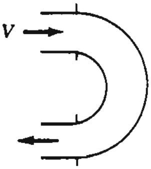
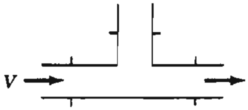
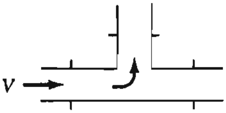
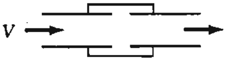


$$\uparrow \frac{\varepsilon}{D} \rightarrow \uparrow K_L \doteq \uparrow h_L$$

■ **FIGURE 8.34** **Character of the flow in a 90° bend and the associated loss coefficient (Ref. 5).**

■ TABLE 8.2

Loss Coefficients for Pipe Components $\left(h_L = K_L \frac{V^2}{2g}\right)$ **(Data from Refs. 5, 10, 27)**

Component	K_L	
a. Elbows		
Regular 90°, flanged	0.3	
Regular 90°, threaded	1.5	
Long radius 90°, flanged	0.2	
Long radius 90°, threaded	0.7	
Long radius 45°, flanged	0.2	
Regular 45°, threaded	0.4	
b. 180° return bends		
180° return bend, flanged	0.2	
180° return bend, threaded	1.5	
c. Tees		
Line flow, flanged	0.2	
Line flow, threaded	0.9	
Branch flow, flanged	1.0	
Branch flow, threaded	2.0	
d. Union, threaded	0.08	
*e. Valves		
Globe, fully open	10	
Angle, fully open	2	
Gate, fully open	0.15	
Gate, $\frac{1}{4}$ closed	0.26	
Gate, $\frac{1}{2}$ closed	2.1	
Gate, $\frac{3}{4}$ closed	17	
Swing check, forward flow	2	
Swing check, backward flow	∞	
Ball valve, fully open	0.05	
Ball valve, $\frac{1}{3}$ closed	5.5	
Ball valve, $\frac{2}{3}$ closed	210	

*See Fig. 8.36 for typical valve geometry

General

K_L threaded > K_L flanged

VALVES

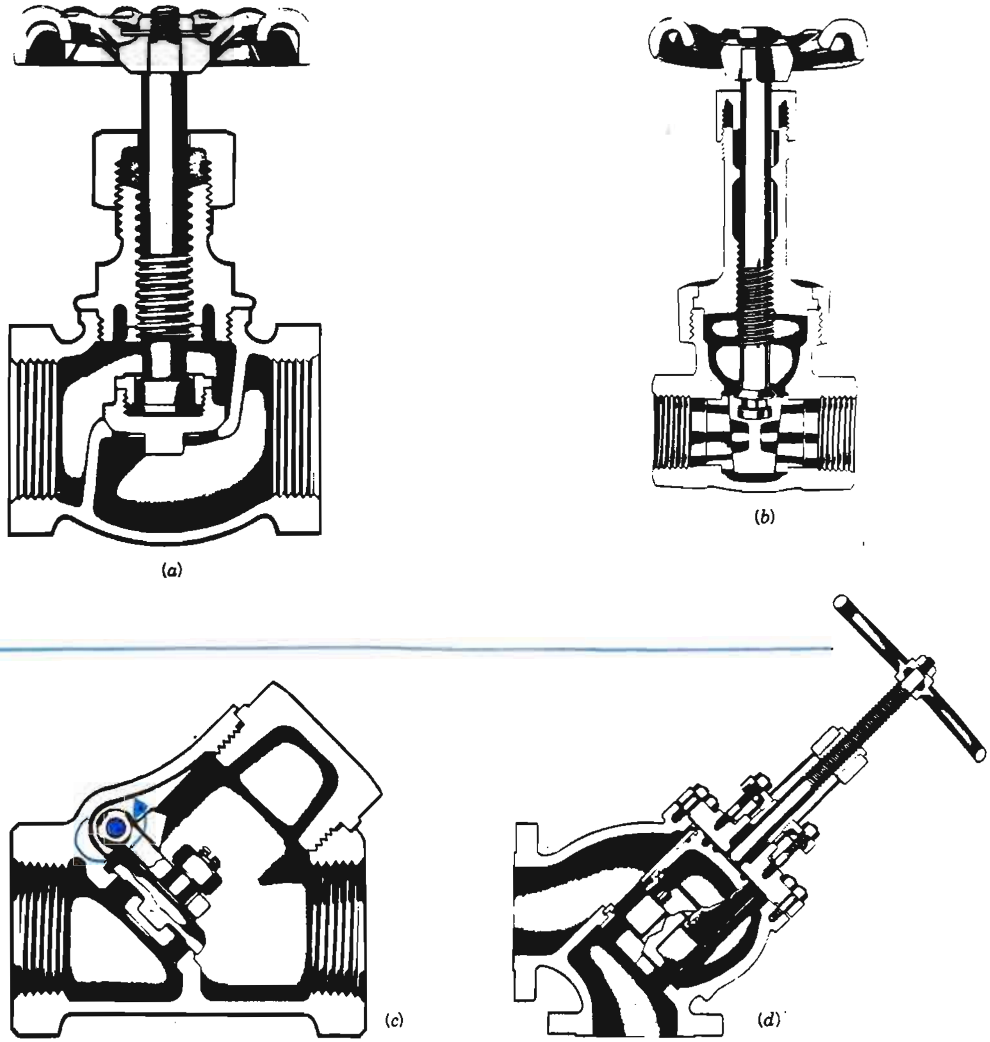
GLOBE

Adjustable q.

CHECK VALVES

ONE WAY FLOW ONLY

NO CONTROL ON q..



■ **FIGURE 8.36** **Internal structure of various valves: (a) globe valve, (b) gate valve, (c) swing check valve, (d) stop check valve (courtesy of Crane Co., Valve Division).**

[11:1] Pipe Flow

Outline

Non-circular sections

Turbulence

Recap.....

Examples

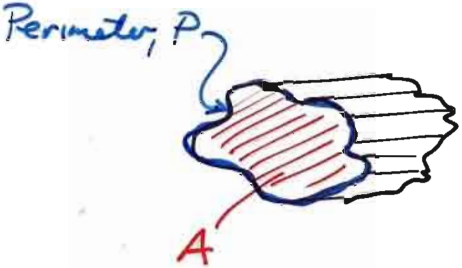
- I Pressure drop
- II Flow rate
- III Geometry

NONCIRCULAR CONDUITS

$C \neq 64$

Noncircular pipe: $$f = \frac{C}{Re_h}$$

$$Re_h = \frac{\rho V D_h}{\mu} \qquad D_h = \frac{4A}{P}$$



D_h = hydraulic diameter $\qquad$ $D_h = D$ for circular pipe

Also include in: $\qquad$ Friction factor: $$h_L = f\left(\frac{L}{D_h}\right)\frac{V^2}{2g}$$

Relative roughness: $$\frac{\varepsilon}{D_h}$$

See table 8.3 for values of C.

TURBULENT FLOW

Use Moody chart with $\frac{\varepsilon}{D_h}$

$$D_h = \frac{4\,\text{Area}}{\text{Perimeter}}$$

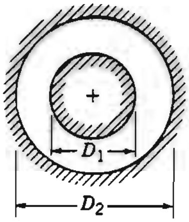
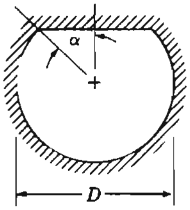
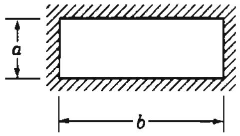
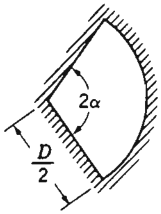
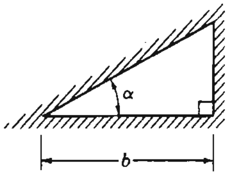
$$Re_h = \frac{\rho V D_h}{\mu}$$

Accurate to ~ 15%

LAMINAR FLOW

TABLE 8.3.

Friction Factors for Laminar Flow in Noncircular Ducts (Data from Ref. 18)

Shape	Parameter	$C = fRe_h$
I. Concentric Annulus $D_h = D_2 - D_1$	D_1/D_2	
	0.0001	71.8
	0.01	80.1
	0.1	89.4
	0.6	95.6
	1.00	96.0
	→ 0	→ 64
II. Circular segment $D_h = D\left[1 + \frac{\sin(2\alpha)}{2(\pi - \alpha)}\right]$	α (degrees)	
	0	64.0 Circular
	60	63.3
	90	63.1
	120	62.8
	180	62.2
		
III. Rectangle $D_h = \frac{2ab}{a + b}$	a/b	
	0	96.0
	0.05	89.9
	0.01	84.7
	0.25	72.9
	0.50	62.2
	0.75	57.9
	1.00	56.9
		
IV. Circular sector $D_h = \frac{\alpha}{1 + \alpha} D$	α (degrees)	
	0	48.0
	30	56.7
	60	60.8
	90	63.1
		
V. Right triangle $D_h = \frac{2\, b \sin\alpha}{(1 + \sin\alpha + \cos\alpha)}$	α (degrees)	
	0	48.0
	10	49.9
	20	51.2
	30	52.0
	40	52.4
	45	52.5
		

EXAMPLE CALCULATIONS

BASIC CALCULATION TYPES

TYPE	GEOMETRY $D, \ell, \varepsilon/D$	FLOWRATE q or V	PRESSURE DROP Δp or h_L
I	—	—	Determine
II	—	Determine	—
III	Determine	—	—

Iterative soln. (II, III) since

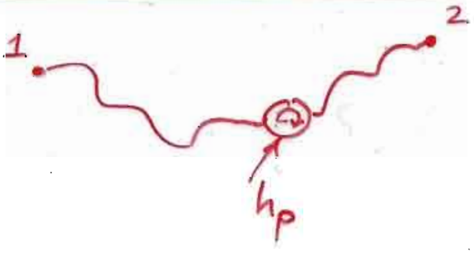
$f = \phi[Re]$?

$Re = \frac{\rho V D}{\mu}$?

All other parameters given/known

BASIC EQUATIONS

①
$$\boxed{\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L}$$



$\alpha_1 = 1$ for turbulent flow

h_p = head provided by pumps

h_L = head loss

②

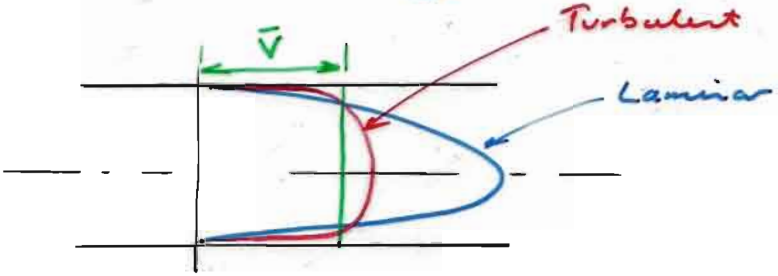
$$h_L = \sum f \frac{\ell}{D} \frac{V^2}{2g}$$ for "major" loss pipe sections

$$h_L = \sum K_L \frac{V^2}{2g}$$ for bends, elbows, valves etc.... "minor" losses

$$\boxed{f = \phi\left(Re ; \frac{\varepsilon}{D}\right)}$$

Recall 'α'

α is a correction factor to allow for changes in "kinetic" energy in pipe flows (and other flows).



$$\alpha = \frac{\int_A (V^2/2)\,\rho\,\underline{V}\cdot\hat{n}\,dA}{\dot{m}\,\bar{V}^2/2}$$

Average velocity, $\bar{V}$

Correction factors: Experimentally: $\alpha = 1 + 2.7f$

For uniform flow $\alpha = 1$

For transitional flow $\alpha \neq 1$

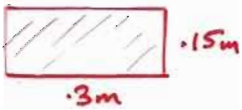
but effect is negligibly small for pipe flows where Δp is typically more important.

Not necessarily neglected in open channel flow.

TYPE I PRESSURE DROP

8.75

8.75 Air flows through a rectangular galvanized iron duct of size 0.30 m by 0.15 m at a rate of 0.068 m³/s. Determine the head loss in 12 m of this duct.



$\nu = \frac{\mu}{\cancel{g}\rho}$

$h_L = f\frac{\ell}{D_h}\frac{V^2}{2g}$, where $D_h = \frac{4A}{P} = \frac{4(0.3m)(0.15m)}{2[0.3m+0.15m]} = 0.2\ m$

and

$V = \frac{Q}{A} = \frac{0.068\frac{m^3}{s}}{(0.3m)(0.15m)} = 1.51\frac{m}{s}$ Also, $Re_h = \frac{VD_h}{\nu} = \frac{(1.51\frac{m}{s})(0.2m)}{1.46\times10^{-5}\frac{m^2}{s}} = 20{,}700$

and from Table 8.2,

$\frac{\varepsilon}{D_h} = \frac{0.15\times10^{-3}m}{0.2\,m} = 7.5\times10^{-4}$ Hence, from Fig. 8.23 $f = 0.027$

so that

$$h_L = (0.027)\left(\frac{12\,m}{0.2\,m}\right)\frac{(1.51\frac{m}{s})^2}{2(9.81\frac{m}{s^2})} = \underline{\underline{0.188m}}$$

8.76

8.76 Air at standard conditions flows through a horizontal 1 ft by 1.5 ft rectangular wooden duct at a rate of 5000 ft³/min. Determine the head loss, pressure drop, and power supplied by the fan to overcome the flow resistance in 500 ft of the duct.

$h_L = f\frac{\ell}{D_h}\frac{V^2}{2g}$, where $V = \frac{Q}{A} = \frac{(5000\frac{ft^3}{min})(\frac{1\,min}{60\,s})}{(1\,ft)(1.5\,ft)} = 55.6\frac{ft}{s}$

and $D_h = \frac{4A}{P} = \frac{4(1\,ft)(1.5\,ft)}{2[1\,ft+1.5\,ft]} = 1.2\ ft$

Also, $Re_h = \frac{VD_h}{\nu} = \frac{(55.6\frac{ft}{s})(1.2\,ft)}{1.57\times10^{-4}\frac{ft^2}{s}} = 4.25\times10^5$ and from Table 8.2

$\varepsilon \approx 0.0006\ ft$ to $0.003\ ft$. Use an "average" $\varepsilon = 0.0018$ ft so that

$\frac{\varepsilon}{D_h} = \frac{0.0018\ ft}{1.2\,ft} = 0.0015$ Thus, from Fig. 8.23 $f = 0.022$, or

$$h_L = (0.022)\left(\frac{500\ ft}{1.2\,ft}\right)\frac{(55.6\frac{ft}{s})^2}{2(32.2\frac{ft}{s^2})} = \underline{\underline{440\ ft}}$$

For this horizontal pipe $\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$, where $z_1 = z_2$ and $V_1 = V_2$.

Thus, $p_1 - p_2 = \gamma h_L = (7.65\times10^{-2}\frac{lb}{ft^3})(440ft) = 33.7\frac{lb}{ft^2} = \underline{\underline{0.234psi}}$

$\mathcal{P} = \gamma Q h_L = Q(p_1 - p_2) = (5000\frac{ft^3}{min})(\frac{1\,min}{60\,s})(33.7\frac{lb}{ft^2}) = (2810\frac{ft\cdot lb}{s})\left[\frac{1\,hp}{(550\frac{ft\cdot lb}{s})}\right]$

or

$$\mathcal{P} = \underline{\underline{5.11\ hp}}$$

[11:2] Pipe Flow

Outline

Examples

- II Flow rate
- III Geometry

TYPE I

8.77

8.77 When the valve is closed the pressure throughout the horizontal pipe shown in Fig. P8.77 is 400 kPa, and the water level in the closed surge chamber is $h = 0.4$ m. If the valve is fully opened and the pressure at point (1) remains 400 kPa, determine the new level of the water in the surge chamber. Assume the friction factor is $f = 0.02$ and the fittings are threaded fittings.

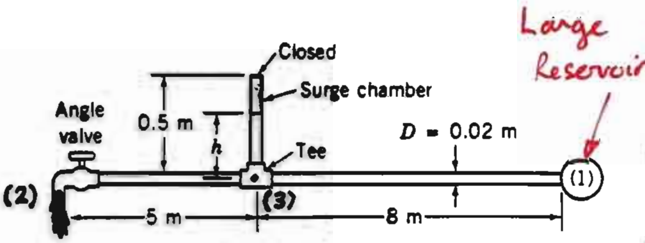


FIGURE P8.77

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+\left(f\frac{\ell}{D}+\Sigma K_L\right)\frac{V^2}{2g}\text{, where } z_1=z_2,\ V_1=0,\ p_2=0$$

Thus, $\frac{p_1}{\gamma}=\frac{V_2^2}{2g}+\left(f\frac{\ell}{D}+\Sigma K_L\right)\frac{V^2}{2g}$ with $V=V_2$.

With $K_L=2$ for an angle valve and $K_L=0.9$ for the tee (see Table 8.3) we obtain

$$\frac{400\frac{kN}{m^2}}{9.80\frac{kN}{m^3}}=\frac{1}{2(9.81\frac{m}{s^2})}\left[1+(0.02)\left(\frac{(8+5)m}{0.02\,m}\right)+2+0.9\right]V^2$$

Find V

or $\boxed{V=6.88\frac{m}{s}}$

Thus, p_3 is determined from

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_3}{\gamma}+\frac{V_3^2}{2g}+z_3+\left(f\frac{\ell}{D}+\Sigma K_L\right)\frac{V^2}{2g}\text{, where } z_1=z_3 \text{ and } V_1=0$$

Also, $V_3=V$ Hence,

Could use $K_L=\frac{1}{2}(0.9)$

$$\frac{p_1}{\gamma}=\frac{p_3}{\gamma}+\left(1+f\frac{\ell}{D}+\Sigma K_L\right)\frac{V^2}{2g}\text{, where } \ell=8m \text{ and } K_L=0$$

Thus,

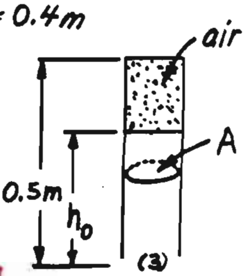
$$p_3=p_1-\left(1+f\frac{\ell}{D}\right)\frac{V^2}{2g}\gamma=p_1-\left(1+f\frac{\ell}{D}\right)\frac{1}{2}\rho V^2$$

$$=400\,kP-\left(1+(0.02)\left(\frac{8m}{0.02m}\right)\right)\frac{1}{2}\left(999\frac{kg}{m^3}\right)\left(6.88\frac{m}{s}\right)^2=400\,kP-2.13\times10^5\frac{N}{m^2}$$

Find p_3

or $\boxed{p_3=187\,kPa}$

Thus, $p_3=400\,kPa$ with the valve closed when $h=0.4m$ and $p_3=187\,kPa$ with the valve open and $h=h_o$



M = mass of air in surge chamber $=\rho\forall$ = constant, where $\forall=A(0.5m-h)$ and $p=\rho RT$, or $\rho=\frac{p}{RT}$

Thus, with $(\ )_c$ denoting the closed valve condition,

$\rho\forall=\frac{P}{RT}\forall$

Assume T = constant (Isothermal).

$\rho_c\forall_c=\rho_o\forall_o$, or $\boxed{\frac{p_c}{RT_c}A(0.5-0.4)=\frac{p_o}{RT_o}A(0.5-h_o)}$ Assume $T_c=T_o$

or

$0.1\,p_c=(0.5-h_o)\,p_o$, where $\boxed{p_c=400kPa-\gamma h_c}=400kPa-9.80\frac{kN}{m^3}(0.4m)=396\,kPa$

and $p_o=187kPa-\gamma h_o=187kPa-9.80\frac{kN}{m^2}h_o=(187-9.8h_o)\,kPa$

Thus, $0.1(396)=(0.5-h_o)(187-9.8h_o)$, or $\boxed{h_o=0.285\ m}$

TYPE II - DETERMINE FLOW RATE

8.114

8.114 The pump shown in Fig. P8.114 adds 25 kW to the water and causes a flowrate of 0.04 m³/s. Determine the flowrate expected if the pump is removed from the system. Assume $f = 0.016$ for either case and neglect minor losses.

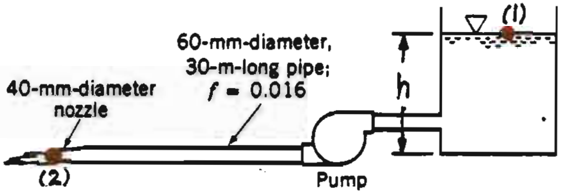


FIGURE P8.114

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+\overset{?\;\;h.}{\boxed{z_1}}+\underline{h_p}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+f\frac{\ell}{D}\frac{V^2}{2g}$$, where $p_1=p_2=0$, $z_1=h$, $z_2=0$,

$V_1=0$, $V_2=\frac{Q}{A_2}=\frac{0.04\,\frac{m^3}{s}}{\frac{\pi}{4}(0.04m)^2}=\underset{②}{\underline{31.8\,\frac{m}{s}}}$, $V=\frac{Q}{A}=\frac{0.04\,\frac{m^3}{s}}{\frac{\pi}{4}(0.06m)^2}=\underset{①}{\underline{14.15\,\frac{m}{s}}}$

Thus,

$$h+h_p=\frac{(31.8\,\frac{m}{s})^2}{2(9.81\,\frac{m}{s^2})}+0.016\left(\frac{30m}{0.06m}\right)\frac{(14.15\,\frac{m}{s})^2}{2(9.81\,\frac{m}{s^2})}=133.2\,m$$

but,

$$h_p=\frac{P}{\gamma Q}=\frac{25\times10^3\,\frac{N\cdot m}{s}}{(9.80\times10^3\,\frac{N}{m^3})(0.04\,\frac{m^3}{s})}=63.8\,m$$

[$h_p=\frac{w}{g}$; $w=\frac{\dot{W}}{\dot{m}}$]

Hence,

$$h=133.2m-63.8\,m=69.5m$$

$\boxed{h=69.5m}$

Without the pump $h_p=0$ and $z_1=\frac{V_2^2}{2g}+f\frac{\ell}{D}\frac{V^2}{2g}$ where $h=69.5m=z_1$

and

$$V_2=\frac{AV}{A_2}=\left(\frac{D}{D_2}\right)^2V \quad\text{or}\quad V_2=\left(\frac{60mm}{40mm}\right)^2V=2.25V$$

Thus,

$$69.5\,m=\frac{(2.25V)^2+0.016\left(\frac{30m}{0.06m}\right)V^2}{2(9.81\,\frac{m}{s^2})}\quad\text{or}\quad V=10.22\,\frac{m}{s}$$

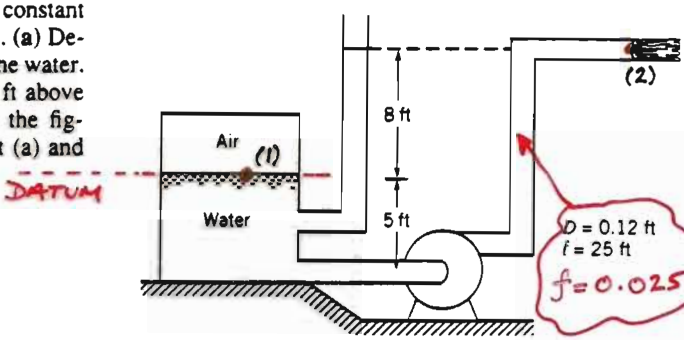
so that

$$Q=AV=\frac{\pi}{4}(0.06m)^2(10.22\,\frac{m}{s})=\underline{\underline{0.0289\,\frac{m^3}{s}}}$$

TYPE II — DETERMINE FLOWRATE

8.117

8.117 Water is pumped from a large, closed, pressurized tank as shown in Fig. P8.117. The friction factor for the constant diameter pipe is 0.025, and minor losses are negligible. (a) Determine the flowrate if the pump adds 1 horsepower to the water. (b) Repeat the problem if the outlet of the pipe is 10 ft above the air–water interface rather than 8 ft as shown in the figure. Comment on the differences between that of Part (a) and Part (b).



■ FIGURE P8.117

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\overset{?}{\frac{V_2^2}{2g}}+z_2+f\frac{\ell}{D}\overset{?}{\frac{V^2}{2g}} \qquad (1)$$

1 hp ≐ 550 ft·lb/s

where $\frac{p_1}{\gamma}=8\text{ ft}$, $z_1=0$, $V_1=0$, $p_2=0$, $z_2=8\text{ ft}$, $V=V_2$, and $\mathcal{P}=\text{power}=1\text{hp}$ so that

$$h_p=\frac{\mathcal{P}}{\gamma Q}=\frac{\mathcal{P}}{\gamma A V}=\frac{(1\text{ hp})(550\frac{\text{ft}\cdot\text{lb}}{\text{s}\cdot\text{hp}})}{(62.4\frac{\text{lb}}{\text{ft}^3})\frac{\pi}{4}(0.12\text{ft})^2V}=\frac{779}{V}\text{ ft, where } V\sim\frac{\text{ft}}{\text{s}}$$

Thus, Eq. (1) becomes

$$\frac{779}{V}\text{ ft}=\frac{V^2}{2(32.2\frac{\text{ft}}{\text{s}^2})}\left[1+0.025\left(\frac{25\text{ ft}}{0.12\text{ ft}}\right)\right]\text{, or } V=20.1\frac{\text{ft}}{\text{s}}$$

Hence,

$$Q=AV=\frac{\pi}{4}(0.12\text{ft})^2(20.1\frac{\text{ft}}{\text{s}})=\underline{\underline{0.227\frac{\text{ft}^3}{\text{s}}}}$$

If the pipe outlet is raised to 10 ft, the Eq. (1) becomes

$$8+\frac{779}{V}=10+\left[1+0.025\left(\frac{25}{0.12}\right)\right]\frac{V^2}{64.4}$$

or $0.0964V^3+2V-779=0$

The only real, positive root is $V=19.72\frac{\text{ft}}{\text{s}}$ so that

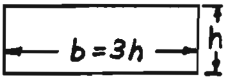
$$Q=AV=\frac{\pi}{4}(0.12\text{ft})^2(19.72\frac{\text{ft}}{\text{s}})=\underline{\underline{0.223\frac{\text{ft}^3}{\text{s}}}}$$

Note that with $z_1 \neq z_2$ it was necessary to solve a cubic equation. With $z_1=z_2$ the equation is a quadratic equation.

TYPE III DETERMINE GEOMETRY

8.89

8.89 Air is to flow through a smooth horizontal rectangular duct at a rate of 100 m³/s with a pressure drop of not more than 40 mm of water per 50 m of duct. If the aspect ratio (width to height) is 3 to 1, determine the size of the duct.



$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+f\frac{\ell}{D_h}\frac{V^2}{2g}$, where $z_1=z_2$ and $V_1=V_2=V$ (1)

with

$V=\frac{Q}{A}=\frac{100\frac{m^3}{s}}{(3h)h}=\frac{100}{3h^2}$, where $h \sim m$, $V \sim \frac{m}{s}$

Also, $D_h=\frac{4A}{P}=\frac{4(3h^2)}{2(3h+h)}=1.5h$

Hence, from Eq. (1)

$p_1-p_2=f\frac{\ell}{D_h}\frac{1}{2}\rho V^2$, or with $p_1-p_2=\gamma_{H_2O}H=9800\frac{N}{m^3}(0.04m)=392\frac{N}{m^2}$,

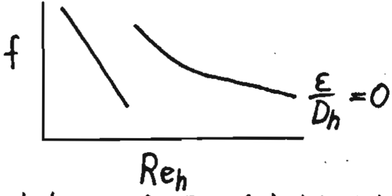
$392\frac{N}{m^2}=f\left(\frac{50m}{1.5h}\right)\left(\frac{1}{2}\right)\left(1.23\frac{kg}{m^3}\right)\left(\frac{100}{3h^2}\frac{m}{s}\right)^2$

Thus,

$f=0.0172\,h^5$ (2)

Also, $Re=\frac{VD_h}{\nu}=\frac{\left(\frac{100}{3}h^{-2}\right)(1.5h)}{1.46\times10^{-5}\frac{m^2}{s}}$ or $Re_h=\frac{3.42\times10^6}{h}$ (3)

and from Fig. 8.23



(4)

Trial and error solution of Eqs. (2), (3), (4) for f, Re, and V:

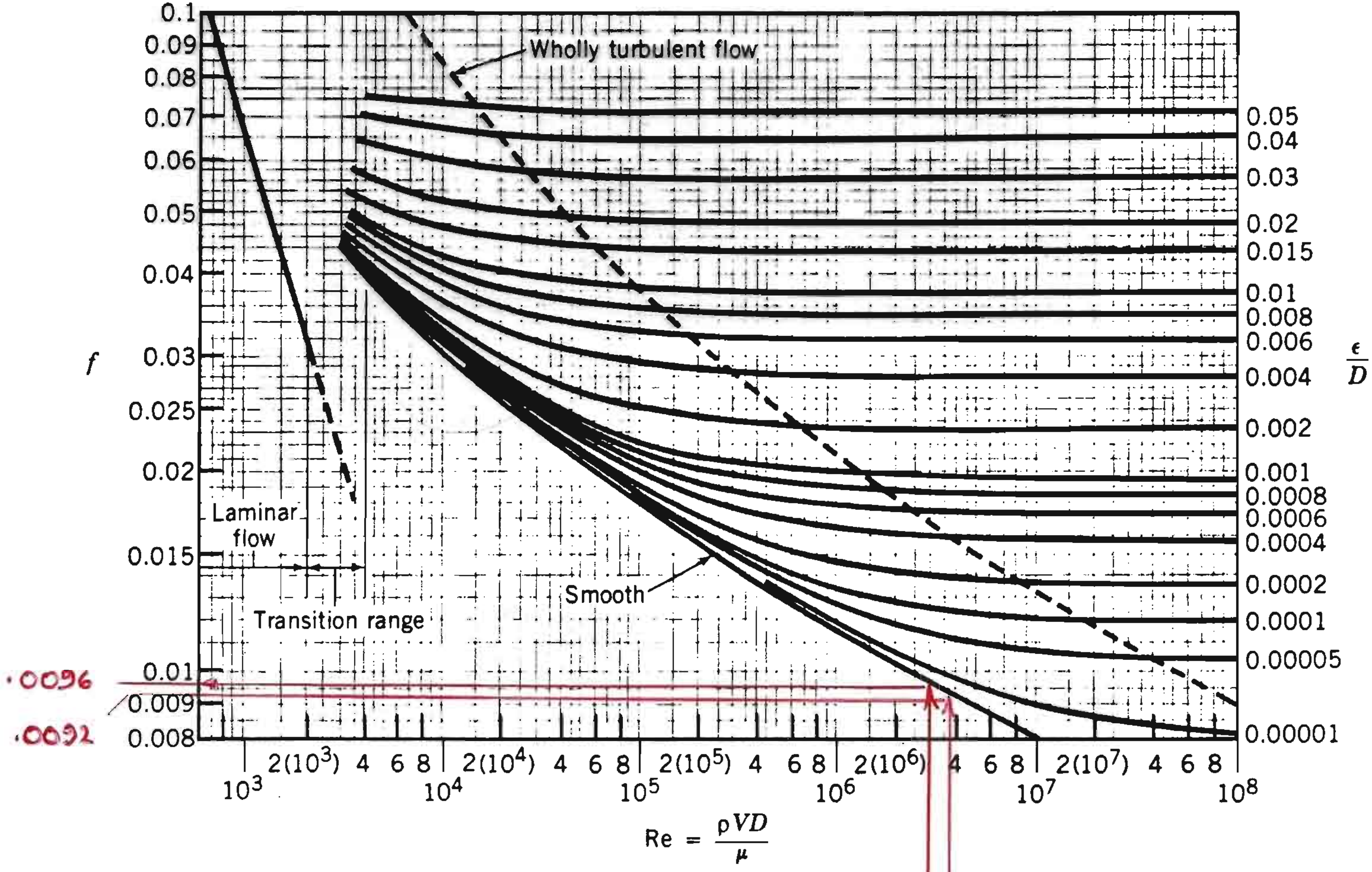
Assume $f=0.02$ so that $0.02=0.0172h^5$ or $h=1.03m$. From Eq. (3),

$Re_h=\frac{3.42\times10^6}{1.03}=3.32\times10^6$ which from Fig. 8.23 gives $f=0.0096\neq0.02$

Assume $f=0.0096$ which gives $h=0.890m$. Thus, $Re_h=3.84\times10^6$ or $f=0.0093\neq0.0096$

Assume $f=0.0093$, or $h=0.884m$. Thus, $Re_h=3.87\times10^6$, of $f=0.0093$ which agrees with the assumed value.

Thus, the duct is $\underline{h=0.884\,m \text{ by } 3h=2.65\,m \text{ in size.}}$



■ **FIGURE 8.23** **Friction factor as a function of Reynolds number and relative roughness for round pipes—the Moody chart (Data from Ref. 7 with permission).**

$Re = 3.84 \times 10^6$

3.4×10^6

[11:3] Pipe Flow

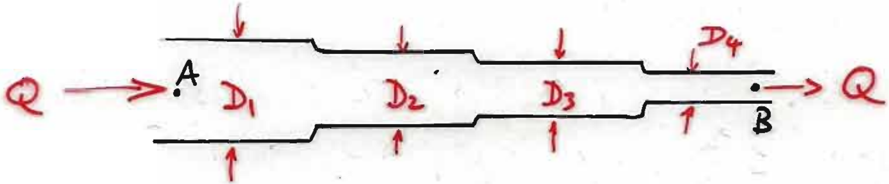
Outline

Pipe networks

Flowrate measurement

MULTIPLE PIPE SYSTEMS

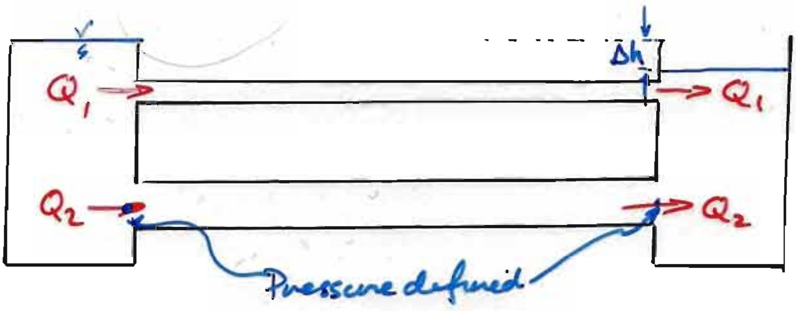
SERIES



$$Q_1 = Q_2 = Q_3 = Q_4 = Q$$

$$h_{L_{A-B}} = h_{L_1} + h_{L_2} + h_{L_3} + h_{L_4}$$

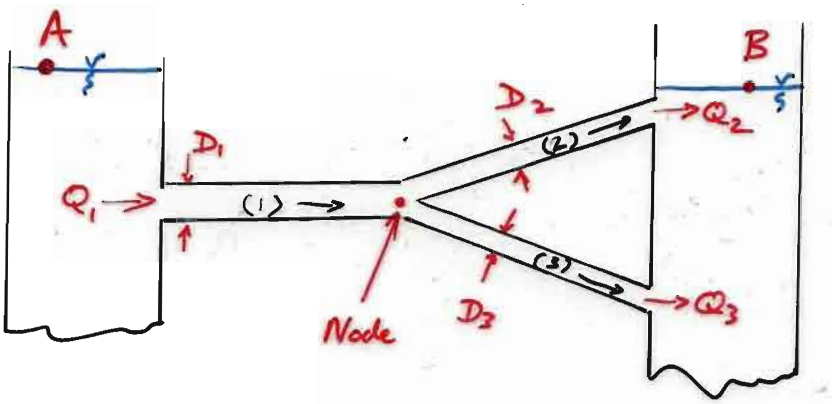
PARALLEL



$$Q = Q_1 + Q_2 - - - -$$

$$h_{L_1} = h_{L_2} = - - - \cdot \text{etc.} = \Delta h$$

PIPE "LOOP" SYSTEMS



Pipes (1) and (2):

$$\frac{P_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{P_B}{\gamma} + \frac{V_B^2}{2g} + z_B + h_{L_1} + h_{L_2} \quad ①$$

Pipes (1) and (3):

$$\frac{P_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{P_B}{\gamma} + \frac{V_B^2}{2g} + z_B + h_{L_1} + h_{L_3} \quad ②$$

From these, the equations are identical ∴ $\boxed{h_{L_2} = h_{L_3}}$

Physically: Energy conditions at Node are a single value $\underline{HGL_N}$?
Flowing to final energy in tank $\underline{B}$ they are
also at the same energy (but different
from (HGL_N)) ie HGL_B.

From (2) energy equations (above) have 3 unknowns V_1, V_2, V_3

need extra equation, continuity

$$Q_1 = Q_2 + Q_3$$

$$V_1 A_1 = V_2 A_2 + V_3 A_3 \quad ③$$

3 equations and 3 unknowns!!

8.125

8.125 The flowrate between tank A and tank B shown in Fig. P8.125 is to be increased by 30% (i.e., from Q to $1.30Q$) by the addition of a second pipe (indicated by the dotted lines) running from node C to tank B. If the elevation of the free surface in tank A is 25 ft above that in tank B, determine the diameter, D, of this new pipe. Neglect minor losses and assume that the friction factor for each pipe is 0.02.

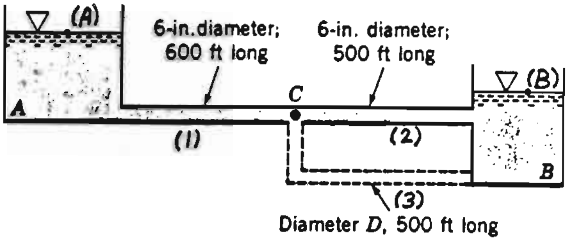


FIGURE P8.125

With the single pipe: $$\frac{p_A}{\gamma}+\frac{V_A^2}{2g}+z_A=\frac{p_B}{\gamma}+\frac{V_B^2}{2g}+z_B+f_1\frac{\ell_1}{D_1}\frac{V_1^2}{2g}+f_2\frac{\ell_2}{D_2}\frac{V_2^2}{2g} \qquad (1)$$

where $p_A=p_B=0$, $V_A=V_B=0$, $z_A=25\text{ ft}$, $z_B=0$,

and $V_1=V_2$ (since $D_1=D_2$).

Thus, $z_A = f_1\frac{(\ell_1+\ell_2)}{D_1}\frac{V_1^2}{2g}$, or $25\text{ ft} = (0.02)\frac{(600+500)\text{ft}}{(\frac{6}{12}\text{ft})}\frac{V_1^2}{2(32.2\frac{\text{ft}}{\text{s}^2})}$

or $V_1 = 6.05\frac{\text{ft}}{\text{s}}$ Hence, $Q=A_1V_1=\frac{\pi}{4}(\frac{6}{12}\text{ft})^2(6.05\frac{\text{ft}}{\text{s}})=1.188\frac{\text{ft}^3}{\text{s}}$ ONE PIPE

$Q_1 = Q_2 + Q_3$

With the second pipe $Q = 1.30(1.188\frac{\text{ft}^3}{\text{s}}) = 1.54\frac{\text{ft}^3}{\text{s}}$

Thus, $Q_1 = 1.54\frac{\text{ft}^3}{\text{s}} = Q_2+Q_3$ or $V_1=\frac{Q_1}{A_1}=\frac{1.54\frac{\text{ft}^3}{\text{s}}}{\frac{\pi}{4}(\frac{6}{12}\text{ft})^2}=7.84\frac{\text{ft}}{\text{s}}$

For fluid flowing from A to B through pipes 1 and 2,

$z_A = h_{L_1} + h_{L_2}$

$$z_A = h_{L_1}+h_{L_2} = f_1\frac{\ell_1}{D_1}\frac{V_1^2}{2g}+f_2\frac{\ell_2}{D_2}\frac{V_2^2}{2g} \quad \text{(see Eq.(1))}$$

or

$$25\text{ ft} = (0.02)\frac{600\text{ ft}}{(\frac{6}{12}\text{ft})}\frac{(7.84\frac{\text{ft}}{\text{s}})^2}{2(32.2\frac{\text{ft}}{\text{s}^2})}+(0.02)\frac{500\text{ft}}{(\frac{6}{12}\text{ft})}\frac{V_2^2}{2(32.2\frac{\text{ft}}{\text{s}^2})}$$

Hence, $V_2 = 2.60\frac{\text{ft}}{\text{s}}$

and

$$Q_2 = A_2V_2 = \frac{\pi}{4}(\frac{6}{12}\text{ft})^2(2.60\frac{\text{ft}}{\text{s}}) = 0.511\frac{\text{ft}^3}{\text{s}}$$

Thus, $Q_3 = Q_1 - Q_2 = 1.54\frac{\text{ft}^3}{\text{s}} - 0.511\frac{\text{ft}^3}{\text{s}} = 1.03\frac{\text{ft}^3}{\text{s}}$

For fluid flowing from A to B through pipes 1 and 3,

$z_A = h_{L_1} + h_{L_3}$

$$z_A = h_{L_1}+h_{L_3} = f_1\frac{\ell_1}{D_1}\frac{V_1^2}{2g}+f_3\frac{\ell_3}{D_3}\frac{V_3^2}{2g}, \text{ where } V_3=\frac{Q_3}{A_3}=\frac{1.03\frac{\text{ft}^3}{\text{s}}}{\frac{\pi}{4}D_3^2}=\frac{1.31}{D_3^2}$$

Thus,

$$25\text{ft} = (0.02)\frac{600\text{ ft}}{(\frac{6}{12}\text{ft})}\frac{(7.84\frac{\text{ft}}{\text{s}})^2}{2(32.2\frac{\text{ft}}{\text{s}^2})}+(0.02)\frac{500\text{ft}}{D_3}\frac{(\frac{1.31}{D_3^2})^2}{2(32.2\frac{\text{ft}}{\text{s}^2})}$$

or

$$\underline{\underline{D_3 = 0.662\text{ft}}}$$

Note: With the parameters given, the solution is quite sensitive to roundoff errors in the calculations

"THREE RESERVOIR" PROBLEM

8.130

8.130 The three tanks shown in Fig. P8.130 are connected by pipes with friction factors of 0.03 for each pipe. Determine the water velocity in each pipe. Neglect minor losses.

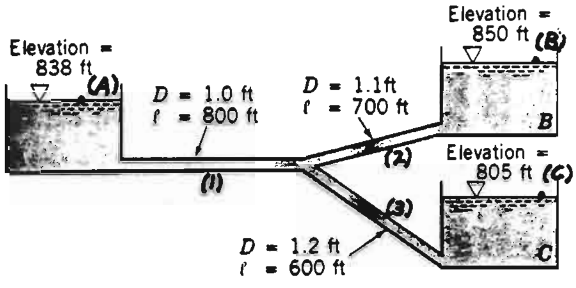


FIGURE P8.130

CONTINUITY

Assume the flow from both tanks A and B is into tank C, or $Q_3 = Q_1 + Q_2$

Thus, $\frac{\pi}{4}D_3^2 V_3 = \frac{\pi}{4}D_1^2 V_1 + \frac{\pi}{4}D_2^2 V_2$, or $1.2^2 V_3 = 1.0^2 V_1 + 1.1^2 V_2$

Hence, $\boxed{V_3 = 0.694\,V_1 + 0.840\,V_2}$ (1)

For the flow from A to C, with $p_A = p_C = 0$, $V_A = V_C = 0$, we obtain

ENERGY A -> C

$$z_A = z_C + f_1 \frac{\ell_1}{D_1}\frac{V_1^2}{2g} + f_3 \frac{\ell_3}{D_3}\frac{V_3^2}{2g}, \text{ or } 838\text{ ft} = 805\text{ ft} + \frac{0.03}{2(32.2\frac{\text{ft}}{\text{s}^2})}\left[\frac{800\text{ft}}{1\text{ft}}V_1^2 + \frac{600\text{ft}}{1.2\text{ft}}V_3^2\right]$$

or

$$\boxed{33 = 0.373\,V_1^2 + 0.233\,V_3^2} \quad (2)$$ (2)

ENERGY B -> C

Similarly for the flow from B to C, with $p_B = p_C = 0$, $V_B = V_C = 0$, we obtain

$$z_B = z_C + f_2 \frac{\ell_2}{D_2}\frac{V_2^2}{2g} + f_3 \frac{\ell_3}{D_3}\frac{V_3^2}{2g}, \text{ or } 850\text{ft} = 805\text{ ft} + \frac{0.03}{2(32.2\frac{\text{ft}}{\text{s}^2})}\left[\frac{700\text{ft}}{1.1\text{ft}}V_2^2 + \frac{600\text{ft}}{1.2\text{ft}}V_3^2\right]$$

or

$$\boxed{45 = 0.296\,V_2^2 + 0.233\,V_3^2} \quad (3)$$ (3)

Thus, 3 equations ((1),(2), and (3)) for V_1, V_2, and V_3. Solve as follows:

Eliminate V_3 →

Subtract (2) from (3) to obtain

$\boxed{12 = 0.296\,V_2^2 - 0.373\,V_1^2}$ (4)

From (2): $V_3 = \sqrt{141.6 - 1.6\,V_1^2}$, or when combined with (1):

$\sqrt{141.6 - 1.6V_1^2} = 0.694\,V_1 + 0.840\,V_2$, or $\boxed{V_2 = \sqrt{200 - 2.27V_1^2} - 0.826\,V_1}$ (5)

Two equations in two unknowns; V_1 &

Combine Eqs. (4) and (5) to obtain:

$$\frac{12}{0.296} = \left[\sqrt{200 - 2.27V_1^2} - 0.826\,V_1\right]^2 - \frac{0.373}{0.296}, \text{ which can be simplified to}$$

$V_1\sqrt{200 - 2.27V_1^2} = 96.5 - 1.725\,V_1^2$ By squaring this equation we (6)

obtain (after simplification):

$$V_1^4 - 101.5\,V_1^2 + 1774 = 0 \quad \text{Hence: } V_1^2 = \frac{101.5 \pm \sqrt{101.5^2 - 4(1774)}}{2} = \begin{matrix}79.1\\ \text{or}\\ 22.4\end{matrix}$$

Check roots →

✓ 4.73 ok.

✗ 8.89 doesn't satisfy (6)

Thus, $\boxed{V_1 = 8.89\,\frac{\text{ft}}{\text{s}} \text{ or } V_1 = 4.73\,\frac{\text{ft}}{\text{s}}}$

Note: The $V_1 = 8.89$ solution is an extra root introduced by squaring Eq. (6). It is not a solution of the original Eqs. (1), (2), (3). For this value, Eq. (6) becomes $8.89\sqrt{200 - 2.27(8.89^2)} \overset{?}{=} 96.5 - 1.725(8.89)^2$, or "$40 = -40$"

Thus $\underline{\underline{V_1 = 4.73\,\frac{\text{ft}}{\text{s}}}}$, from Eq. (2) $V_3 = \left[\frac{33 - 0.373(4.73)^2}{0.233}\right]^{1/2} = \underline{\underline{10.3\,\frac{\text{ft}}{\text{s}}}}$,

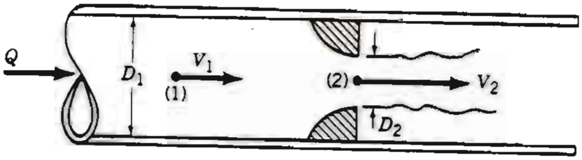
and from Eq. (1) $V_2 = \frac{10.3 - 0.694(4.73)}{0.840} = \underline{\underline{8.35\,\frac{\text{ft}}{\text{s}}}}$

NOTE: If solution of equations is not possible with real positive roots, then change direction of flow assumption.

PIPE FLOWRATE MEASUREMENT

- ☐ Constrict flow and measure pressure difference (Venturi type)
- ☐ Balance forces / momentum (Turbine or suspended plug)
- ☐ Reservoir volume measurement (Bucketfull per-unit-time).

ORIFICE METER



■ **FIGURE 8.43** Typical pipe flowmeter geometry.

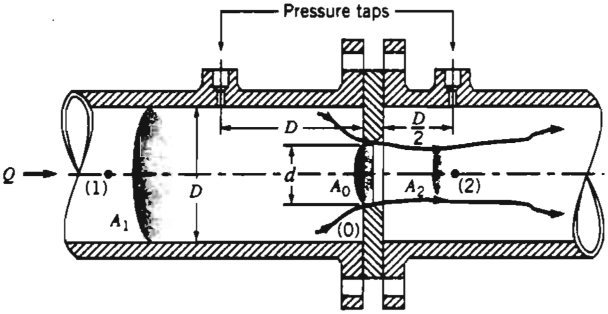
$$Q_{ideal} = A_2V_2 = A_2\sqrt{\frac{2(p_1-p_2)}{\rho(1-\beta^4)}}$$

$$\beta = \frac{D_2}{D_1}$$

But viscous head losses generate an effect

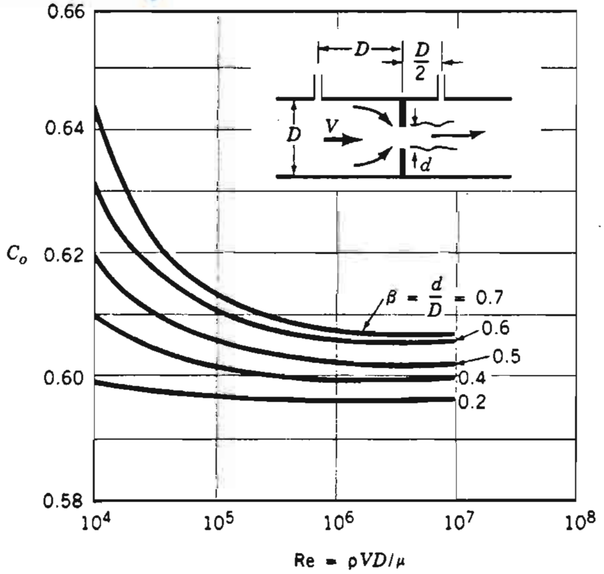
$$Q = C_o Q_{ideal}$$

$$Q = C_oA_o\sqrt{\frac{2(p_1-p_2)}{\rho(1-\beta^4)}}$$



■ **FIGURE 8.44** Typical orifice meter construction.

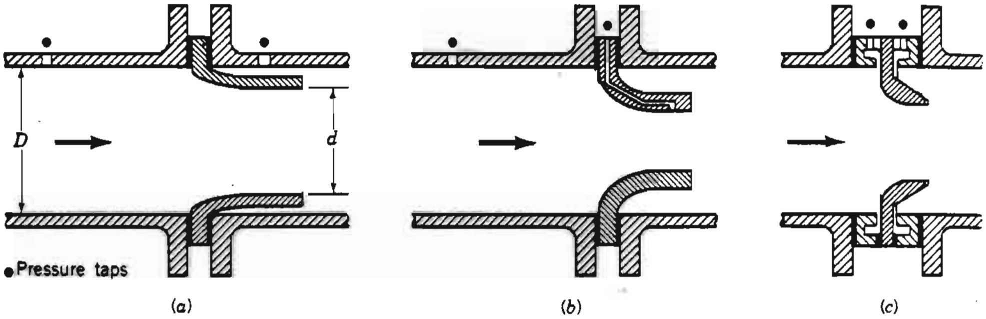
i.e. function[Geometry, Re]



■ **FIGURE 8.45** Orifice meter discharge coefficient (Ref. 24).

Nozzle orifice is closer to "ideal" than orifice mete as it reduces viscous losses and prevents separation.

NOZZLE METERS



■ **FIGURE 8.46** Typical nozzle meter construction.

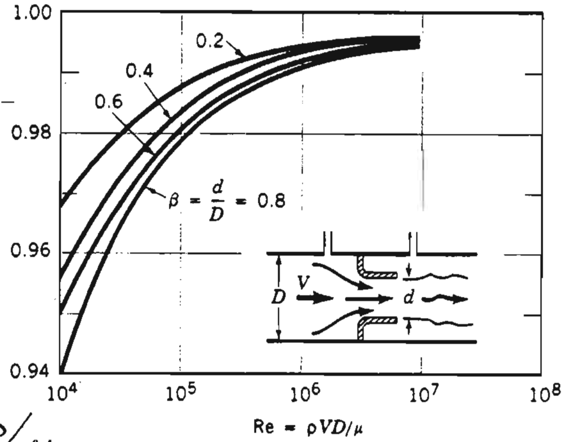
$Q = C_n Q_{ideal}$

$= C_n A_n \sqrt{\dfrac{2(p_1 - p_2)}{\rho(1-\beta^4)}}$

Note, since closer to ideal, $C_n \rightarrow 1$!!

$A_n = \dfrac{\pi d^2}{4}$

$\beta = d/D \qquad Re = \rho VD/\mu$

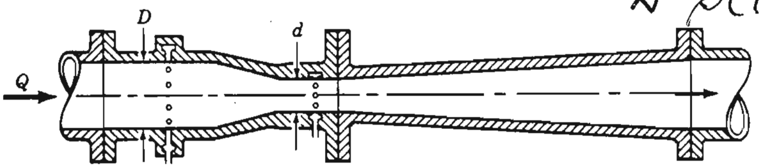


■ **FIGURE 8.47** Nozzle meter discharge coefficient (Ref. 24).

VENTURI METERS

$Q = C_v Q_{ideal} = C_v A_T \sqrt{\dfrac{2(p_1 - p_2)}{\rho(1-\beta^4)}}$

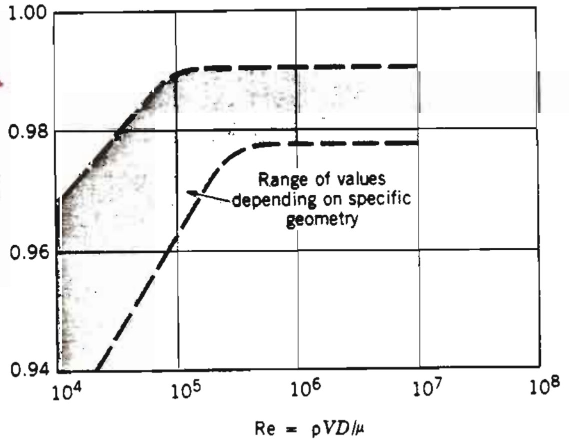
$A_T = \dfrac{\pi d^2}{4}$ = throat area



■ **FIGURE 8.48** Typical Venturi meter construction.

CLOSEST TO IDEAL

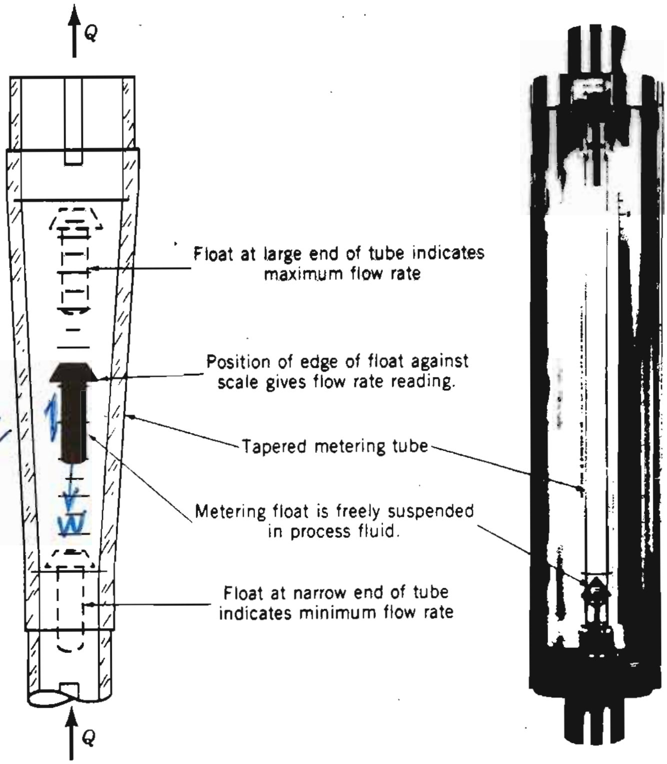
.04 range !!



■ **FIGURE 8.49** Venturi meter discharge coefficient (Ref. 23).

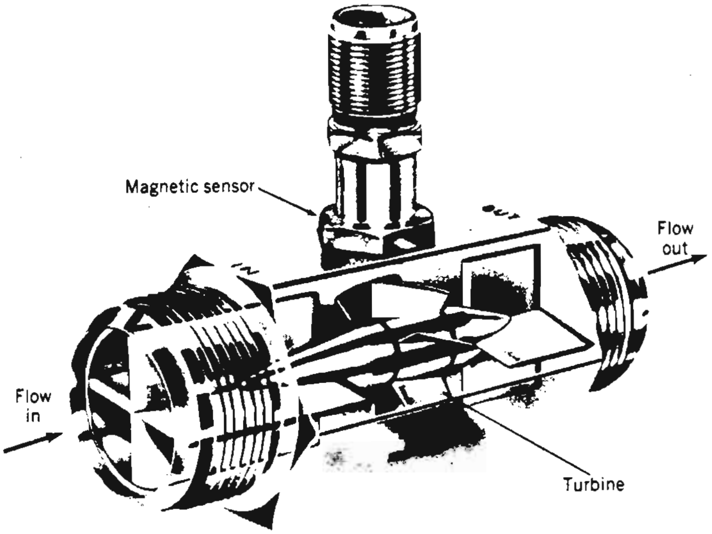
FORCE - BALANCE FLOWMETERS

LINEAR MOMENTUM



■ **FIGURE 8.50**
Rotameter-type flowmeter. (Courtesy of Fischer & Porter Co.).

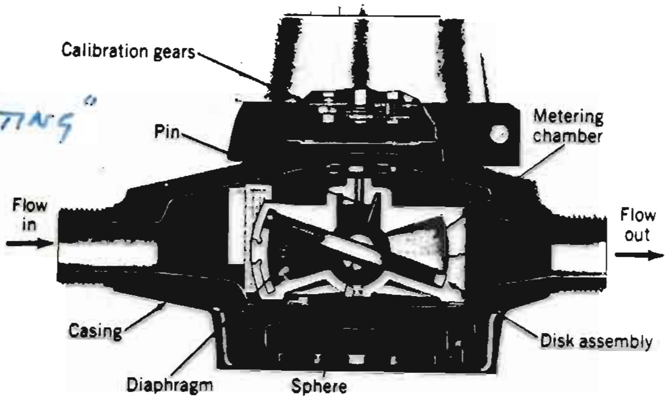
MOMENT-OF-MOMENTUM



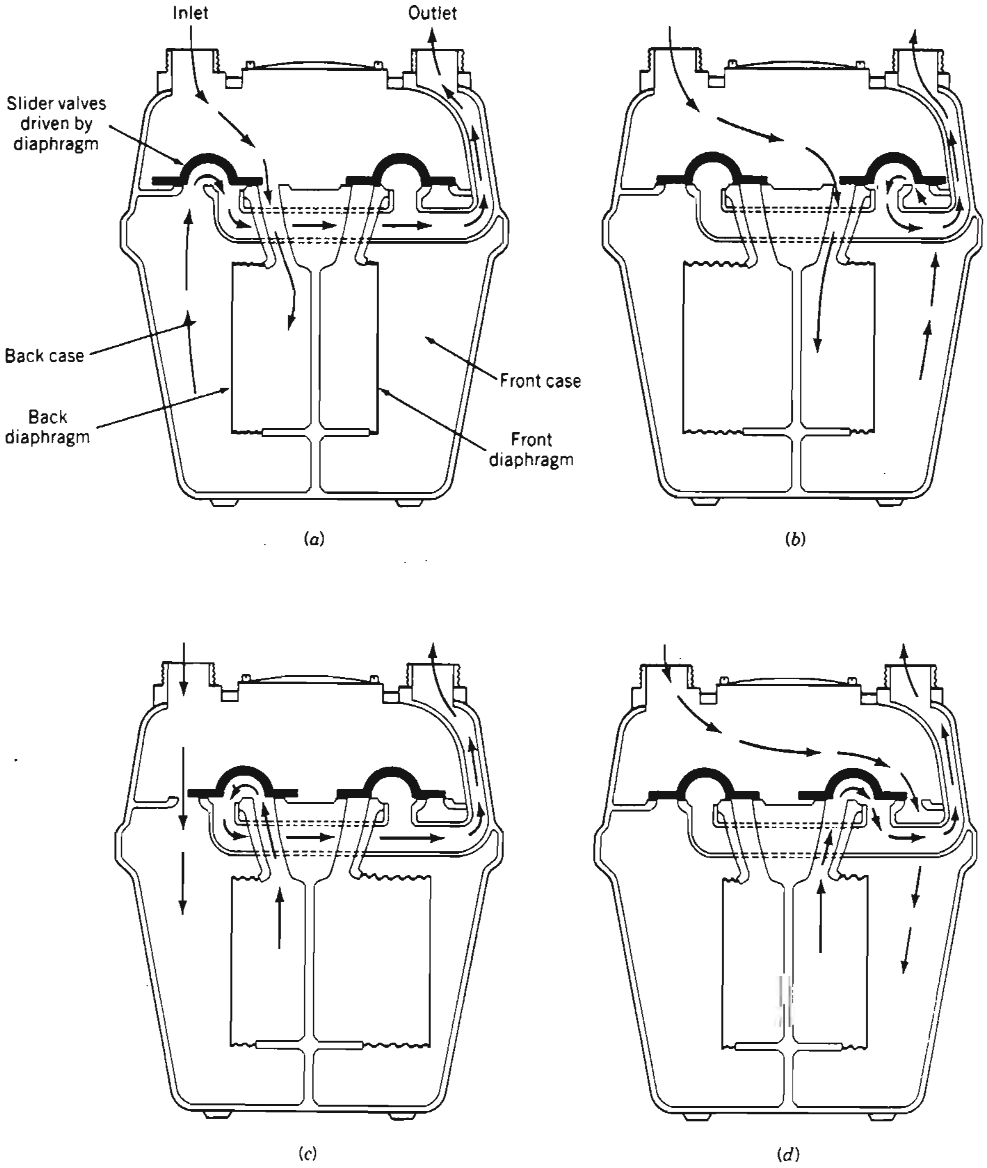
■ **FIGURE 8.51**
Turbine-type flowmeter. (Courtesy of E G & G Flow Technology, Inc.).

VOLUME FLOWMETERS

GAS-PUMP – "NUTATING" FLOWMETER



■ **FIGURE 8.52** Nutating disk flowmeter. (Courtesy of Badger Meter, Inc.).



■ **FIGURE 8.53** Bellows-type flowmeter. (Courtesy of BTR—Rockwell Gas Products). (*a*) Back case emptying, back diaphragm filling. (*b*) Front diaphragm filling, front case emptying. (*c*) Back case filling, back diaphragm emptying. (*d*) Front diaphragm emptying, front case filling.

[12]

External Flows

External Flows [12]

$$D = \int dF_x = \int p\cos\theta dA + \int \tau_w \sin\theta dA$$

$$L = \int dF_y = -\int p\sin\theta dA + \int \tau_w \cos\theta dA$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}; \qquad C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

[12:1] External Flows

Outline

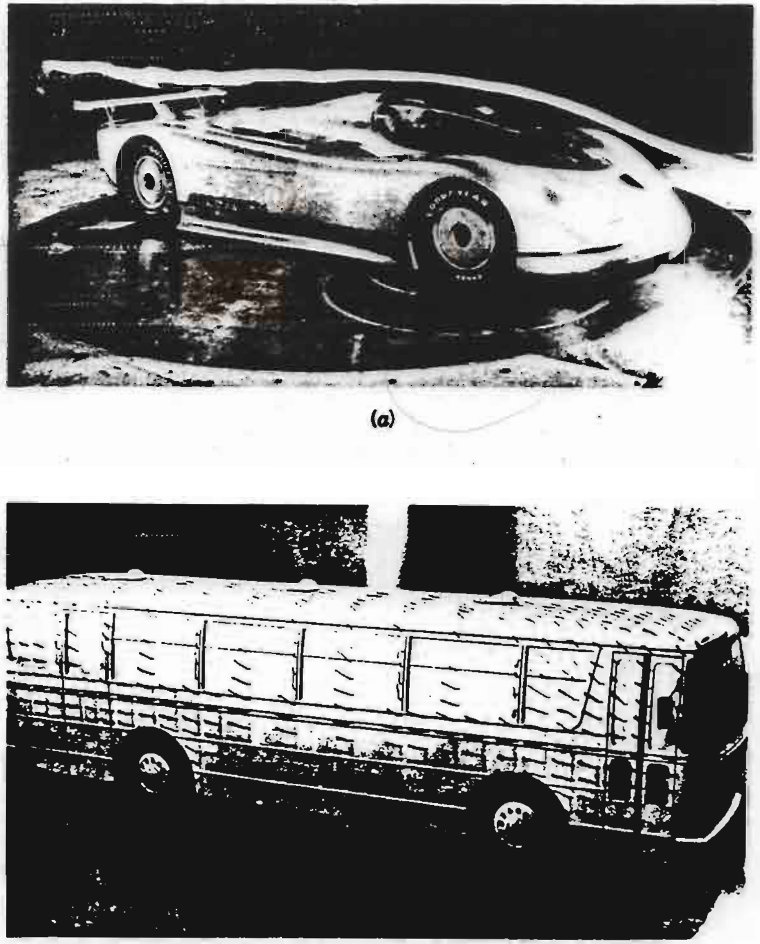
Drag and Lift

$$D = \int dF_x = \int p \cos\theta dA + \int \tau_w \sin\theta dA$$

$$L = \int dF_y = -\int p \sin\theta\, dA + \int \tau_w \cos\theta dA$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}; \qquad C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

EXTERNAL FLOWS



(a)

(b)

■ FIGURE 9 (*a*) Flow past a ful sized streamlined hicle in the GM ae dynamics laborato wind tunnel, an 1 by 34-ft test sectio facility driven by 4000-hp, 43-ft dia ter fan. (Photogra courtesy of Gener Motors Corporati (*b*) Surface flow o model vehicle as i cated by tufts attached to the surf (Reprinted with p mission from Soci of Automotive En neers, Ref. 28.)

GENERAL ASPECTS

Applied Forces –
- ☐ Lift 
- ☐ Drag 

Evaluate Forces

- ☐ Analytical means
  - – equations
- ☐ Dimensional analyses

↓

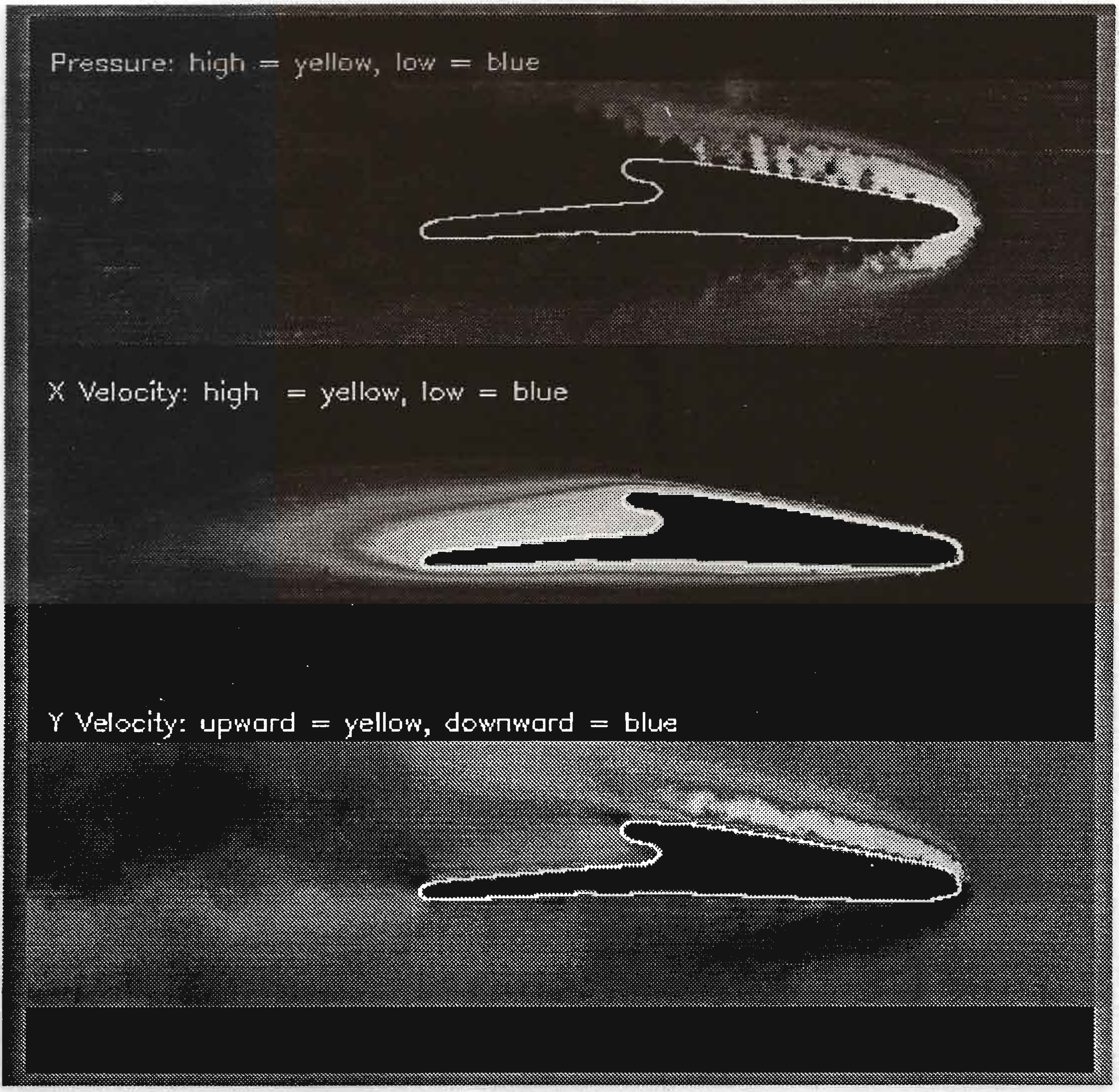
Physical models.

Applications:

- ☐ Vehicle drag
- ☐ Flight structures
- ☐ Sedimentation
- ☐ Buildings
- ☐ Tacoma Narrows Br.

AIRFOIL

- External flow is uniform away from structure
- Static object and moving air/fluid velocity
- Boundary layer develops @ solid/fluid boundary.
  (allows external flow to neglect viscous effects)



- Wind tunnel & tow tank ⟶
  evaluate drag/lift forces.

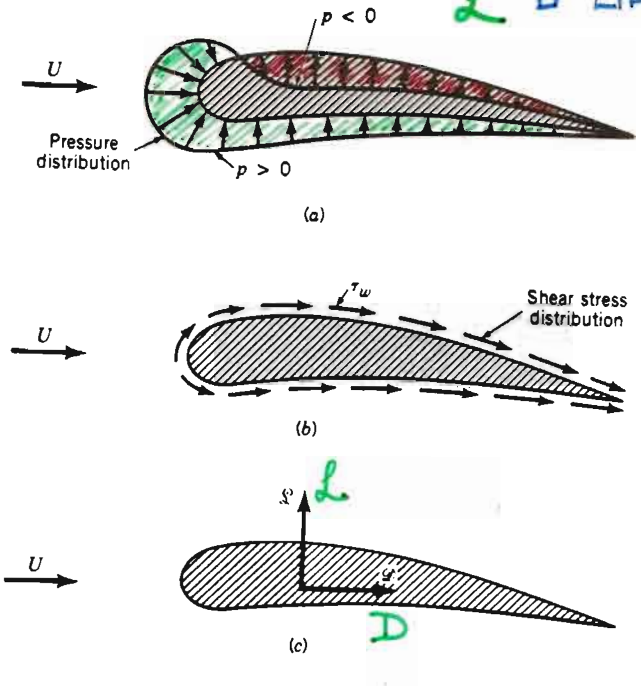
LIFT & DRAG

$\mathcal{D}$ ☐ DRAG – FORCE TANGENTIAL TO FLOW.

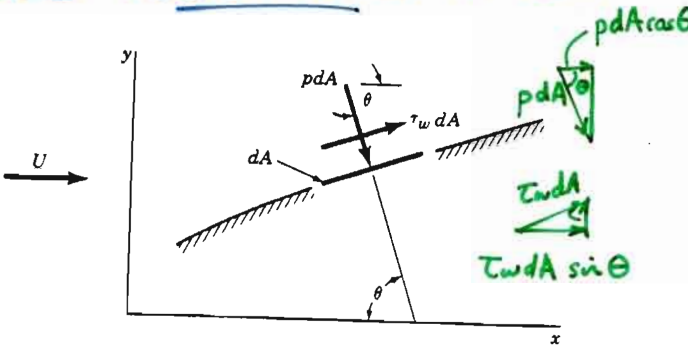
☐ SHEAR DRAG

☐ PRESSURE DRAG

$\mathcal{L}$ ☐ LIFT – FORCE PERPENDICULAR TO FLOW.



■ FIGURE 9.3 Forces from the surrounding fluid on a two-dimensional object: (a) pressure force, (b) viscous force, (c) resultant force (lift and drag).



DRAG:
$$dF_x = (p\,dA)\cos\theta + (\tau_w\,dA)\sin\theta$$

$$\boxed{\mathcal{D} = \int dF_x = \int p\cos\theta\,dA + \int \tau_w \sin\theta\,dA}$$

LIFT:
$$dF_y = -(p\,dA)\sin\theta + (\tau_w\,dA)\cos\theta$$

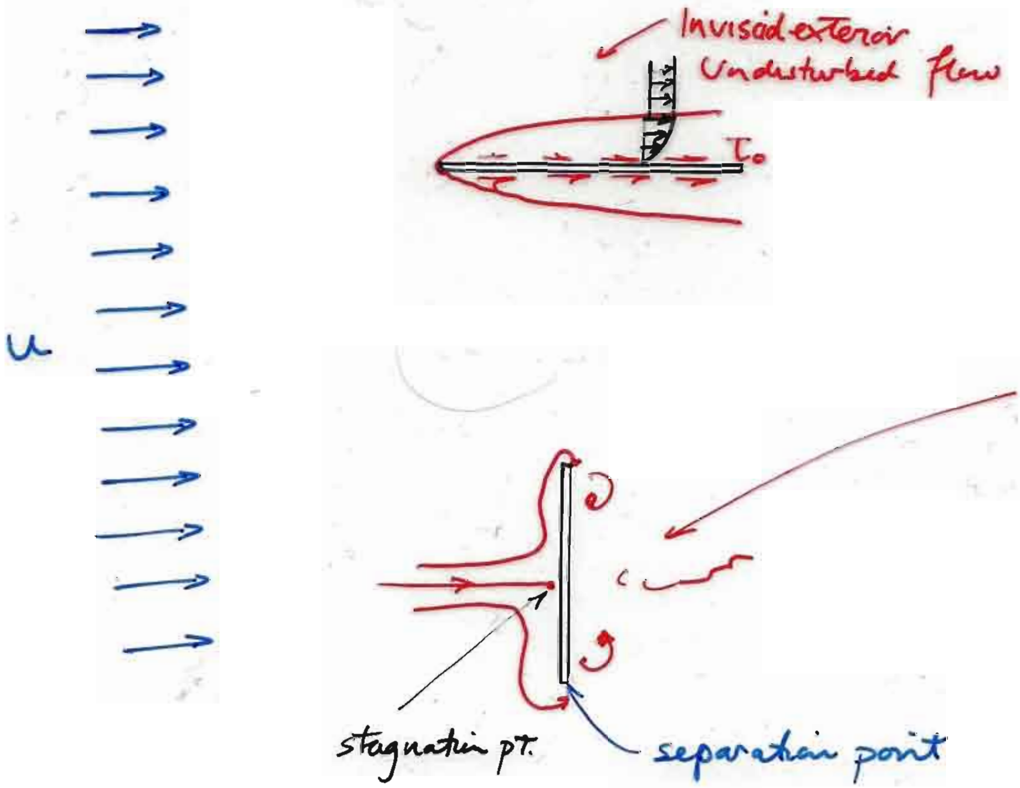
$$\boxed{\mathcal{L} = \int dF_y = -\int p\sin\theta\,dA + \int \tau_w\cos\theta\,dA}$$

NOTE: p and τ_w difficult to determine for 'real' shapes ∴ experimentally determined.

Exception is flat plate.

FORCES ON IMMERSED BODIES

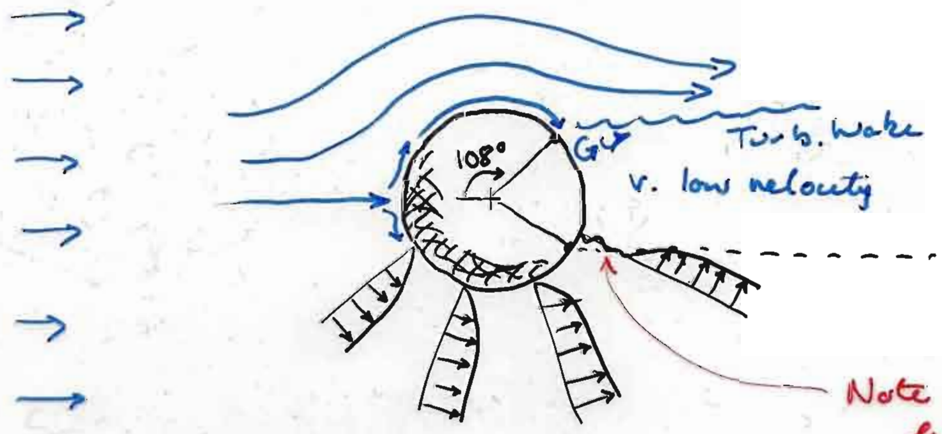
Influenced by flow direction:



Friction Drag Predominates

Pressure Drag Predominates.

Separation pt. has large influence on drag:
eg. for flow around cylinder



Note stagnant flow ∴ no velocity

$\therefore \ \tau = \mu \cancel{\frac{du}{dy}}^{0}$

[12:2] External Flows

Outline

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}; \qquad C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

- Drag
  - Evaluation
  - Re dependence
    - Friction drag
    - Pressure drag
  - Shape dependence
  - Evaluating motion
  - Composite drag
- Lift
  - Evaluation
  - Rotating objects

EXAMPLE 9.1

Air at standard conditions flows past a flat plate as is indicated in Fig. E9.1. In case (a) the plate is parallel to the upstream flow, and in case (b) it is perpendicular to the upstream flow. If the pressure and shear stress distributions on the surface are as indicated (obtained either by experiment or theory), determine the lift and drag on the plate.

NOTE:
Need p and τ_w distribution. Unlikely avail in reality.

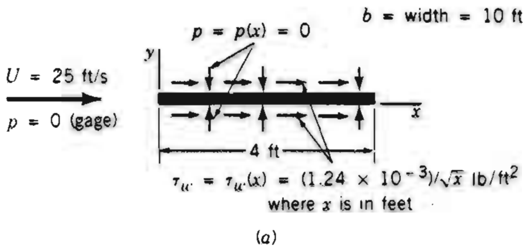


FIGURE E9.1

$\mathcal{L}=0$ since $P_{top}=P_{bot}$.

$D\neq0$ due to shear drag

SOLUTION

For either orientation of the plate, the lift and drag are obtained from Eqs. 9.1 and 9.2. With the plate parallel to the upstream flow we have $\theta = 90°$ on the top surface and $\theta = 270°$ on the bottom surface so that the lift and drag are given by

$$\mathscr{L} = -\int_{\text{top}} p\, dA + \int_{\text{bottom}} p\, dA = 0$$

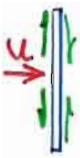
and

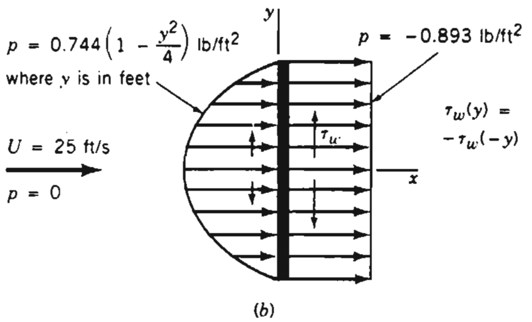
$$\mathscr{D} = \int_{\text{top}} \tau_w\, dA + \int_{\text{bottom}} \tau_w\, dA = 2\int_{\text{top}} \tau_w\, dA \qquad (1)$$

Shear drag only.

Pressure drag only.

Shear and pressure drag





$\mathcal{L}=0$ since τ_w balanced

$D\neq0$ since $P_{front}\neq P_{back}$ $\therefore$ pressure drag.

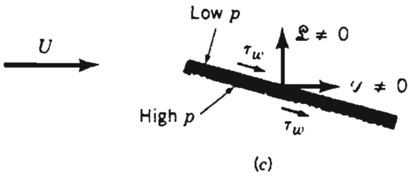


FIGURE E9.1 *(Continued)*

where we have used the fact that because of symmetry the shear stress distribution is the same on the top and the bottom surfaces, as is the pressure also [whether we use gage ($p = 0$) or absolute ($p = p_{atm}$) pressure]. There is no lift generated—the plate does not know up from down. With the given shear stress distribution, Eq. 1 gives

$$\mathscr{D} = 2\int_{x=0}^{4\text{ ft}} \left(\frac{1.24 \times 10^{-3}}{x^{1/2}}\ \text{lb/ft}^2\right)(10\text{ ft})\, dx$$

or

$$\mathscr{D} = 0.0992\ \text{lb} \qquad \textbf{(Ans)}$$

With the plate perpendicular to the upstream flow, we have $\theta = 0°$ on the front and $\theta = 180°$ on the back. Thus, from Eqs. 9.1 and 9.2

$$\mathscr{L} = \int_{\text{front}} \tau_w \, dA - \int_{\text{back}} \tau_w \, dA = 0$$

and

$$\mathscr{D} = \int_{\text{front}} p \, dA - \int_{\text{back}} p \, dA$$

Again there is no lift because the pressure forces act parallel to the upstream flow (in the direction of $\mathscr{D}$ not $\mathscr{L}$) and the shear stress is symmetrical about the center of the plate. With the given relatively large pressure on the front of the plate (the center of the plate is a stagnation point) and the negative pressure (less than the upstream pressure) on the back of the plate, we obtain the following drag

$$\mathscr{D} = \int_{y=-2}^{2 \text{ ft}} \left[0.744 \left(1 - \frac{y^2}{4} \right) \text{lb/ft}^2 - (-0.893) \text{ lb/ft}^2 \right] (10 \text{ ft}) \, dy$$

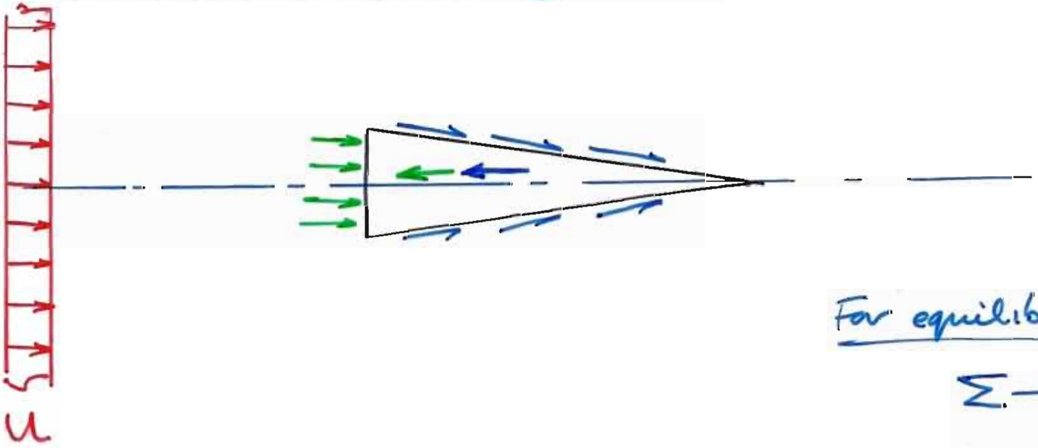
or

$$\mathscr{D} = 55.6 \text{ lb} \qquad \textbf{(Ans)}$$

Clearly there are two mechanisms responsible for the drag. On the ultimately streamlined body (a zero thickness flat plate parallel to the flow) the drag is entirely due to the shear stress at the surface and, in this example, is relatively small. For the ultimately blunted body (a flat plate normal to the upstream flow) the drag is entirely due to the pressure difference between the front and back portions of the object and, in this example, is relatively large.

If the flat plate were oriented at an arbitrary angle relative to the upstream flow as indicated in Fig. E9.1*c*, there would be both a lift and a drag, each of which would be dependent on both the shear stress and the pressure. Both the pressure and shear stress distributions would be different for the top and bottom surfaces.

For more complex (but symmetrical) bodies:



For equilibrium

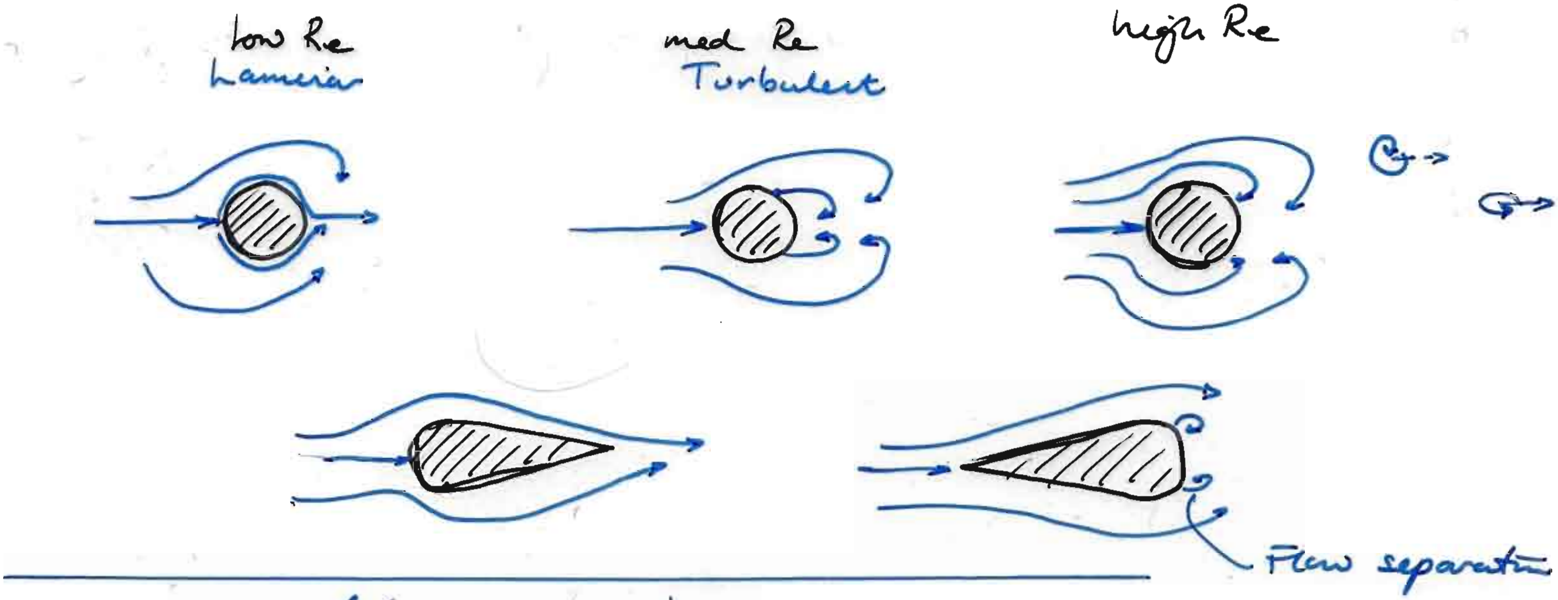
$$\Sigma \rightarrow = \Sigma \leftarrow$$

$$\Sigma \rightarrow = \Sigma \leftarrow$$

$$D = \Sigma \rightarrow + \Sigma \rightarrow$$

DIMENSIONAL ANALYSIS

Complex behavior, difficult to solve analytically



$$F_D = \phi(A, V, \rho, \mu)$$

BUCKINGHAM Π THEOREM

$$\Pi_1 = \phi(A, V, \rho, \mu) \qquad \Pi_1 = \frac{VA^{1/2}\rho}{\mu} = \frac{Vl\rho}{\mu} = Re$$

$$\Pi_2 = \phi(A, V, \rho, F_D) \qquad \Pi_2 = \frac{F_D}{\rho A V^2}$$

Two Π terms ⇒ $$\frac{F_D}{\frac{1}{2}\rho A V^2} = \phi(Re)$$

Drag coeff. C_D.

Note Euler No., $Eu = \frac{P}{\rho V^2} = \frac{F}{A}\frac{1}{\rho V^2}$

$\therefore$ Π_2 is equivalent to Eu !

DRAG AND LIFT COEFFICIENTS

LIFT:

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

lift coefficient, C_L

DRAG:

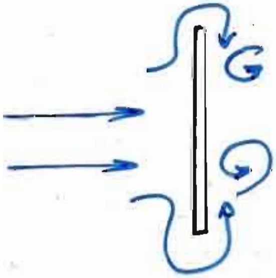
$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$$

drag coefficient, C_D

A is a "characteristic" area:

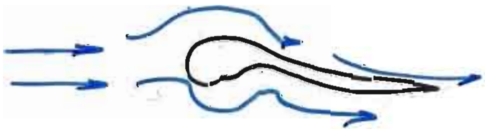
Must be chosen that represents the physical behaviour of system.

FRONTAL AREA



Pressure drag.

PLANFORM AREA



Shear drag

Lift

FLOW CHARACTERISTICS AROUND OBJECTS – BOUNDARY LAYERS

Flow behaviour indexed by:

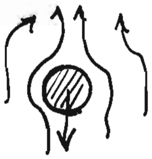
* Re — Fully immersed bodies

Fr — Free surface flows

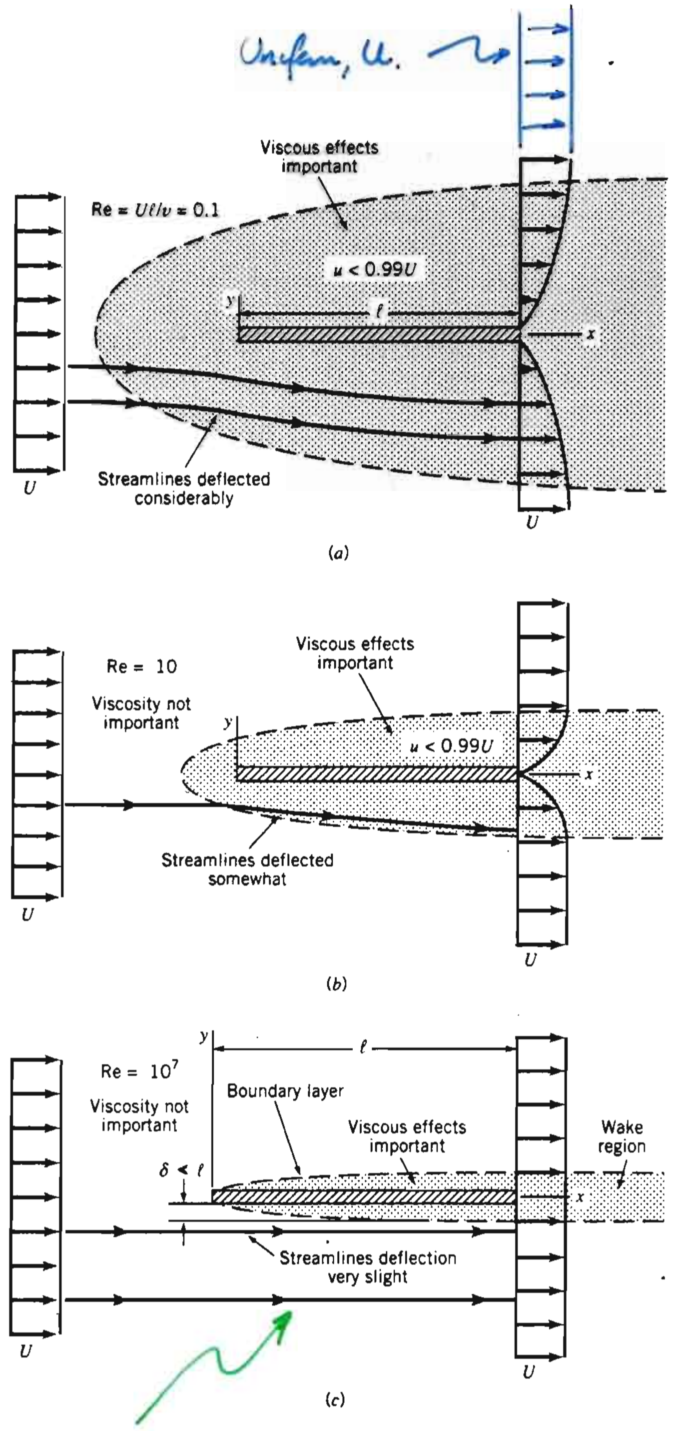
Re < 1 Viscous effects dominate

– see velocity profile

eg. Settling ball in fluid



$$Re = \frac{\rho V l}{\mu}$$



(a)

(b)

(c)

Re > 100 Inertial effects dominate

– see velocity profile

■ FIGURE 9.5
Character of the steady, viscous flow past a flat plate parallel to the upstream velocity: (*a*) low Reynolds number flow, (*b*) moderate Reynolds number flow, (*c*) large Reynolds number flow.

eg Airfoil in air

Boundary layer shrinks to zero thickness

layer thickness, δ $\underline{\delta \ll l}$.

Boundary layer is a 'slip' layer, reducing drag from the laminar form. (Prandtl, 1904)

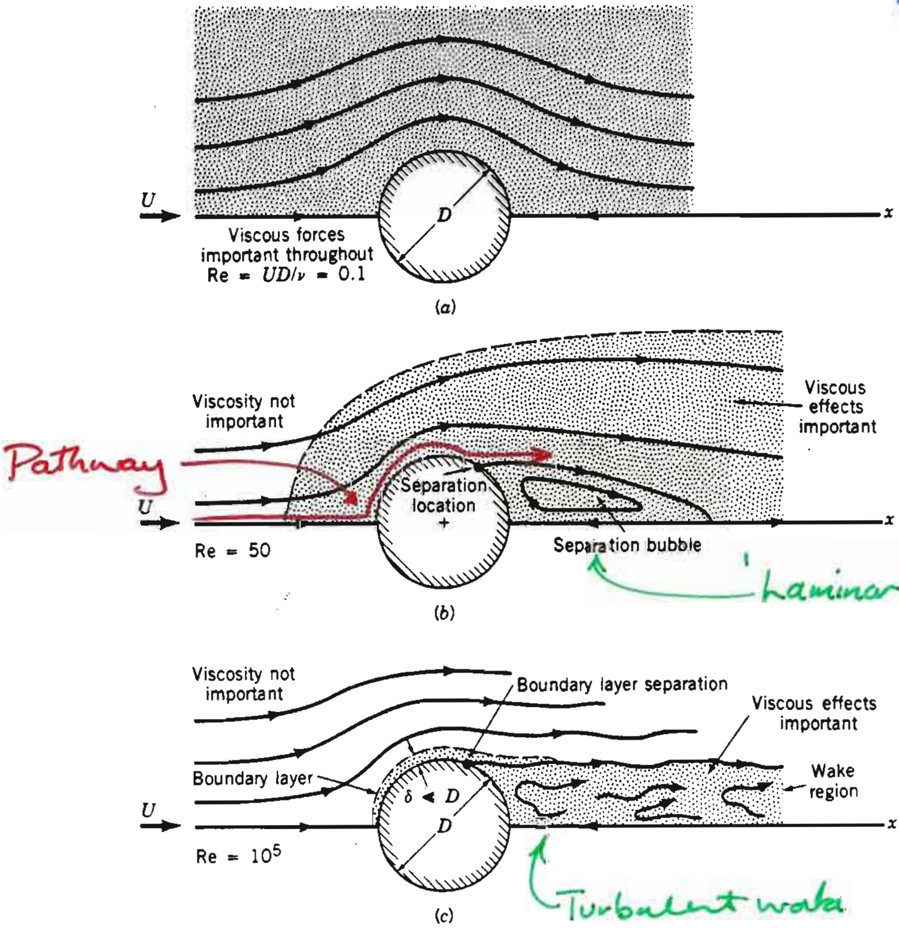
BLUNT BODIES

□ Similar behaviour to streamlined bodies

↑Re ⇒ Volume where viscous forces important ↓

□ Additional behaviour:

Flow separation at large Re.



■ FIGURE 9.6 Character of the steady, viscous flow past a circular cylinder: (*a*) low Reynolds number flow, (*b*) moderate Reynolds number flow, (*c*) large Reynolds number flow.

Boundary layer thickness $\delta \ll D$.

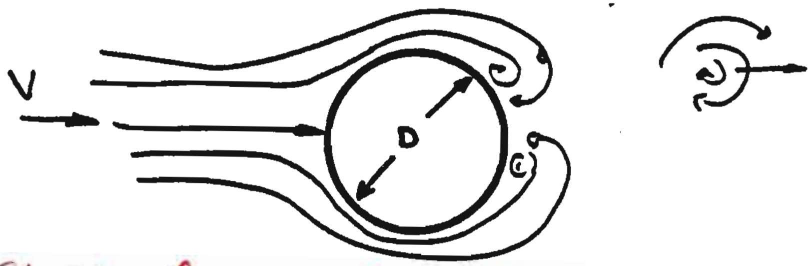
Conclusion from this:

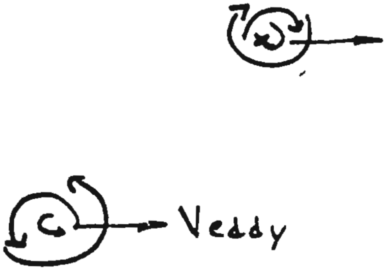
If flow form is $\phi(Re)$ then D and L are also $\phi(Re)$ and other geometric variables.

Formation of 'Karman' vortex sheet and vortex shedding.

KÁRMÁN VORTEX STREET

FIG 10.14





Shedding frequency (Taylor), f.

$$f \approx 0.20 \frac{V}{D}\left(1 - \frac{20}{RE}\right)$$

$RE \approx 60$-~~120~~ 10,000

* "SINGING" OF WIND BLOWING ACROSS WIRES

* TACOMA-NARROWS BRIDGE

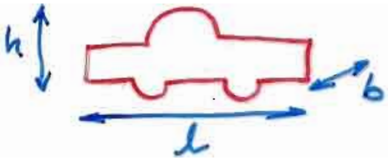
http://www.civeng.carleton.ca/Exhibits/Tacoma_Narrows/

U →
30 mph

USE OF PHYSICAL EXPERIMENTS

EXAMPLE 9.2

It is desired to determine the various characteristics of flow past a car. The following tests could be carried out: (a) $U = 20$ mm/s flow of glycerin past a scale model that is 34 mm tall, 100 mm long and 40 mm wide, (b) $U = 20$ mm/s air flow past the scale model, or (c) $U = 25$ m/s air flow past the actual car, which is 1.7 m tall, 5 m long, and 2 m wide. Would the flow characteristics for these three situations be similar? Explain.



SOLUTION

The characteristics of flow past an object depend on the Reynolds number. For this instance we could pick the characteristic length to be the height, h, width, b, or length, ℓ, of the car to obtain three possible Reynolds numbers, $\text{Re}_h = Uh/\nu$, $\text{Re}_b = Ub/\nu$, and $\text{Re}_\ell = U\ell/\nu$. These numbers will be different because of the different values of h, b, and ℓ. Once we arbitrarily decide on the length we wish to use as the characteristic length, we must stick with it for all calculations when using comparisons between model and prototype.

With the values of kinematic viscosity for air and glycerin obtained from Tables 1.8 and 1.6 as $\nu_{\text{air}} = 1.46 \times 10^{-5}\ \text{m}^2/\text{s}$ and $\nu_{\text{glycerin}} = 1.19 \times 10^{-3}\ \text{m}^2/\text{s}$, we obtain the following Reynolds numbers for the flows described.

Boundary layer form may be strongly influenced by Re.
∴ replicate Re if in range $Re < 10^7$
see Figs 9.5.

Reynolds Number	(a) Model in Glycerin	(b) Model in Air	(c) Car in Air
Re_h	0.571	46.6	2.91×10^6
Re_b	0.672 } low	54.8 } low	3.42×10^6 } high
Re_ℓ	1.68	137.0	8.56×10^6

Clearly, the Reynolds numbers for the three flows are quite different (regardless of which characteristic length we choose). Based on the previous discussion concerning flow past a flat plate or flow past a circular cylinder, we would expect that the flow past the actual car would behave in some way similar to the flows shown in Figs. 9.5*c* or 9.6*c*. That is, we would expect some type of boundary layer characteristic in which viscous effects would be confined to relatively thin layers near the surface of the car and the wake region behind it. Whether the car would act more like a flat plate or a cylinder would depend on the amount of streamlining incorporated into the car's design.

Neither model replicates expected Re.
∴ Drag anticipated to be low Re (viscous) dominated.

Not a good situation!

Remedy?
Run test with faster flows.

Because of the small Reynolds number involved, the flow past the model car in glycerin would be dominated by viscous effects, in some way reminiscent of the flows depicted in Figs. 9.5*a* or 9.6*a*. Similarly, with the moderate Reynolds number involved for the air flow past the model, a flow with characteristics similar to those indicated in Figs. 9.5*b* and 9.6*b* would be expected. Viscous effects would be important—not as important as with the glycerin flow, but more important than with the full-sized car.

It would not be a wise decision to expect the flow past the full-sized car to be similar to the flow past either of the models. The same conclusions result regardless of whether we use Re_h, Re_b, or Re_ℓ. As is indicated in Chapter 7, the flows past the model car and the full-sized prototype will not be similar unless the Reynolds numbers for the model and prototype are the same. It is not always an easy task to ensure this condition. One (expensive) solution is to test full-sized prototypes in very large wind tunnels (see Fig. 9.1).

DRAG

Empirically:
$$C_D = \frac{D}{\frac{1}{2}\rho u^2 A}$$

$$C_D = \phi\,(\text{shape}, Re, Ma, Fr, \varepsilon/\ell)$$

Two components:
- □ Friction drag 
- □ Pressure drag 

FRICTION DRAG

$$D = \int p \cos\theta \, dA + \overset{\text{FRICTIONAL}}{\int \tau_w \sin\theta \, dA}$$

Two components □ τ_w varies with V^2.
□ Orientation of surface θ

Not usually possible to separate friction and pressure drag coeffs. — since both depend on geometry.
Not usually needed.

How about using more viscous fluid to increase τ_w
— but Δp also increases.

$$D_f = \frac{1}{2}\rho u^2 A \, C_{D_f}$$

Variables $Re = \rho u \ell / \mu$, ε/ℓ } see Fig 9.15.

Same character as Moody.

BOUNDARY LAYER CHARACTERISTICS

Fully turbulent regime – boundary layer develops a drag (energy loss to flow) equivalent to pipe friction factor (f) and open channel Manning's coeffs, n.

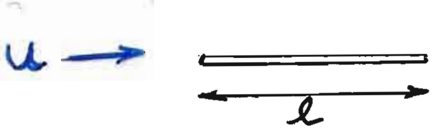
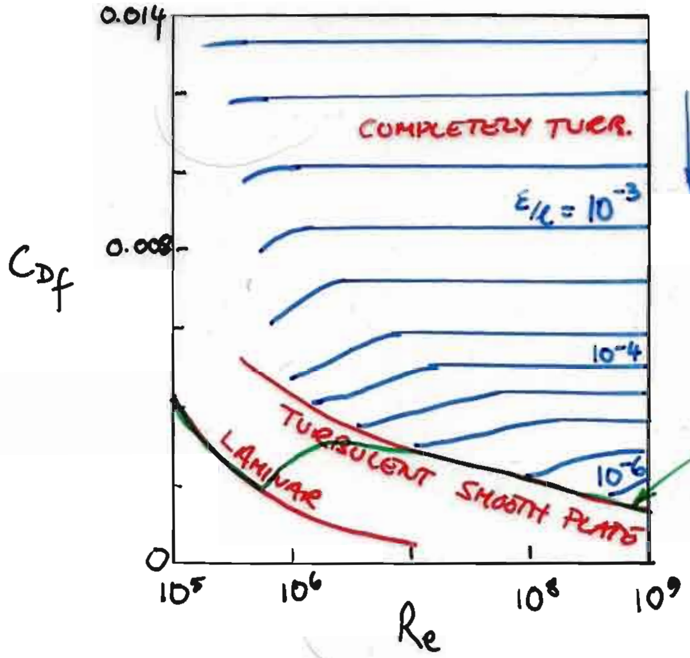


Figure 9.15.



decreasing roughness $\frac{\varepsilon}{l}$

Smooth plate

Like properties of Moody Chart: Fully turbulent

$$\boxed{C_{Df} = \frac{D_f}{\frac{1}{2}\rho u^2 A}}$$

Analog to pipe losses

$$h_L = f \frac{l}{D} \frac{V^2}{2g}$$

Boundary layer analog

$$D_f = C_{Df} \frac{1}{2}\rho u^2 A$$

$$\underbrace{\frac{D_f}{A\gamma}}_{\text{Units of length}} = C_{Df} \frac{u^2}{2g}$$

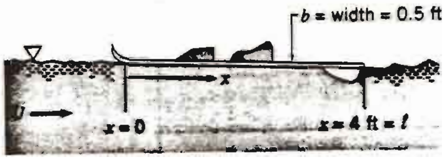
EXAMPLE 9.7

The water ski shown in Fig. E9.7*a* moves through 70 °F water with a velocity U. Estimate the drag caused by the shear stress on the bottom of the ski for $0 < U < 30$ ft/s.

SOLUTION

Clearly the ski is not a flat plate, and it is not aligned exactly parallel to the upstream flow. However, we can obtain a reasonable approximation to the shear force by using the flat plate results. That is, the friction drag, $\mathscr{D}_f$, caused by the shear stress on the bottom of the ski (the wall shear stress) can be determined as

$$\mathscr{D}_f = \tfrac{1}{2}\rho U^2 \ell b C_{Df}$$



(*a*)

With $A = \ell b = 4 \text{ ft} \times 0.5 \text{ ft} = 2 \text{ ft}^2$, $\rho = 1.94 \text{ slugs/ft}^3$, and $\mu = 2.04 \times 10^{-5} \text{ lb}\cdot\text{s/ft}^2$ (see Table B.1) we obtain

$$\mathscr{D}_f = \tfrac{1}{2}(1.94 \text{ slugs/ft}^3)(2.0 \text{ ft}^2)U^2 C_{Df}$$
$$= 1.94\, U^2 C_{Df} \qquad (1)$$

where $\mathscr{D}_f$ and U are in pounds and ft/s, respectively.

The friction coefficient, C_{Df}, can be obtained from Fig. 9.15 or from the appropriate equations given in Table 9.3. As we will see, for this problem, much of the flow lies within the transition regime where both the laminar and turbulent portions of the boundary layer flow occupy comparable lengths of the plate. We choose to use the values of C_{Df} from the table.

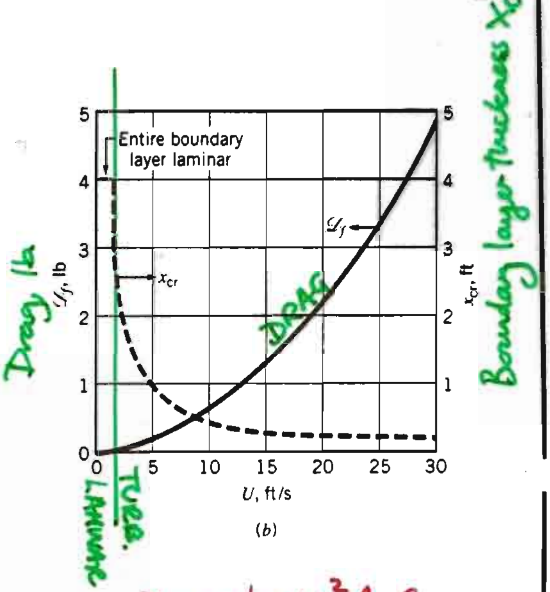
For the given conditions we obtain

$$\text{Re}_\ell = \frac{\rho U \ell}{\mu} = \frac{(1.94 \text{ slugs/ft}^3)(4 \text{ ft})U}{2.04 \times 10^{-5} \text{ lb}\cdot\text{s/ft}^2} = 3.80 \times 10^5\, U$$

where U is in ft/s. With $U = 10$ ft/s, or $\text{Re}_\ell = 3.80 \times 10^6$, we obtain from Table 9.3 $C_{Df} = 0.455/(\log \text{Re}_\ell)^{2.58} - 1700/\text{Re}_\ell = 0.00308$. From Eq. 1 the corresponding drag is

$$\mathscr{D}_f = 1.94(10)^2(0.00308) = 0.598 \text{ lb}$$

By covering the range of upstream velocities of interest we obtain the results shown in Fig. E9.7*b*.



(*b*)

DRAG: $D_f = \frac{1}{2}\rho u^2 A\, C_{Df}$

$C_{Df} = f(Re)$

Use Fig 9.15

$Re = \frac{\rho_w u \ell}{\mu}$

If Re ≲ 1000, the results of boundary layer theory are not valid—inertia effects are not dominant enough and the boundary layer is not thin compared with the length of the plate. For our problem this corresponds to $U = 2.63 \times 10^{-3}$ ft/s. For all practical purposes U is greater than this value, and the flow past the ski is of the boundary layer type.

The approximate location of the transition from laminar to turbulent boundary layer flow as defined by $\text{Re}_{cr} = \rho U x_{cr}/\mu = 5 \times 10^5$ is indicated in Fig. E9.7*b*. Up to $U = 1.31$ ft/s the entire boundary layer is laminar. The fraction of the boundary layer that is laminar decreases as U increases until only the front 0.18 ft is laminar when $U = 30$ ft/s.

For anyone who has water skied, it is clear that it can require considerably more force to be pulled along at 30 ft/s than the 2 × 4.88 lb = 9.76 lb (two skis) indicated in Fig. E9.7*b*. As is discussed in Section 9.3, the total drag on an object such as a water ski consists of more than just the friction drag. Other components, including pressure drag and wave-making drag, add considerably to the total resistance.

FRICTION DRAG

EXAMPLE 9.8

A viscous, incompressible fluid flows past the circular cylinder shown in Fig. E9.8*a*. According to a more advanced theory of boundary layer flow, the boundary layer remains attached to the cylinder up to the separation location at $\theta \sim 108.8°$, with the dimensionless wall shear stress as is indicated in Fig. E9.8*b* (Ref. 1). The shear stress on the cylinder in the wake region, $108.8 < \theta < 180°$, is negligible. Determine C_{Df}, the drag coefficient for the cylinder based on the friction drag only.

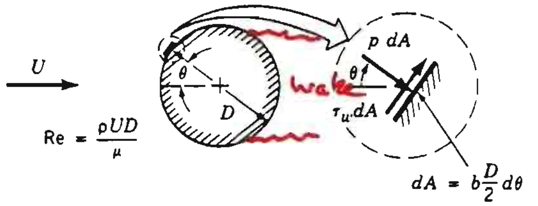
$$dA = \frac{D}{2} d\theta$$

SOLUTION

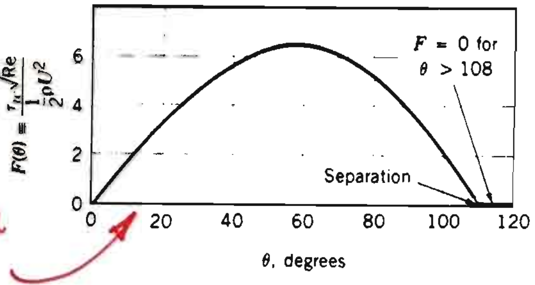
The friction drag, $\mathcal{D}_f$, can be determined from Eq. 9.1 as

$$\mathcal{D}_f = \int \tau_w \sin\theta \, dA = 2\left(\frac{D}{2}\right) b \int_0^{\pi} \tau_w \sin\theta \, d\theta$$

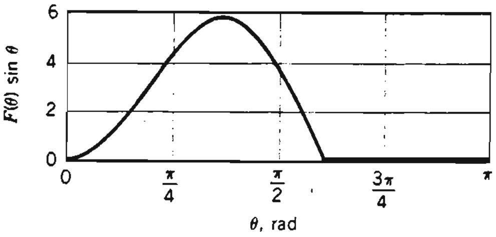
τw defined with θ



(*a*)



(*b*)



(*c*)

where *b* is the length of the cylinder. Note that θ is in radians (not degrees) to ensure proper dimensions of $dA = 2\,(D/2)\,b\,d\theta$. Thus,

$$C_{Df} = \frac{\mathcal{D}_f}{\frac{1}{2}\rho U^2 bD} = \frac{2}{\rho U^2}\int_0^{\pi} \tau_w \sin\theta\, d\theta$$

Frontal area

This can be put into dimensionless form by using the dimensionless shear stress parameter, $F(\theta) = \tau_w\sqrt{\text{Re}}/(\rho U^2/2)$, given in Fig. E9.8*b* as follows:

$$C_{Df} = \int_0^{\pi} \frac{\tau_w}{\frac{1}{2}\rho U^2} \sin\theta\, d\theta = \frac{1}{\sqrt{\text{Re}}}\int_0^{\pi} \frac{\tau_w\sqrt{\text{Re}}}{\frac{1}{2}\rho U^2}\sin\theta\, d\theta$$

$$F(\theta) = \frac{\tau_w \sqrt{Re}}{\frac{1}{2}\rho u^2}$$

$$\therefore \tau_w = \frac{F(\theta)\,\frac{1}{2}\rho u^2}{\sqrt{Re}}$$

where $\text{Re} = \rho UD/\mu$. Thus,

$$C_{Df} = \frac{1}{\sqrt{\text{Re}}}\int_0^{\pi} F(\theta)\sin\theta\, d\theta \qquad \textbf{(1)}$$

The function $F(\theta)\sin\theta$, obtained from Fig. E9.8*b*, is plotted in Fig. E9.8*c*. The necessary integration to obtain C_{Df} from Eq. 1 can be done by an appropriate numerical technique or by an approximate graphical method to determine the area under the given curve.

The result is $\int_0^{\pi} F(\theta)\sin\theta\, d\theta = 5.93$, or

$$C_{Df} = \frac{5.93}{\sqrt{\text{Re}}} \qquad \textbf{(Ans)}$$

This gives friction drag only!!

Note that the total drag must include both the shear stress (friction) drag and the pressure drag. As we will see in Example 9.9, for the circular cylinder most of the drag is due to the pressure force.

The above friction drag result is valid only if the boundary layer flow on the cylinder is laminar. As is discussed in Section 9.3.3, for a smooth cylinder this means that $\text{Re} = \rho UD/\mu < 3 \times 10^5$. It is also valid only for flows that have a Reynolds number sufficiently large to ensure the boundary layer type structure to the flow. For the cylinder this means $\text{Re} > 100$.

PRESSURE DRAG

PRESSURE

$$D = \int p \cos\theta \, dA + \int \tau_w \sin\theta \, dA$$

Two components ☐ p

☐ Inclination angle θ or shape of body.

$$C_{D_p} = \frac{D_p}{\frac{1}{2}\rho U^2 A} = \frac{\int p \cos\theta \, dA}{\frac{1}{2}\rho U^2 A} = \frac{\int C_p \cos\theta \, dA}{A}$$

$$C_p = \frac{(p - p_o)}{\frac{1}{2}\rho U^2} = \text{pressure coefficient.}$$

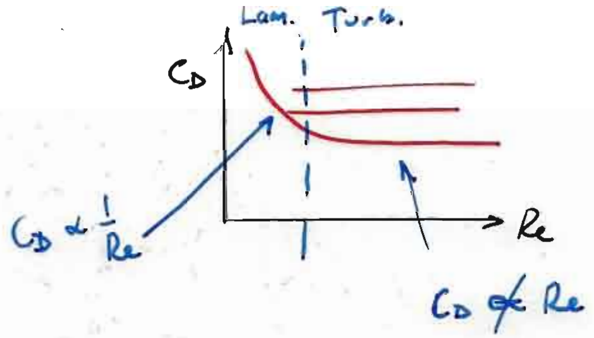
Equivalent to Euler, Eu. No.

Low velocity flows — Low Re → Viscous effects important (no boundary layer)

$C_D \sim \frac{1}{Re}$ (Just like pipe friction factors)

High velocity flows — High Re → Inertial effects important. (boundary layer).

C_D not influenced by Re.



PRESSURE DRAG

EXAMPLE 9.9

A viscous incompressible fluid flows past the circular cylinder shown in Fig. E9.8*a*. The pressure coefficient on the surface of the cylinder (as determined from experimental measurements) is as indicated in Fig. E9.9*a*. Determine the pressure drag coefficient for this flow. Combine the results of Examples 9.8 and 9.9 to determine the drag coefficient for a circular cylinder. Compare your results with those given in Fig. 9.23.

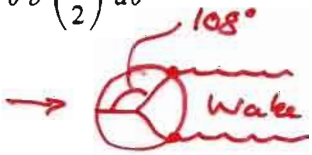
SOLUTION

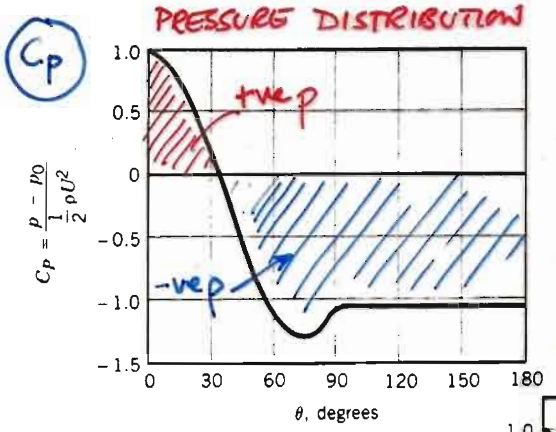
The pressure (form) drag coefficient, C_{Dp}, can be determined from Eq. 9.37 as

$$C_{Dp} = \frac{1}{A}\int C_p \cos\theta \, dA = \frac{1}{bD}\int_0^{2\pi} C_p \cos\theta \, b\left(\frac{D}{2}\right) d\theta$$

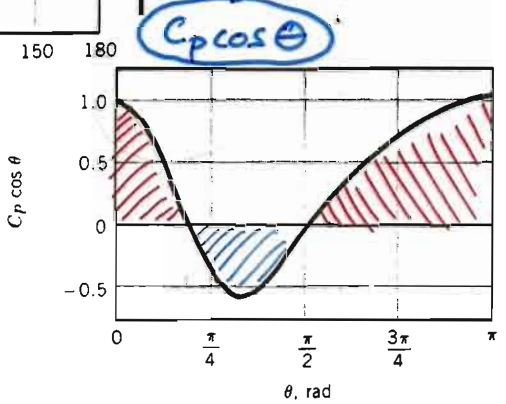
or because of symmetry

$$C_{Dp} = \int_0^{\pi} C_p \cos\theta \, d\theta$$





(*a*)



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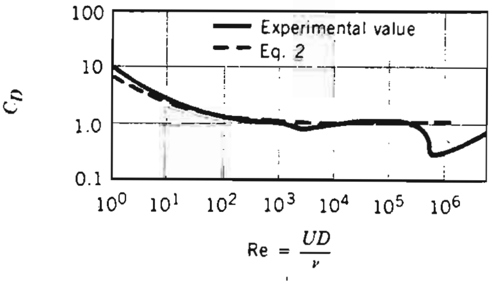
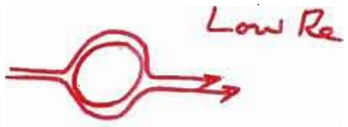


(*c*)

1. $C_{D_p} = \frac{1}{A}\int C_p \cos\theta \, dA$

$C_p = \frac{(P - P_o)}{\frac{1}{2}\rho u^2}$

2. $C_D = C_{D_f} + C_{D_p}$



Compare Expt. with equn (2)

1. Re < 10

Pressure distribution doesn't include absent boundary layer

2. Re > 10^5

Change to turbulent boundary layer.

where *b* and *D* are the length and diameter of the cylinder. To obtain C_{Dp}, we must integrate the $C_p \cos\theta$ function from $\theta = 0$ to $\theta = \pi$ radians. Again, this can be done by some numerical integration scheme or by determining the area under the curve shown in Fig. E9.9*b*. The result is

$$C_{Dp} = 1.17 \qquad \textbf{(1)} \quad \textbf{(Ans)}$$

Note that the positive pressure on the front portion of the cylinder ($0 \leq \theta \leq 34°$) and the negative pressure (less than the upstream value) on the rear portion ($90 \leq \theta \leq 180°$) produce positive contributions to the drag. The negative pressure on the front portion of the cylinder ($34 < \theta < 90°$) reduces the drag by pulling on the cylinder in the upstream direction. The positive area under the $C_p \cos\theta$ curve is greater than the negative area—there is a net pressure drag. In the absence of viscosity, these two contributions would be equal—there would be no pressure (or friction) drag.

The net drag on the cylinder is the sum of friction and pressure drag. Thus, from Eq. 1 of Example 9.8 and Eq. 1 of this example, we obtain the drag coefficient

$$\boxed{C_D = C_{Df} + C_{Dp} = \frac{5.93}{\sqrt{\text{Re}}} + 1.17} \qquad \textbf{(2)} \quad \textbf{(Ans)}$$

This result is compared with the standard experimental value (obtained from Fig. 9.23) in Fig. E9.9*c*. The agreement is very good over a wide range of Reynolds numbers. For Re < 10 the curves diverge because the flow is not a boundary layer type flow—the shear stress and pressure distributions used to obtain Eq. 2 are not valid in this range. The drastic divergence in the curves for Re > 3×10^5 is due to the change from a laminar to turbulent boundary layer, with the corresponding change in the pressure distribution. This is discussed in Section 9.3.3.

It is of interest to compare the friction drag to the total drag on the cylinder. That is,

$$\frac{\mathfrak{D}_f}{\mathfrak{D}} = \frac{C_{Df}}{C_D} = \frac{5.93/\sqrt{\text{Re}}}{(5.93/\sqrt{\text{Re}}) + 1.17} = \frac{1}{1 + 0.197\sqrt{\text{Re}}}$$

For Re = 10^3, 10^4, and 10^5 this ratio is 0.138, 0.0483, and 0.0158, respectively. Most of the drag on the blunt cylinder is pressure drag—a result of the boundary layer separation.

DRAG COEFFICIENT DATA

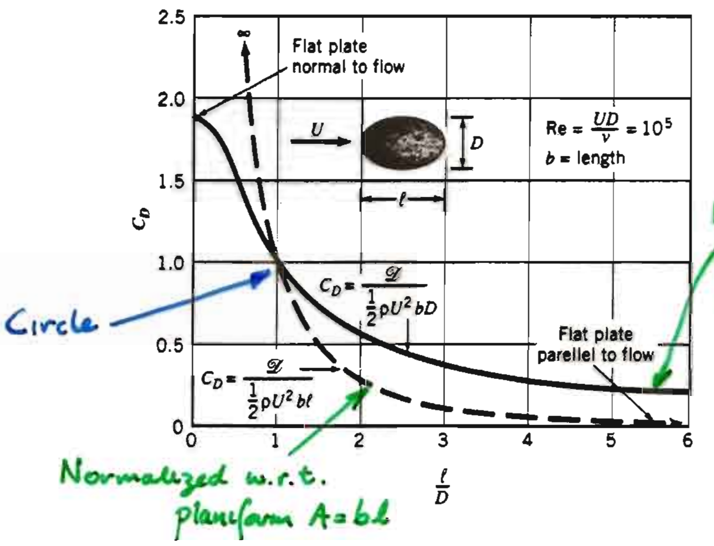
Generally use $C_D = C_{D_f} + C_{D_p}$ $\qquad$ C_D available.

Experimental results show influencing factors $\quad C_D = \phi\left[\text{Shape}, Re, \overset{Ma}{Ca}, \frac{\varepsilon}{l}, Fr \ldots\right]$

1. Shape	geometry
2. Flow velocity	Re
3. Compressibility	Ma, Ca
4. Surface roughness	$\frac{\varepsilon}{l}$
5. Froude No	Fr
6. Composite Body Drag	

SHAPE DEPENDENCE

Streamlined → Blunt

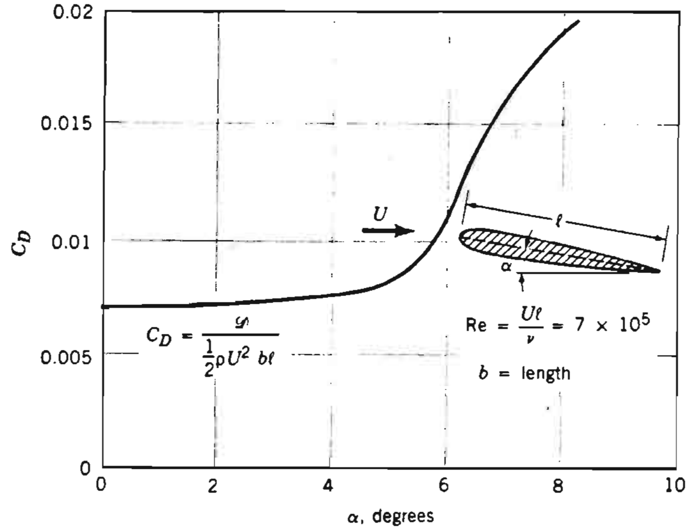


Normalized w.r.t frontal $A = Db$

■ FIGURE 9.19 Drag coefficient for an ellipse with the characteristic area either the frontal area, $A = bD$, or the planform area, $A = b\ell$ (Ref. 5).

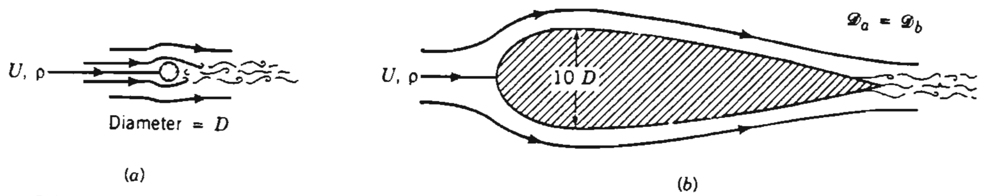
1. Changing shape changes drag.

C_D changes with reference area, $A = b\ell$
$A = bD$.



■ FIGURE 9.20 A typical variation of drag coefficient as a function of angle of attack for an airfoil.

2. Rotating object of same shape changes 'effective' shape.

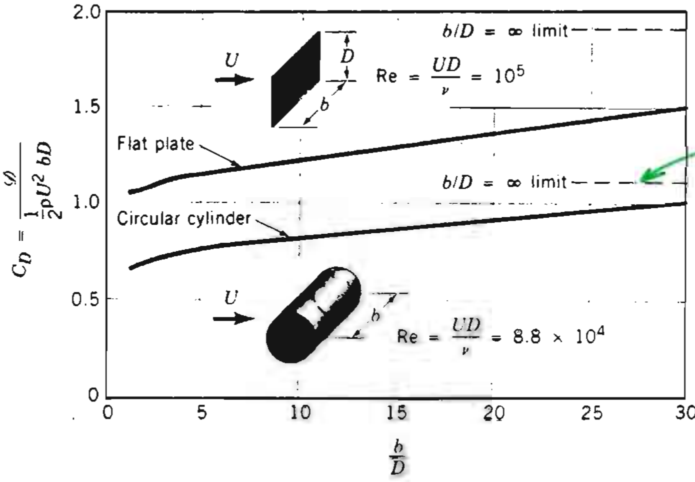


(a)

(b)

3. Streamlining reduces drag.

■ FIGURE 9.21 Two objects of considerably different size that have the same drag force: (a) circular cylinder $C_D = 1.2$, (b) streamlined strut $C_D = 0.12$.



2-D object.

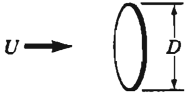
4. 3rd dimension also influences drag

Sphere-vs- cylinder etc.

■ FIGURE 9.22 Drag coefficient as a function of aspect ratio for a flat plate normal to the upstream flow and a circular cylinder (Ref. 5).

■ TABLE 9.4

Low Reynolds Number Drag Coefficients (Ref. 7)
($\mathrm{Re} = \rho UD/\mu$, $A = \pi D^2/4$)

Object	$C_D = \mathcal{D}/(\rho U^2 A/2)$ (for $\mathrm{Re} \leq 1$)
a. Circular disk normal to flow 	20.4/Re
b. Circular disk parallel to flow 	13.6/Re
c. Sphere 	24.0/Re
d. Hemisphere 	22.2/Re

[12:3] External Flows

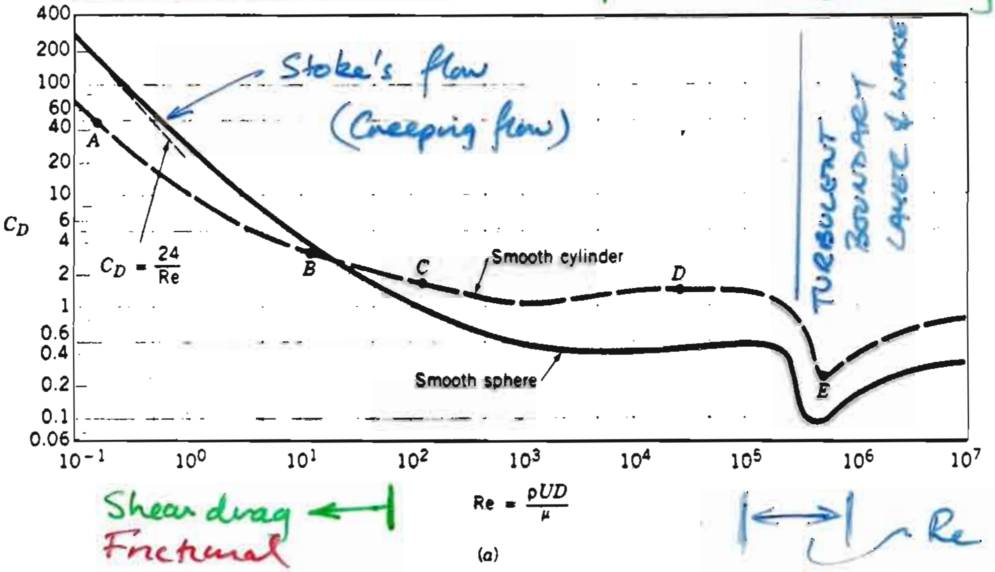
Outline

- Drag
    - Evaluating motion
    - Composite drag
- Lift
    - Evaluation
    - Rotating objects

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}; \qquad C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$$

RE DEPENDENCE

⟼ Pressure drag



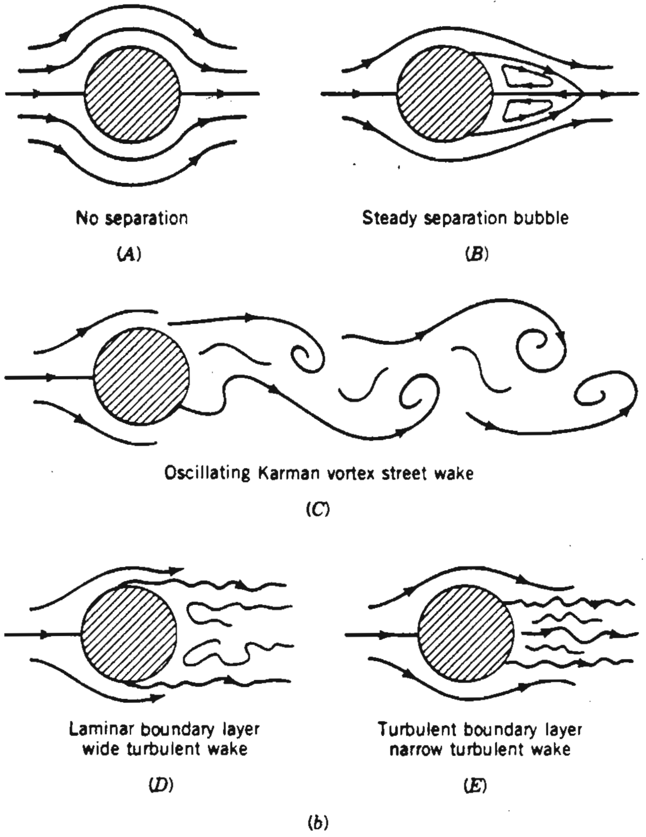
Stoke's flow (Creeping flow)

TURBULENT BOUNDARY LAYER & WAKE

Shear drag ⟻ Frictional

$|\leftrightarrow| \sim Re \quad 10^5 < Re < 10^6$

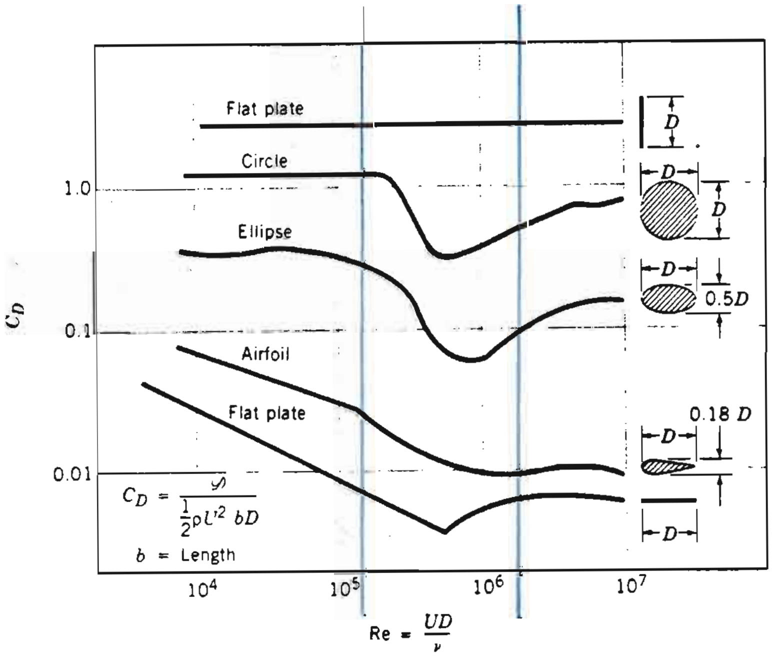
■ FIGURE 9.23 (*a*) Drag coefficient as a function of Reynolds number for a smooth circular cylinder and a smooth sphere.



■ FIGURE 9.23 (*Continued*) (*b*) Typical flow patterns for flow past a circular cylinder at various Reynolds numbers as indicated in (*a*).

Note effect of boundary layer turning turbulent @ $10^5 < Re < 10^6$

Boundary layer turns turbulent



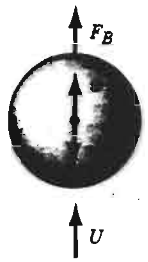
■ FIGURE 9.24 Character of the drag coefficient as a function of Reynolds number for objects with various degrees of streamlining, from a flat plate normal to the upstream flow to a flat plate parallel to the flow (two-dimensional flow) (Ref. 5).

EXAMPLE 9.10

A small grain of sand diameter $D = 0.10$ mm and specific gravity $SG = 2.3$ settles to the bottom of a lake after having been stirred up by a passing boat. Determine how fast it falls through the still water.

SOLUTION

A free-body diagram of the particle (relative to the moving particle) is shown in Fig. E9.10. The particle moves downward with a constant velocity U that is governed by a balance between the weight of the particle, $\mathcal{W}$, the buoyancy force of the surrounding water, F_B, and the drag of the water on the particle, $\mathcal{D}$.



■ FIGURE E9.10

From the free-body diagram we obtain

$$\boxed{\mathcal{W} = \mathcal{D} + F_B}$$

where

$$\mathcal{W} = \gamma_{\text{sand}} \forall = SG\ \gamma_{H_2O} \frac{\pi}{6} D^3 \qquad (1)$$

and

$$F_B = \gamma_{H_2O} \forall = \gamma_{H_2O} \frac{\pi}{6} D^3 \qquad (2)$$

CREEPING FLOW Table 9.4

$C_D = \frac{24}{Re}$ $Re \leq 1$

$C_D = \frac{D}{\frac{1}{2}\rho u^2 A}$

We assume (because of the smallness of the object) that the flow will be creeping flow (Re < 1) with $C_D = 24/\text{Re}$ (see Table 9.4) so that

$$\mathcal{D} = \frac{1}{2}\rho_{H_2O}U^2 \frac{\pi}{4} D^2 C_D = \frac{1}{2}\rho_{H_2O}U^2 \frac{\pi}{4} D^2 \left(\frac{24}{\rho_{H_2O}UD/\mu_{H_2O}}\right)$$

or

$$\mathcal{D} = 3\pi\mu_{H_2O}UD \qquad (3)$$

$C_D = \frac{24}{Re}$

We must eventually check to determine if this assumption is valid or not. Equation 3 is called Stokes law in honor of G. G. Stokes (1819–1903), a British mathematician and physicist. By combining Eqs. 1, 2, and 3, we obtain

$$SG\ \gamma_{H_2O} \frac{\pi}{6} D^3 = 3\pi\mu_{H_2O}UD + \gamma_{H_2O} \frac{\pi}{6} D^3$$

or, since $\gamma = \rho g$,

$$U = \frac{(SG\rho_{H_2O} - \rho_{H_2O})gD^2}{18\mu} \qquad (4)$$

From Table 1.6 for water at 15.6 °C we obtain $\rho_{H_2O} = 999\ \text{kg/m}^3$ and $\mu_{H_2O} = 1.12 \times 10^{-3}\ \text{N·s/m}^2$. Thus, from Eq. 4 we obtain

Evaluate, u.

$$U = \frac{(2.3 - 1)(999\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.10 \times 10^{-3}\ \text{m})^2}{18(1.12 \times 10^{-3}\ \text{N·s/m}^2)}$$

or

$$U = 6.32 \times 10^{-3}\ \text{m/s} \qquad \text{(Ans)}$$

Check $Re \leq 1$.

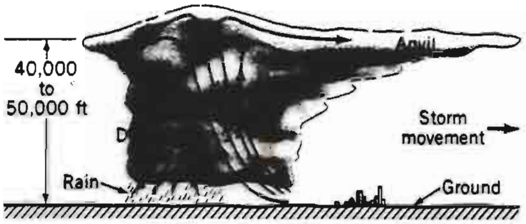
Since

$$\text{Re} = \frac{\rho D U}{\mu} = \frac{(999\ \text{kg/m}^3)(0.10 \times 10^{-3}\ \text{m})(0.00632\ \text{m/s})}{1.12 \times 10^{-3}\ \text{N·s/m}^2} = 0.564$$

we see that Re < 1, and the form of the drag coefficient used is valid.

EXAMPLE 9.11

Hail is produced by the repeated rising and falling of ice particles in the updraft of a thunderstorm, as is indicated in Fig. E9.11. When the hail becomes large enough, the aerodynamic drag from the updraft can no longer support the weight of the hail, and it falls from the storm cloud. Estimate the velocity, U, of the updraft needed to make $D = 1.5$-in.-diameter (i.e., "golf ball-sized") hail.



■ FIGURE E9.11

SOLUTION

As is discussed in Example 9.10, for steady state conditions a force balance on an object falling through a fluid gives

$$\mathcal{W} = \mathcal{D} + F_B$$

$W = \frac{\pi D^3}{6} \rho_{ice} g$

where $F_B = \gamma_{\text{air}} \mathcal{V}$ is the buoyant force of the air on the particle, $\mathcal{W} = \gamma_{\text{ice}} \mathcal{V}$ is the particle weight, and $\mathcal{D}$ is the aerodynamic drag. This equation can be rewritten as

$F_B \ll W$

$$\tfrac{1}{2}\rho_{\text{air}} U^2 \frac{\pi}{4} D^2 C_D = \mathcal{W} - F_B \tag{1}$$

With $\mathcal{V} = \pi D^3/6$ and since $\gamma_{\text{ice}} \gg \gamma_{\text{air}}$ (i.e., $\mathcal{W} \gg F_B$), Eq. 1 can be simplified to

$\downarrow$

$$U = \left(\frac{4}{3}\frac{\rho_{\text{ice}}}{\rho_{\text{air}}}\frac{gD}{C_D}\right)^{1/2} \tag{2}$$

By using $\rho_{\text{ice}} = 1.84 \text{ slugs/ft}^3$, $\rho_{\text{air}} = 2.37 \times 10^{-3} \text{ slugs/ft}^3$, and $D = 1.5 \text{ in.} = 0.125 \text{ ft}$, Eq. 2 becomes

$$U = \left[\frac{4(1.84 \text{ slugs/ft}^3)(32.2 \text{ ft/s}^2)(0.125 \text{ ft})}{3(2.37 \times 10^{-3} \text{ slugs/ft}^3)C_D}\right]^{1/2}$$

$U = \frac{64.5}{\sqrt{C_D}}$

or

$$U = \frac{64.5}{\sqrt{C_D}} \tag{3}$$

High Re...

$C_D \rightsquigarrow 0.5$

Evaluate U.

where U is in ft/s. To determine U, we must know C_D. Unfortunately, C_D is a function of the Reynolds number (see Fig. 9.23), which is not known unless U is known. Thus, we must use an iterative technique similar to that done with the Moody chart for certain types of pipe flow problems (see Section 8.5).

From Fig. 9.23 we expect that C_D is on the order of 0.5. Thus, we assume $C_D = 0.5$ and from Eq. 3 obtain

$$U = \frac{64.5}{\sqrt{0.5}} = 91.2 \text{ ft/s}$$

The corresponding Reynolds number (assuming $\nu = 1.57 \times 10^{-4} \text{ ft}^2/\text{s}$) is

$$\text{Re} = \frac{UD}{\nu} = \frac{91.2 \text{ ft/s } (0.125 \text{ ft})}{1.57 \times 10^{-4} \text{ ft}^2/\text{s}} = 7.26 \times 10^4$$

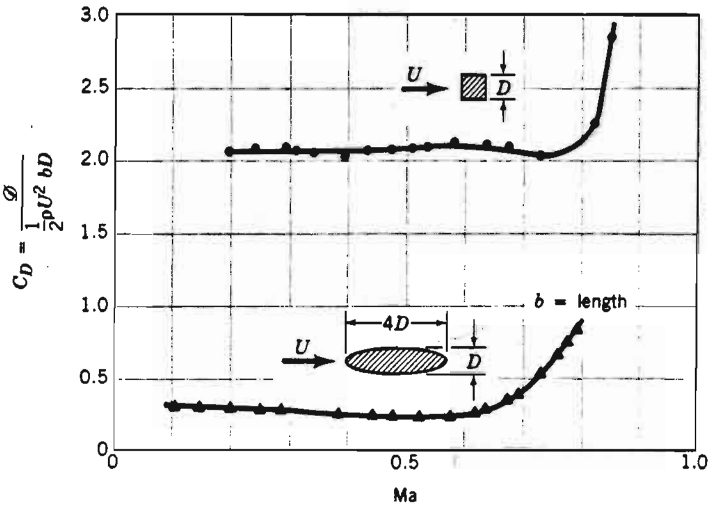
For this value of Re we obtain from Fig. 9.23, $C_D = 0.5$. Thus, our assumed value of $C_D = 0.5$ was correct. The corresponding value of U is

$$U = 91.2 \text{ ft/s} = 62.2 \text{ mph} \qquad \textbf{(Ans)}$$

This result was obtained by using standard sea-level properties for the air. If conditions at 20,000 ft altitude are used (i.e., from Table C.1, $\rho_{\text{air}} = 1.267 \times 10^{-3} \text{ slugs/ft}^3$ and $\mu = 3.324 \times 10^{-7} \text{ lb}\cdot\text{s/ft}^2$), the corresponding result is $U = 125 \text{ ft/s} = 85.2 \text{ mph}$.

Clearly, an airplane flying through such an updraft would feel its effects (even if it were able to dodge the hail). As seen from Eq. 2, the larger the hail, the stronger the necessary updraft. Hailstones greater than 6 in. in diameter have been reported. In reality, a hailstone is seldom spherical and often not smooth. However, the calculated updraft velocities are in agreement with measured values.

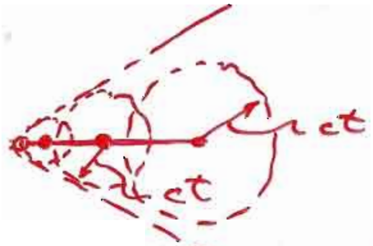
COMPRESSIBILITY EFFECTS



■ FIGURE 9.25 Drag coefficient as a function of Mach number for two-dimensional objects in subsonic flow (Ref. 5).

1. Only imp. for Ma > 0.7

Corresponds with Ma of 0.3 for Bernoulli expression. (Chap. 3)

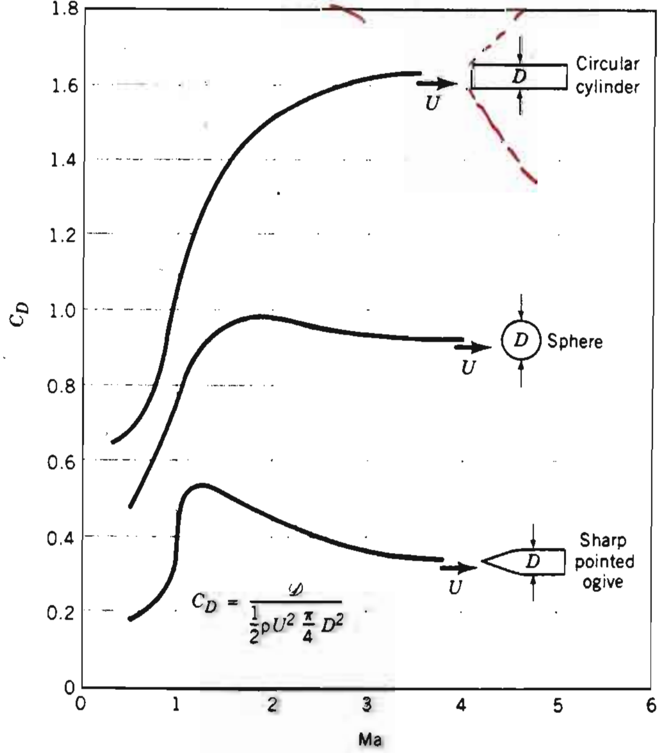


Mach cone

2. Large change @ Ma = 1

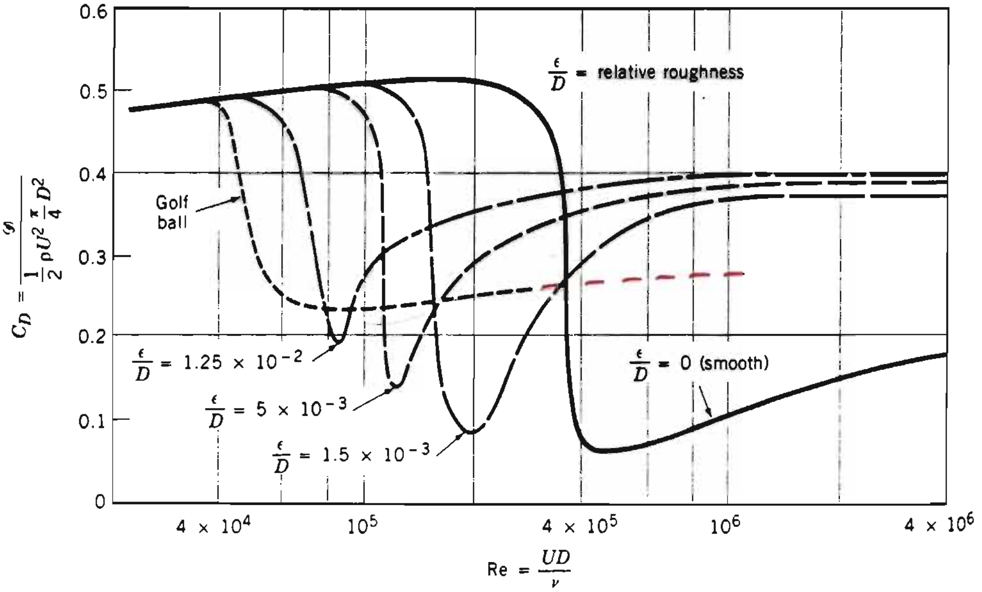
☐ Drag due to shock wave — eg. Bow wave of boat ↑ drag.

☐ Sharp objects give C_D max @ Ma = 1 and decrease.



■ FIGURE 9.26 Drag coefficient as a function of Mach number for supersonic flow (adapted from Ref. 19).

SURFACE ROUGHNESS



■ **FIGURE 9.27** The effect of surface roughness on the drag coefficient of a sphere in the Reynolds number range for which the laminar boundary layer becomes turbulent (Ref. 5).

↑ Roughness speeds the onset of turbulent separation

Streamlined bodies

↑ Drag ↑ surface roughness

Blunt bodies

↓ Pressure drag with ↑ surface roughness

since boundary layer is tripped to turbulence at a lower Re.

eg. Golf ball.

EXAMPLE 9.12

A well-hit golf ball (diameter $D = 1.69$ in., weight $\mathcal{W} = 0.0992$ lb) can travel at $U = 200$ ft/s as it leaves the tee. A well-hit table tennis ball (diameter $D = 1.50$ in., weight $\mathcal{W} = 0.00551$ lb) can travel at $U = 60$ ft/s as it leaves the paddle. Determine the drag on a standard golf ball, a smooth golf ball, and a table tennis ball for the conditions given. Also determine the deceleration of each ball for these conditions.

SOLUTION

For either ball, the drag can be obtained from

$$\mathcal{D} = \frac{1}{2}\rho U^2 \frac{\pi}{4} D^2 C_D \tag{1}$$

where the drag coefficient, C_D, is given in Fig. 9.27 as a function of the Reynolds number and surface roughness. For the golf ball in standard air

$$\mathrm{Re} = \frac{UD}{\nu} = \frac{(200 \text{ ft/s})(1.69/12 \text{ ft})}{1.57 \times 10^{-4} \text{ ft}^2/\text{s}} = 1.79 \times 10^5$$

while for the table tennis ball

$$\mathrm{Re} = \frac{UD}{\nu} = \frac{(60 \text{ ft/s})(1.50/12 \text{ ft})}{1.57 \times 10^{-4} \text{ ft}^2/\text{s}} = 4.78 \times 10^4$$

The corresponding drag coefficients are $C_D = 0.25$ for the standard golf ball, $C_D = 0.51$ for the smooth golf ball, and $C_D = 0.50$ for the table tennis ball. Hence, from Eq. 1 for the standard golf ball

$$\mathcal{D} = \frac{1}{2}(2.38 \times 10^{-3} \text{ slugs/ft}^3)(200 \text{ ft/s})^2 \frac{\pi}{4}\left(\frac{1.69}{12} \text{ ft}\right)^2 (0.25) = 0.185 \text{ lb} \qquad \text{(Ans)}$$

for the smooth golf ball

$$\mathcal{D} = \frac{1}{2}(2.38 \times 10^{-3} \text{ slugs/ft}^3)(200 \text{ ft/s})^2 \frac{\pi}{4}\left(\frac{1.69}{12} \text{ ft}\right)^2 (0.51) = 0.378 \text{ lb} \qquad \text{(Ans)}$$

and for the table tennis ball

$$\mathcal{D} = \frac{1}{2}(2.38 \times 10^{-3} \text{ slugs/ft}^3)(60 \text{ ft/s})^2 \frac{\pi}{4}\left(\frac{1.50}{12} \text{ ft}\right)^2 (0.50) = 0.0263 \text{ lb} \qquad \text{(Ans)}$$

The corresponding decelerations are $a = \mathcal{D}/m = g\mathcal{D}/\mathcal{W}$, where m is the mass of the ball. Thus, the deceleration relative to the acceleration of gravity, a/g (i.e., the number of g's deceleration) is $a/g = \mathcal{D}/\mathcal{W}$ or

$$\frac{a}{g} = \frac{0.185 \text{ lb}}{0.0992 \text{ lb}} = 1.86 \text{ for the standard golf ball} \qquad \text{(Ans)}$$

$$\frac{a}{g} = \frac{0.378 \text{ lb}}{0.0992 \text{ lb}} = 3.81 \text{ for the smooth golf ball} \qquad \text{(Ans)}$$

and

$$\frac{a}{g} = \frac{0.0263 \text{ lb}}{0.00551 \text{ lb}} = 4.77 \text{ for the table tennis ball} \qquad \text{(Ans)}$$

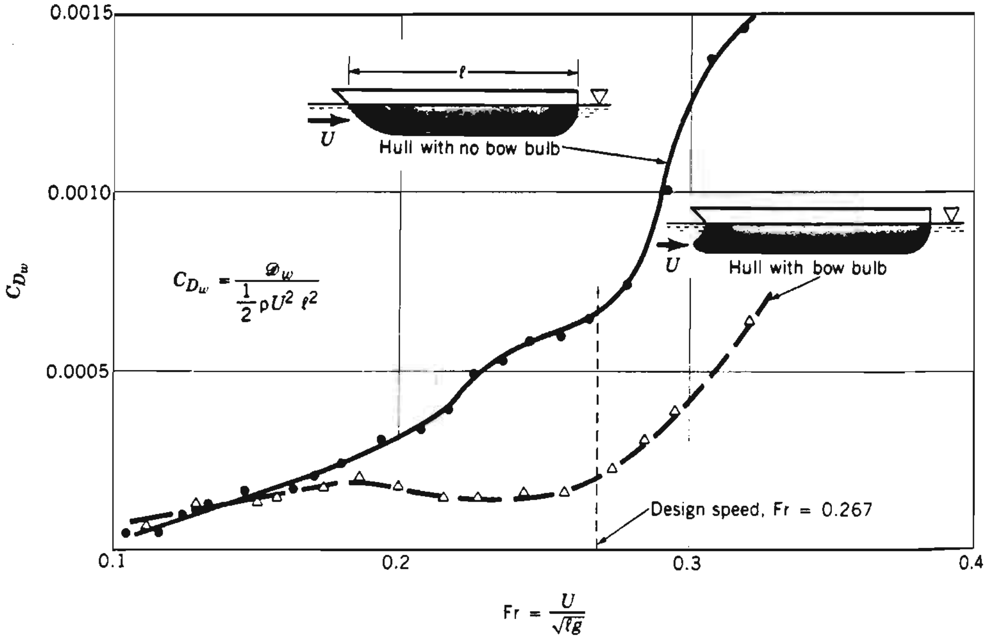
Note that there is a considerably smaller deceleration for the rough golf ball than for the smooth one. Because of its much larger drag-to-mass ratio, the table tennis ball slows down relatively quickly and does not travel as far as the golf ball. (Note that with $U = 60$ ft/s the standard golf ball has a drag of $\mathcal{D} = 0.0200$ lb and a deceleration of $a/g = 0.202$, considerably less than the $a/g = 4.77$ of the table tennis ball. Conversely, a table tennis ball hit from a tee at 200 ft/s would decelerate at a rate of $a = 1740 \text{ ft}^3/\text{s}$, or $a/g = 54.1$. It would not travel nearly as far as the golf ball.)

The Reynolds number range for which a rough golf ball has smaller drag than a smooth one (i.e., 4×10^4 to 4×10^5) corresponds to a flight velocity range of $45 < U < 450$ ft/s. This is comfortably within the range of most golfers. As is discussed in Section 9.4.4, the dimples (roughness) on a golf ball also help produce a lift (due to the spin of the ball) that allows the ball to travel farther than a smooth ball.

FROUDE No EFFECTS

$Fr = \frac{u}{\sqrt{gl}}$

For bodies traversing free surface $C_D = \phi[Re, Fr]$



■ **FIGURE 9.28** **Typical drag coefficient data as a function of Froude number and hull characteristics for that portion of the drag due to the generation of waves (adapted from Ref. 25).**

PROBLEM: Difficult to run tests with Re and Fr similar for prototype and model $\left\{ Re = \frac{\rho u l}{\mu} \quad Fr = \frac{u}{\sqrt{gl}} \right\}$ since both include u.

SOLUTION: Separate out viscous (Re) and wave (Fr) drag effects.

$C_D = C_{D_v} + C_{D_w}$.

eg. Baidarkas

COMPOSITE DRAG

EXAMPLE 9.13

CARE IN NOTING 3-D AND INTERACTION EFFECTS

A 60-mph (i.e., 88-fps) wind blows past the water tower shown in Fig. E9.13*a*. Estimate the moment (torque), M, needed at the base to keep the tower from tipping over.

SOLUTION

We treat the water tower as a sphere resting on a circular cylinder and assume that the total drag is the sum of the drag from these parts. The free-body diagram of the tower is shown in Fig. E.9.13*b*. By summing moments about the base of the tower, we obtain

$$M = \mathcal{D}_s \left(b + \frac{D_s}{2} \right) + \mathcal{D}_c \left(\frac{b}{2} \right) \tag{1}$$

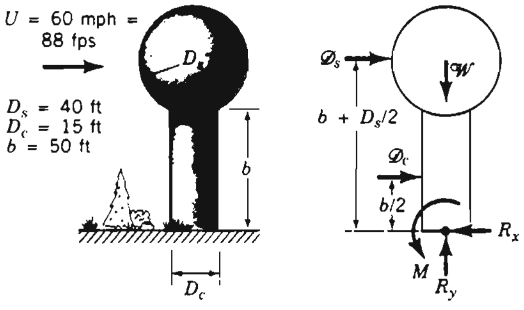
where

$$\mathcal{D}_s = \frac{1}{2} \rho U^2 \frac{\pi}{4} D_s^2 C_{Ds} \tag{2}$$

and

$$\mathcal{D}_c = \frac{1}{2} \rho U^2 b D_c C_{Dc} \tag{3}$$

Evaluate Drag components →



(*a*) (*b*) ■ FIGURE E9.13

are the drag on the sphere and cylinder, respectively. For standard atmospheric conditions, the Reynolds numbers are

Need Re. Fig 9.23

$$\text{Re}_s = \frac{UD_s}{\nu} = \frac{(88\text{ ft/s})(40\text{ ft})}{1.57 \times 10^{-4}\text{ ft}^2/\text{s}} = 2.24 \times 10^7$$

and

$$\text{Re}_c = \frac{UD_c}{\nu} = \frac{(88\text{ ft/s})(15\text{ ft})}{1.57 \times 10^{-4}\text{ ft}^2/\text{s}} = 8.41 \times 10^6$$

The corresponding drag coefficients, C_{Ds} and C_{Dc}, can be approximated from Fig. 9.23 as

$$C_{Ds} \approx 0.3 \quad \text{and} \quad C_{Dc} \approx 0.7$$

Note that the value of C_{Ds} was obtained by an extrapolation of the given data to Reynolds numbers beyond those given (a potentially dangerous practice!). From Eqs. 2 and 3 we obtain

$$\mathcal{D}_s = 0.5(2.38 \times 10^{-3}\text{ slugs/ft}^3)(88\text{ ft/s})^2 \frac{\pi}{4}(40\text{ ft})^2(0.3) = 3470\text{ lb}$$

and

$$\mathcal{D}_c = 0.5(2.38 \times 10^{-3}\text{ slugs/ft}^3)(88\text{ ft/s})^2(50\text{ ft} \times 15\text{ ft})(0.7) = 4840\text{ lb}$$

From Eq. 1 the corresponding moment needed to prevent the tower from tipping is

$$M = 3470\text{ lb}\left(50\text{ ft} + \frac{40}{2}\text{ ft}\right) + 4840\text{ lb}\left(\frac{50}{2}\text{ ft}\right) = 3.64 \times 10^5\text{ ft·lb} \qquad \text{(Ans)}$$

The above result is only an estimate because (a) the wind is probably not uniform from the top of the tower to the ground, (b) the tower is not exactly a combination of a smooth sphere and a circular cylinder, (c) the cylinder is not of infinite length, (d) there will be some interaction between the flow past the cylinder and that past the sphere so that the net drag is not exactly the sum of the two, and (e) a drag coefficient value was obtained by extrapolation of the given data. However, such approximate results are often quite accurate.

eg. AUTOMOBILES

Recipie

$$C_D = 0.16 + 0.0095 \sum_{i=A}^{H} N_i$$

■ TABLE 9.5

Values of N_i (Ref. 12). See Fig. 9.29

A. Plan view, front end

N_A =	1	*A*-1	Approximately semicircular
	2	*A*-2	Well-rounded outer quarters
	3	*A*-3	Rounded corners without protuberances
	4	*A*-4	Rounded corners with protuberances
	5	*A*-5	Squared tapering-in corners
	6	*A*-6	Squared constant width front

B. Plan view, windshield

N_B =	1	*B*-1	Full wraparound (approximately semicircular)
	2	*B*-2	Wraparound ends
	3	*B*-3	Bowed
	4	*B*-4	Flat

C. Plan view, roof

N_C =	1	*C*-1	Well- or medium-tapered to rear
	2	*C*-2	Tapering to front and rear or approximately constant width
	3	*C*-3	Tapering to front (maximum width at rear)

D. Plan view, lower rear end

N_C =	1	*D*-1	Well- or medium-tapered to rear
	2	*D*-2	Small taper to rear or constant width
	3	*D*-3	Outward taper (or flared-out fins)

E. Side elevation, front end

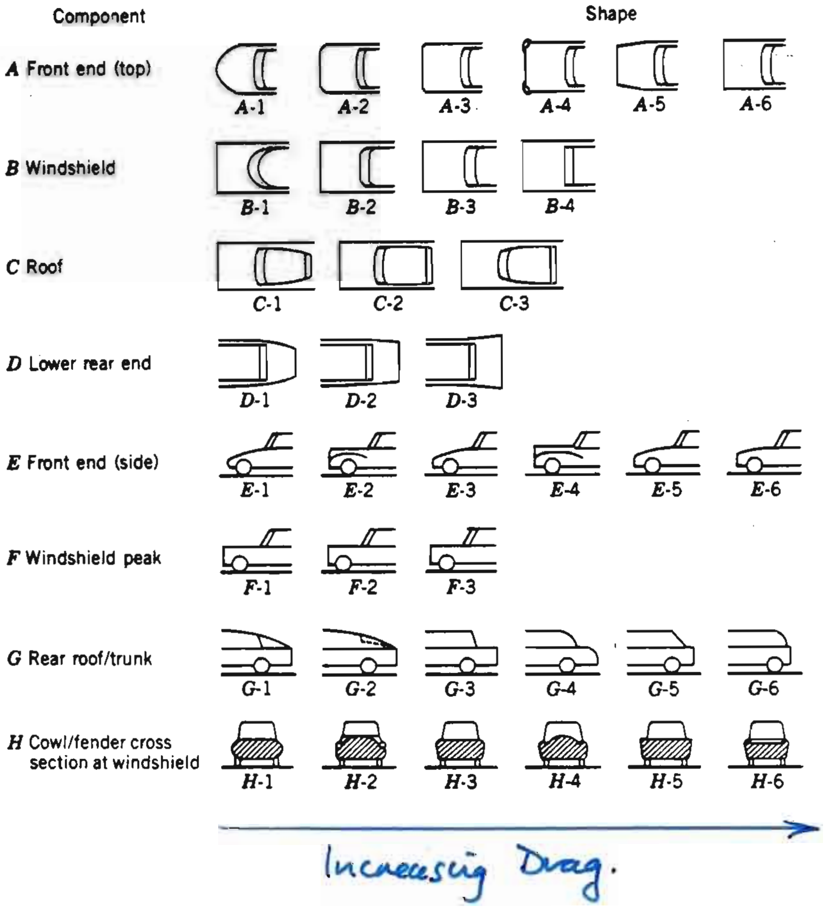
N_E =	1	*E*-1	Low, rounded front, sloping up
	1	*E*-2	High, tapered, rounded hood
	2	*E*-3	Low, squared front, sloping up
	2	*E*-4	High, tapered, squared hood
	3	*E*-5	Medium-height, rounded front, sloping up
	4	*E*-6	Medium-height, squared front, sloping up
	4	*E*-7	High, rounded front, with horizontal hood
	5	*E*-8	High, squared front, with horizontal hood

F. Side elevation, windshield peak

N_F =	1	*F*-1	Rounded
	2	*F*-2	Squared (including flanges or gutters)
	3	*F*-3	Forward-projecting peak

G. Side elevation, rear roof/trunk

N_G =	1	*G*-1	Fastback (roofline continuous to tail)
	2	*G*-2	Semifastback (with discontinuity in line to tail)
	3	*G*-3	Squared roof with trunk rear edge squared
	4	*G*-4	Rounded roof with rounded trunk

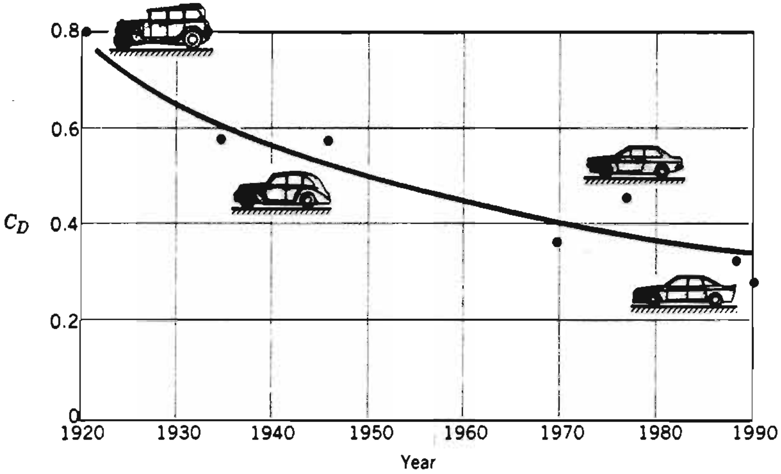


■ TABLE 9.5 (continued)

	4	*G*-5	Squared roof with short or no trunk
	5	*G*-6	Rounded roof with short or no trunk

H. Front elevation, cowl and fender cross section at windshield

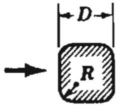
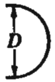
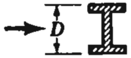
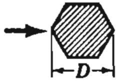
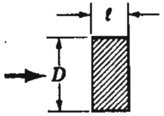
N_H =	1	*H*-1	Flush hood and fenders, well-rounded body sides
	2	*H*-2	High cowl, low fenders
	3	*H*-3	Hood flush with rounded-top fenders
	3	*H*-4	High cowl with rounded-top fenders
	4	*H*-5	Hood flush with square-edged fenders
	5	*H*-6	Depressed hood with high squared-edged fenders



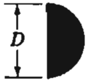
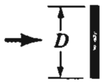
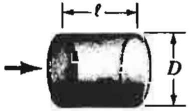
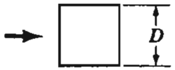
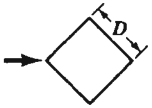
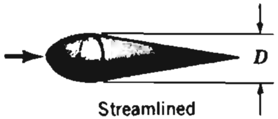
■ **FIGURE 9.30** **The historical trend of streamlining automobiles to reduce their aerodynamic drag and increase their miles per gallon (adapted from Ref. 5).**

Faster than 30 mph — Drag effect overtakes rolling resistance.

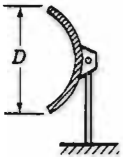
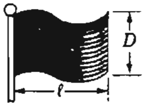
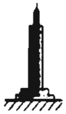
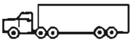
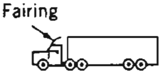
∴ ↓ Drag ↑ gas mileage

Shape		Reference area A (b = length)	Drag coefficient $C_D = \dfrac{\mathcal{D}}{\frac{1}{2}\rho U^2 A}$	Reynold number $\mathrm{Re} = \rho U D/\nu$
	Square rod with rounded corners	$A = bD$	R/D C_D 0 2.2 0.02 2.0 0.17 1.2 0.33 1.0	$\mathrm{Re} = 10^5$
	Rounded equilateral triangle	$A = bD$	R/D C_D → ← 0 1.4 2.1 0.02 1.2 2.0 0.08 1.3 1.9 0.25 1.1 1.3	$\mathrm{Re} = 10^5$
	Semicircular shell	$A = bD$	→ 2.3 ← 1.1	$\mathrm{Re} = 2 \times 10^4$
	Semicircular cylinder	$A = bD$	→ 2.15 ← 1.15	$\mathrm{Re} > 10^4$
	T-beam	$A = bD$	→ 1.80 ← 1.65	$\mathrm{Re} > 10^4$
	I-Beam	$A = bD$	2.05	$\mathrm{Re} > 10^4$
	Angle	$A = bD$	→ 1.98 ← 1.82	$\mathrm{Re} > 10^4$
	Hexagon	$A = bD$	1.0	$\mathrm{Re} > 10^4$
	Rectangle	$A = bD$	ℓ/D C_D ≤0.1 1.9 0.5 2.5 0.65 2.9 1.0 2.2 2.0 1.6 3.0 1.3	$\mathrm{Re} = 10^5$

■ **FIGURE 9.31** **Typical drag coefficients for regular two-dimensional objects (Refs. 5 and 6).**

Shape		Reference area A	Drag coefficient C_D	Reynolds number
	Solid hemisphere	$A = \frac{\pi}{4}D^2$	→ 1.17 ← 0.42	$\mathrm{Re} > 10^4$
	Hollow hemisphere	$A = \frac{\pi}{4}D^2$	→ 1.42 ← 0.38	$\mathrm{Re} > 10^4$
	Thin disk	$A = \frac{\pi}{4}D^2$	1.1	$\mathrm{Re} > 10^3$
	Circular rod parallel to flow	$A = \frac{\pi}{4}D^2$	ℓ/D C_D 0.5 1.1 1.0 0.93 2.0 0.83 4.0 0.85	$\mathrm{Re} > 10^5$
	Cone	$A = \frac{\pi}{4}D^2$	θ, degrees C_D 10 0.30 30 0.55 60 0.80 90 1.15	$\mathrm{Re} > 10^4$
	Cube	$A = D^2$	1.05	$\mathrm{Re} > 10^4$
	Cube	$A = D^2$	0.80	$\mathrm{Re} > 10^4$
 Streamlined body		$A = \frac{\pi}{4}D^2$	0.04	$\mathrm{Re} > 10^5$

■ **FIGURE 9.32** **Typical drag coefficients for regular three-dimensional objects (Ref. 5).**

Shape	Reference area	Drag coefficient C_D
 Parachute	Frontal area $A = \frac{\pi}{4}D^2$	1.4
 Porous parabolic dish	Frontal area $A = \frac{\pi}{4}D^2$	Porosity: 0 / 0.2 / 0.5 → : 1.42 / 1.20 / 0.82 ← : 0.95 / 0.90 / 0.80 Porosity = open area/total area
 Average person	Standing Sitting Crouching	$C_DA = 9\ \text{ft}^2$ $C_DA = 6\ \text{ft}^2$ $C_DA = 2.5\ \text{ft}^2$
 Fluttering flag	$A = \ell D$	ℓ/D : C_D 1 : 0.07 2 : 0.12 3 : 0.15
 Empire State Building	Frontal area	1.4
 Six-car passenger train	Frontal area	1.8
Bikes		
 Upright commuter	$A = 5.5\ \text{ft}^2$	1.1
 Racing	$A = 3.9\ \text{ft}^2$	0.88
 Drafting	$A = 3.9\ \text{ft}^2$	0.50
 Streamlined	$A = 5.0\ \text{ft}^2$	0.12
Tractor-trailer trucks		
 Standard	Frontal area	0.96
 With fairing	Frontal area	0.76
 With fairing and gap seal	Frontal area	0.70
 U = 10 m/s U = 20 m/s U = 30 m/s	Frontal area	0.43 0.26 0.20
 Dolphin	Wetted area	0.0036 at Re = 6×10^6 (flat plate has $C_{D_f} = 0.0031$)
 Large Birds	Frontal area	0.40

■ **FIGURE 9.33** **Typical drag coefficients for objects of interest (Refs. 5, 6, 15, and 20).**

LIFT

Coefficient of lift, $C_L = \dfrac{\mathcal{L}}{\frac{1}{2}\rho u^2 A}$

$$C_L = \phi\left[\text{shape},\ Re,\ Ma,\ Fr,\ \epsilon/\ell\right]$$

Shape – largest effect

Re – some effect

Ma – only for very high velocities

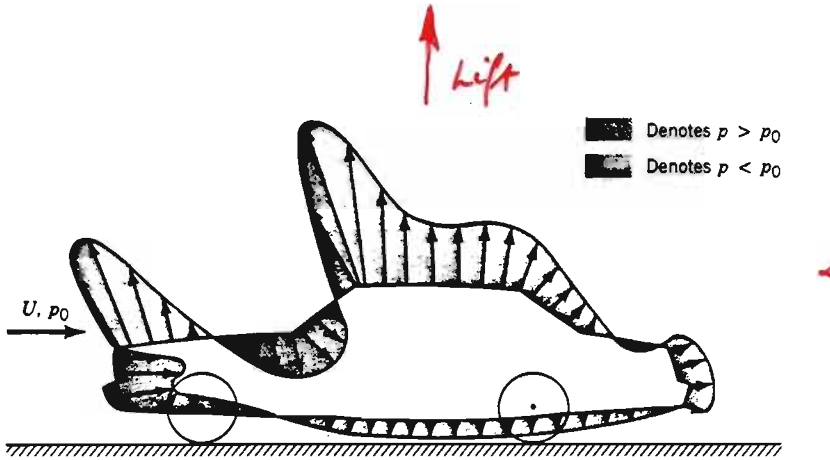
Fr – only if free surface

Two components of lift:

- ☐ Pressure lift – largest influence
  predominant
- ☐ Shear lift (τ_w) – only significant for $Re < 1$
  where shear lift may be equivalent to pressure lift.

PRINCIPLE OF LIFT

$$\frac{P}{\gamma} + \frac{V^2}{2g} + z = \text{const}$$



← Pressure lift

■ FIGURE 9.34 Pressure distribution on the surface of an automobile.

Large Re ⟶ Boundary layer is thin and surrounds body

Viscous (τ_w) effects are limited to this layer and turbulent wake

∴ pressure lift predominates.

ELEMENTS OF LIFT

- Determine lift from pressure distribution as

$$\mathcal{L} = -\int p \sin\theta \, dA + \int \tau_w \cos\theta \, dA$$

For most applications pressure lift predominates

$$\therefore \mathcal{L} \propto -\int p \sin\theta \, d\theta$$

- Determine $p(\theta)$ from inviscid flow calculations

Inviscid solns. ok since at high velocities the boundary layer is very thin.

Airfoils

→ Circulation

→ Rotation

EXAMPLE 9.14

When a uniform wind of velocity U blows past the semicircular building shown in Fig. E9.14*a*, the wall shear stress and pressure distributions on the outside of the building are as given previously in Figs. E9.8*b* and E9.9*a*, respectively. If the pressure in the building is atmospheric (i.e., the value, p_0, far from the building), determine the lift coefficient and the lift on the roof.

SOLUTION

From Eq. 9.2 we obtain the lift as

$$\mathscr{L} = -\int p \sin\theta \, dA + \int \tau_w \cos\theta \, dA \qquad (1)$$

As is indicated in Fig. E9.14*a*, we assume that on the inside of the building the pressure is uniform, $p = p_0$, and that there is no shear stress. Thus, Eq. 1 can be written as

$$\mathscr{L} = -\int_0^\pi (p - p_0) \sin\theta \, b\left(\frac{D}{2}\right) d\theta + \int_0^\pi \tau_w \cos\theta \, b\left(\frac{D}{2}\right) d\theta$$

or

$$\mathscr{L} = \frac{bD}{2}\left[-\int_0^\pi (p - p_0)\sin\theta \, d\theta + \int_0^\pi \tau_w \cos\theta \, d\theta\right] \qquad (2)$$

where b and D are the length and diameter of the building, respectively, and $dA = b(D/2)d\theta$. Equation 2 can be put into dimensionless form by using the dynamic pressure, $\rho U^2/2$, planform area, $A = bD$, and dimensionless shear stress

$$F(\theta) = \tau_w(\text{Re})^{1/2}/(\rho U^2/2)$$

to give

$$\mathscr{L} = \frac{1}{2}\rho U^2 A\left[-\frac{1}{2}\int_0^\pi \frac{(p - p_0)}{\frac{1}{2}\rho U^2}\sin\theta \, d\theta + \frac{1}{2\sqrt{\text{Re}}}\int_0^\pi F(\theta)\cos\theta \, d\theta\right] \qquad (3)$$

The values of the two integrals in Eq. 3 can be obtained by determining the area under the curves of $[(p - p_0)/(\rho U^2/2)] \sin\theta$ versus θ and $F(\theta) \cos\theta$ versus θ plotted in Figs. E9.14*b*

1. Determine p, τ_w
   - expt.
   - Euler equations
2. Change integration as

$dA = \frac{D}{2} d\theta$

3. Evaluate $\mathscr{L}$

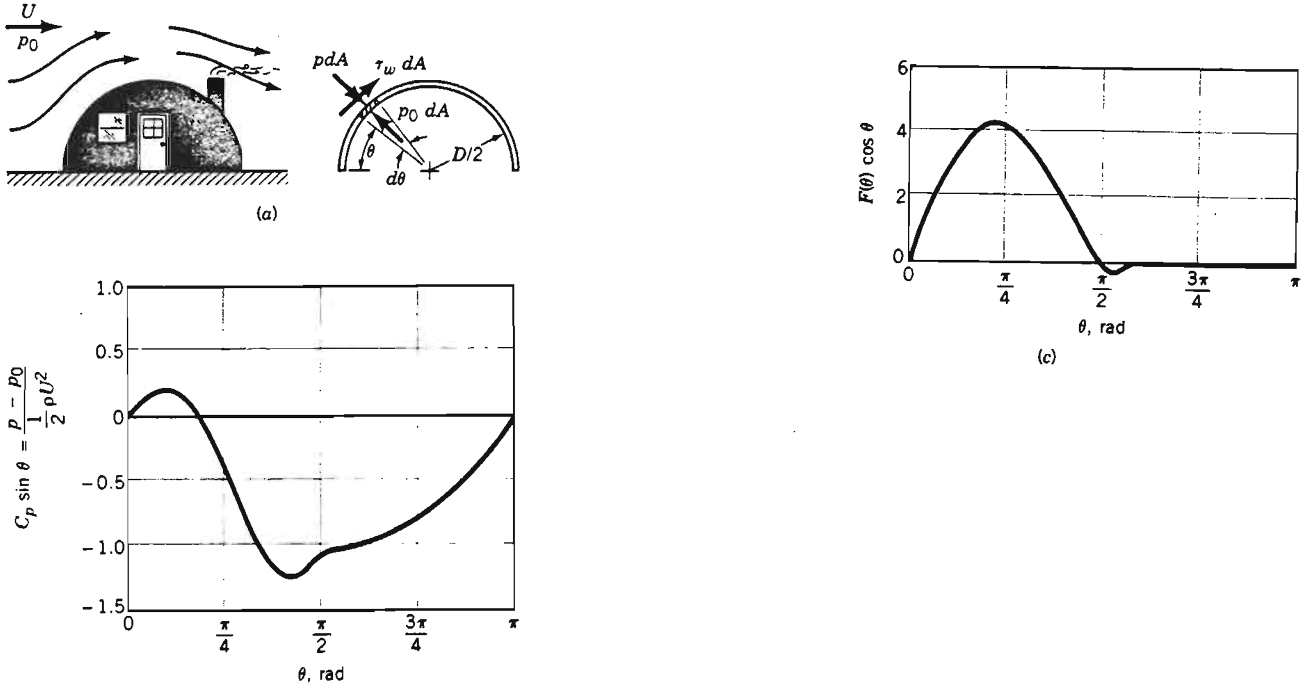
$$\mathscr{L} = \left(0.88 + \frac{1.96}{\sqrt{Re}}\right)\left(\tfrac{1}{2}\rho U^2 A\right)$$

OR

$$C_L = \frac{\mathscr{L}}{\frac{1}{2}\rho U^2 A} = 0.88 + \frac{1.96}{\sqrt{Re}}$$

Pressure lift

Shear lift.



(*a*)

(*b*)

(*c*)

and E9.14*c*. The results are

$$\int_0^{\pi} \frac{(p - p_0)}{\frac{1}{2}\rho U^2} \sin \theta \, d\theta = -1.76$$

and

$$\int_0^{\pi} F(\theta) \cos \theta \, d\theta = 3.92$$

Thus, the lift is

$$\mathscr{L} = \frac{1}{2}\rho U^2 A \left[\left(-\frac{1}{2}\right)(-1.76) + \frac{1}{2\sqrt{\text{Re}}}(3.92) \right]$$

or

$$\mathscr{L} = \left(0.88 + \frac{1.96}{\sqrt{\text{Re}}}\right)\left(\frac{1}{2}\rho U^2 A\right) \qquad \textbf{(Ans)}$$

4. Determine appropriate Reynold's No.

and

$$C_L = \frac{\mathscr{L}}{\frac{1}{2}\rho U^2 A} = 0.88 + \frac{1.96}{\sqrt{\text{Re}}} \qquad \textbf{(4)} \quad \textbf{(Ans)}$$

Consider a typical situation with $D = 20$ ft, $U = 30$ ft/s, $b = 50$ ft, and standard atmospheric conditions ($\rho = 2.38 \times 10^{-3}$ slugs/ft^3 and $\nu = 1.57 \times 10^{-4}$ ft^2/s), which gives a Reynolds number of

$$\text{Re} = \frac{UD}{\nu} = \frac{(30 \text{ ft/s})(20 \text{ ft})}{1.57 \times 10^{-4} \text{ ft}^2/\text{s}} = 3.82 \times 10^6$$

NOTE: Result not sensitiv to Re since pressure lift dominat

Hence, the lift coefficient is

$$C_L = 0.88 + \frac{1.96}{(3.82 \times 10^6)^{1/2}} = 0.88 + 0.001 = 0.881$$

NOTE: Lift for a circular cylinder (symmetrical) is zero.

Note that the pressure contribution to the lift coefficient is 0.88 whereas that due to the wall shear stress is only $1.96/(\text{Re}^{1/2}) = 0.001$. The Reynolds number dependency of C_L is quite minor. The lift is pressure dominated. Recall from Example 9.9 that this is also true for the drag on a similar shape.

From Eq. 4, we obtain the lift for the assumed conditions as

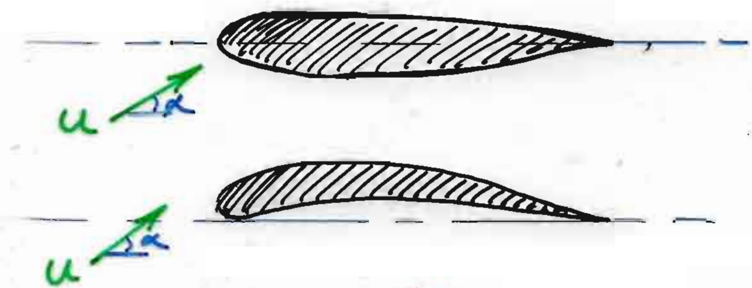
$$\mathscr{L} = \tfrac{1}{2}\rho U^2 A C_L = \tfrac{1}{2}(2.38 \times 10^{-3} \text{ slugs/ft}^3)(30 \text{ ft/s})^2(20 \text{ ft} \times 50 \text{ ft})(0.881)$$

or

$$\mathscr{L} = 944 \text{ lb}$$

There is a considerable tendency for the building to lift off the ground. Clearly this is due to the object being nonsymmetrical. The lift force on a complete circular cylinder is zero, although the fluid forces do tend to pull the upper and lower halves apart.

AIRFOILS

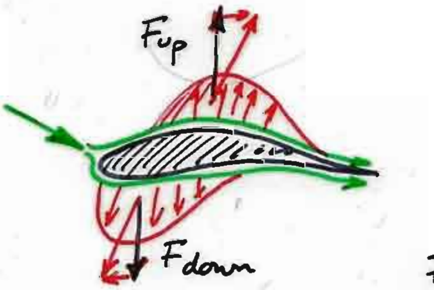


Symmetrical

No lift unless oblique angle of attack

Non-symmetrical

Lift even for $\alpha = 0$.
(Note a stall angle for -ve small α)



$F_{up} = F_{down}$ @ same small α.

Dependence of lift on Re

For  hut.

$$C_L = \frac{\mathcal{L}}{\frac{1}{2}\rho U^2 A} = \underset{\text{pressure lift}}{\underset{\uparrow}{0.88}} + \underset{\text{shear lift}}{\underset{\uparrow}{\frac{1.96}{\sqrt{Re}}}}$$

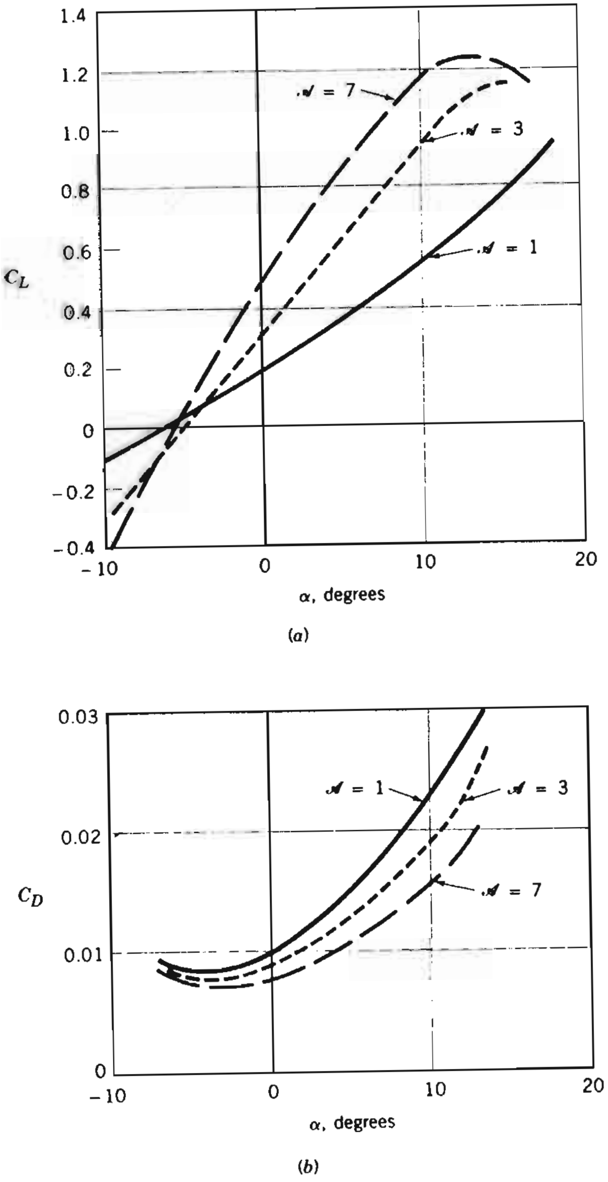
- ☐ C_L always +ve
- ☐ Shear effect diminishes for large Re.
  ∴ lift not strongly influenced by Re.

Drag also has U^2 dependence

- ☐ Power dependence of lift magnitude $\mathcal{L} = C_L \frac{1}{2}\rho U^2 A$

Double speed → quadruple lift.

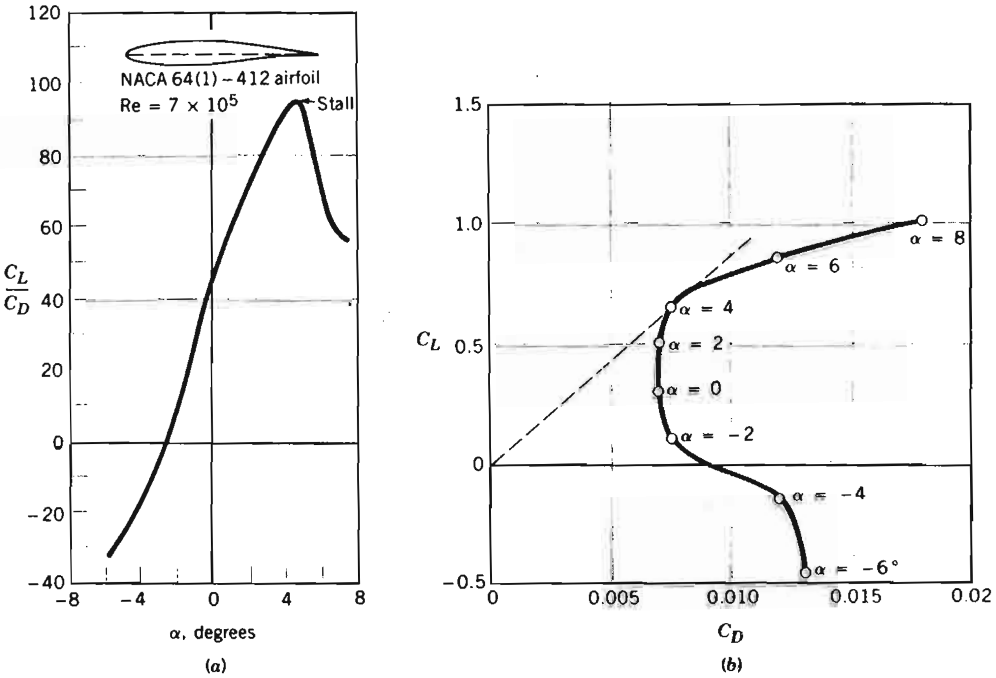
Changing angle of attack, α, changes effective geometry of wing. → Changes ratio $\frac{C_L}{C_D}$



(*a*)

(*b*)

■ FIGURE 9.36 Typical lift and drag coefficient data as a function of angle of attack and the aspect ratio of the airfoil: (*a*) lift coefficient, (*b*) drag coefficient.

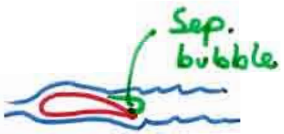


(*a*)

(*b*)

■ FIGURE 9.37 Two representations of the same lift and drag data for a typical airfoil: (*a*) lift-to-drag ratio as a function of angle of attack, with the onset of boundary layer separation on the upper surface indicated by the occurrence of stall, (*b*) the lift and drag polar diagram with the angle of attack indicated (Ref. 27).

Stall ≡ shedding of boundary layer



→ turbulent wake develops

→ drop in lift

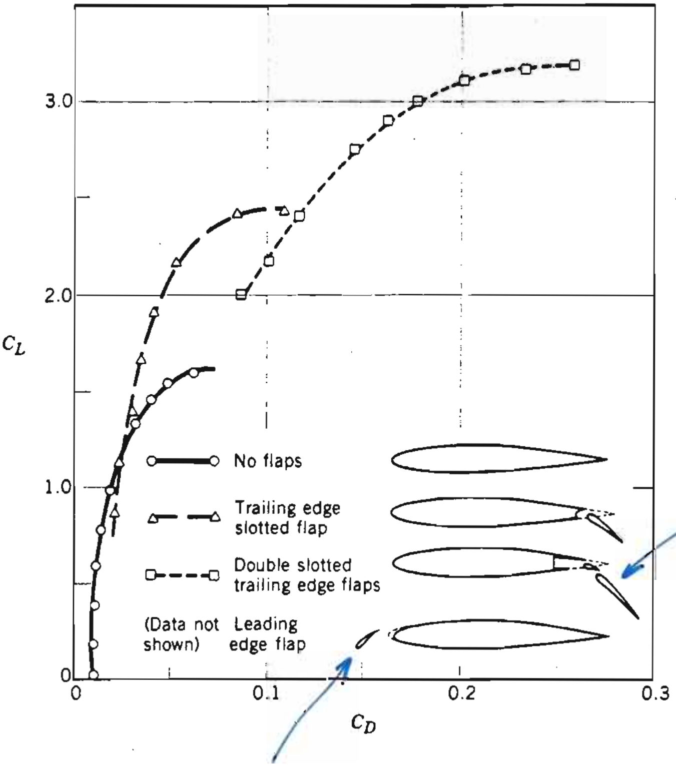
→

Modifying airfoil shape (section) changes lift coefficient.

$L \propto C_L \frac{1}{2} \rho u^2 A$

But drag also similarly related

$D \propto C_D \frac{1}{2} \rho u^2 A.$



Trailing edge

Leading edge.

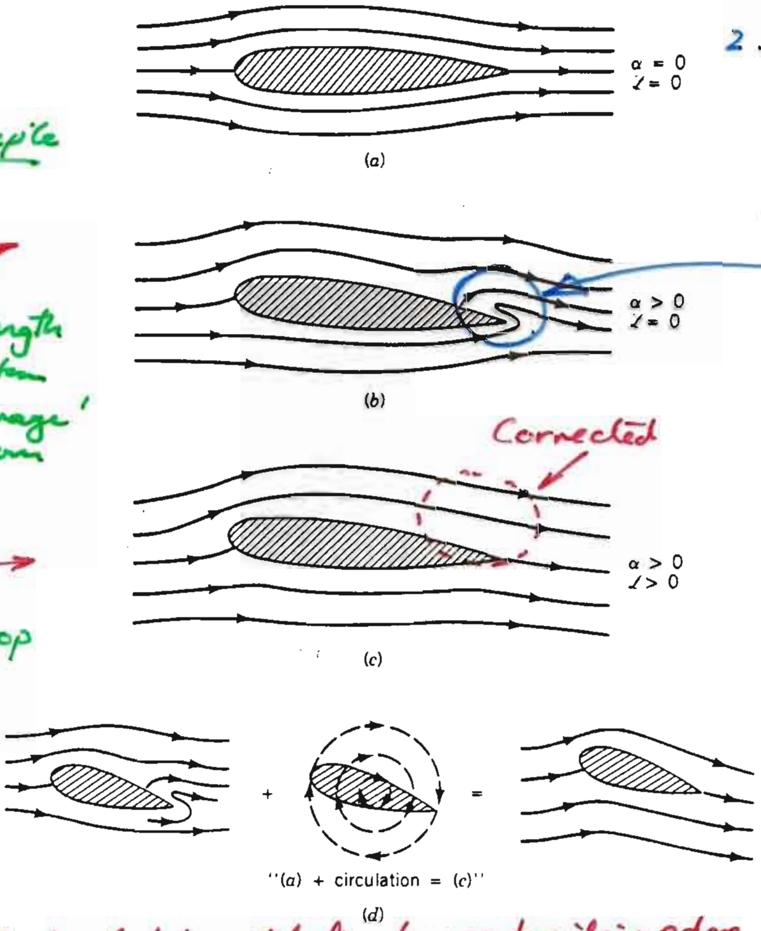
■ **FIGURE 9.38** Typical lift and drag alterations possible with the use of various types of flap designs (Ref. 21).

CIRCULATION

Bernoulli Principle

Same path length top and bottom ∴ same 'average' pressure

Longer path top than base ∴ +ve lift.



■ **FIGURE 9.39** **Inviscid flow past an airfoil: (*a*) symmetrical flow past the symmetrical airfoil at a zero angle of attack, (*b*) same airfoil at a nonzero angle of attack—no lift, flow near trailing edge not realistic, (*c*) same conditions as for (*b*) except circulation has been added to the flow–nonzero lift, realistic flow, (*d*) superposition of flows to produce the final flow past the airfoil.**

1. Viscous effects are very small when evaluating lift
2. Therefore use inviscid soln. to determine flow.

but

3. Inviscid soln. gives incorrect flow field @ trailing edge

therefore

4. Correct by superposing "imaginary" circulation.

enables

5. $\mathcal{L} = \int p \sin\theta \, dA$

Add circulation until flow leaves trailing edge.

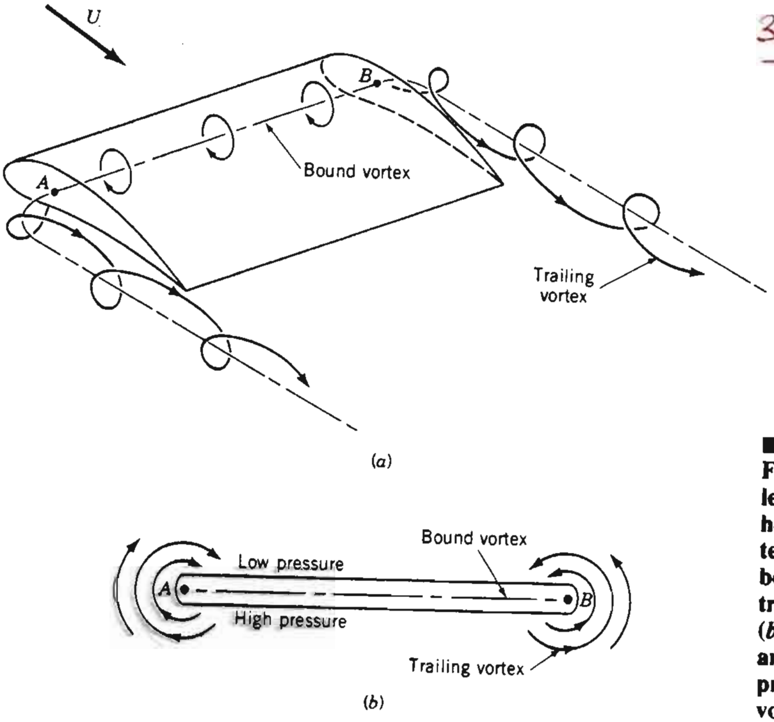
3-d EFFECTS

Leakage off wing ends:

1. Prevent with winglets
2. Wake lengths

5 mi. 747/757

1 mi 737/727 etc.



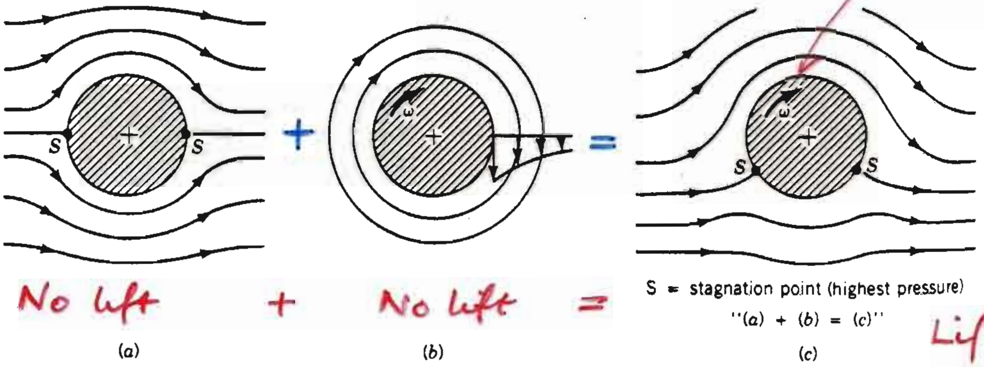
■ **FIGURE 9.40** **Flow past a finite length wing: (*a*) the horseshoe vortex system produced by the bound vortex and the trailing vortices; (*b*) the leakage of air around the wing tips produces the trailing vortices.**

ROTATING OBJECTS "MAGNUS" EFFECT

Linear flow

Static but rotating flow

Rotation effectively increases velocity in direction of rotation



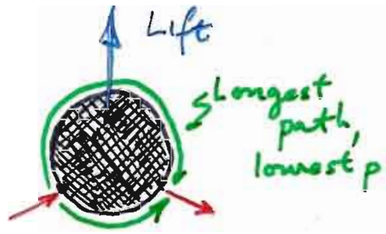
No lift + No lift =

S = stagnation point (highest pressure)

"(a) + (b) = (c)"

Lift

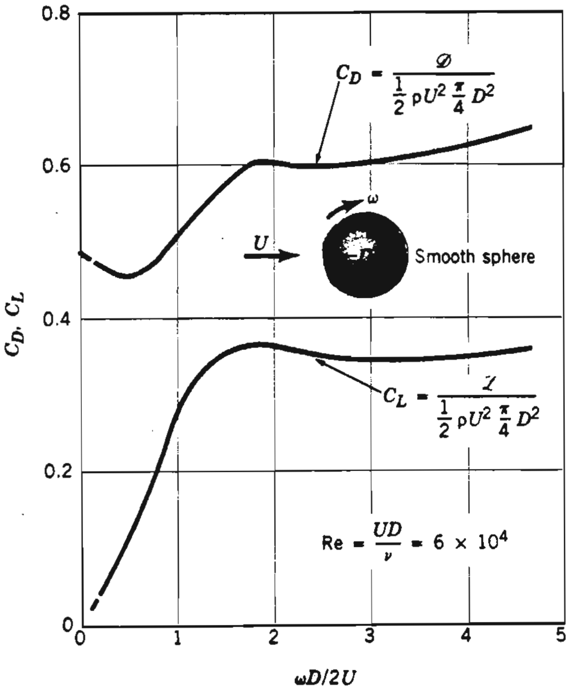
■ FIGURE 9.41 Inviscid flow past a circular cylinder: (*a*) uniform upstream flow without circulation, (*b*) free vortex at the center of the cylinder, (*c*) combination of free vortex and uniform flow past a circular cylinder giving nonsymmetric flow and a lift.



Note: Airflow →

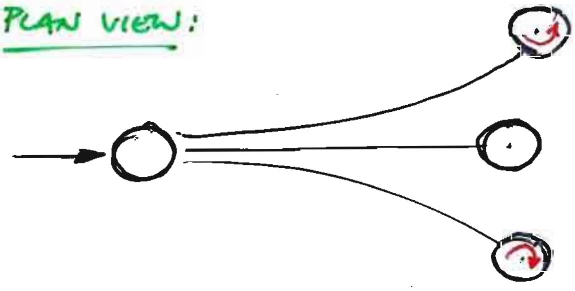
← Object moves

Which way does trajectory arc?



■ FIGURE 9.42 Lift and drag coefficients for a spinning smooth sphere (Ref. 23).

PLAN VIEW:

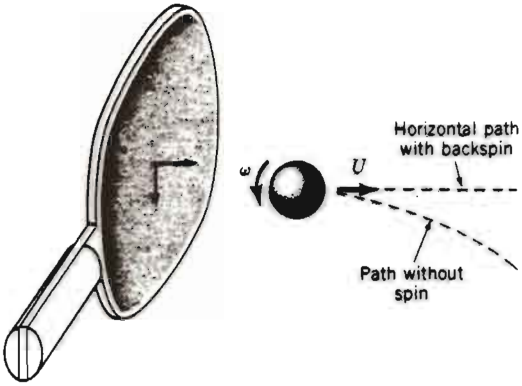


EXAMPLE 9.16

A table tennis ball weighing 2.45×10^{-2} N with diameter $D = 3.8 \times 10^{-2}$ m is hit at a velocity of $U = 12$ m/s with a back spin of angular velocity ω as is shown in Fig. E9.16. What is the value of ω if the ball is to travel on a horizontal path, not dropping due to the acceleration of gravity?

FOR HORIZONTAL FLIGHT

LIFT $\equiv$ WEIGHT



■ FIGURE E9.16

SOLUTION

For horizontal flight, the lift generated by the spinning of the ball must exactly balance the weight, $\mathcal{W}$, of the ball so that

$$\mathcal{W} = \mathcal{L} = \tfrac{1}{2}\rho U^2 A C_L$$

or

$$C_L = \frac{2\mathcal{W}}{\rho U^2(\pi/4)D^2}$$

where the lift coefficient, C_L, can be obtained from Fig. 9.42. For standard atmospheric conditions with $\rho = 1.23\ \text{kg/m}^3$ we obtain

$$C_L = \frac{2(2.45 \times 10^{-2}\ \text{N})}{(1.23\ \text{kg/m}^3)(12\ \text{m/s})^2(\pi/4)(3.8 \times 10^{-2}\ \text{m})^2} = 0.244$$

which, according to Fig. 9.42, can be achieved if

$$\frac{\omega D}{2U} = 0.9$$

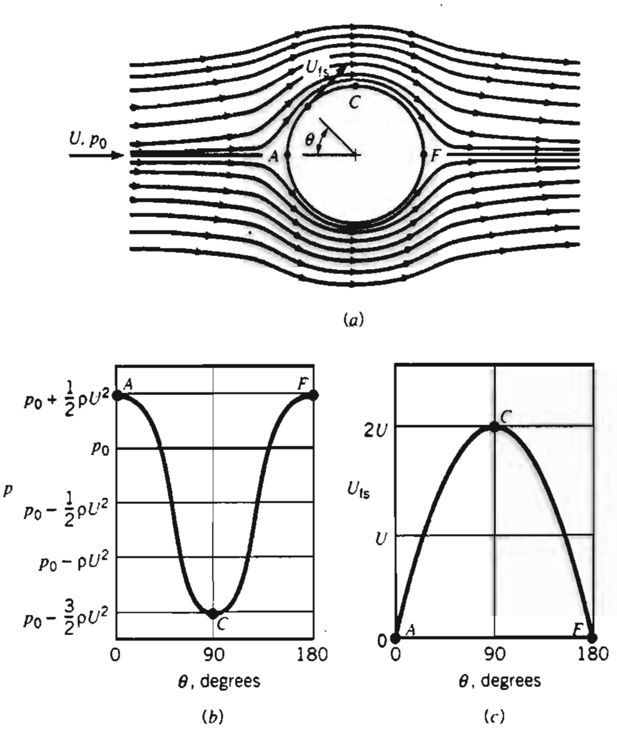
or

$$\omega = \frac{2U(0.9)}{D} = \frac{2(12\ \text{m/s})(0.9)}{3.8 \times 10^{-2}\ \text{m}} = 568\ \text{rad/s}$$

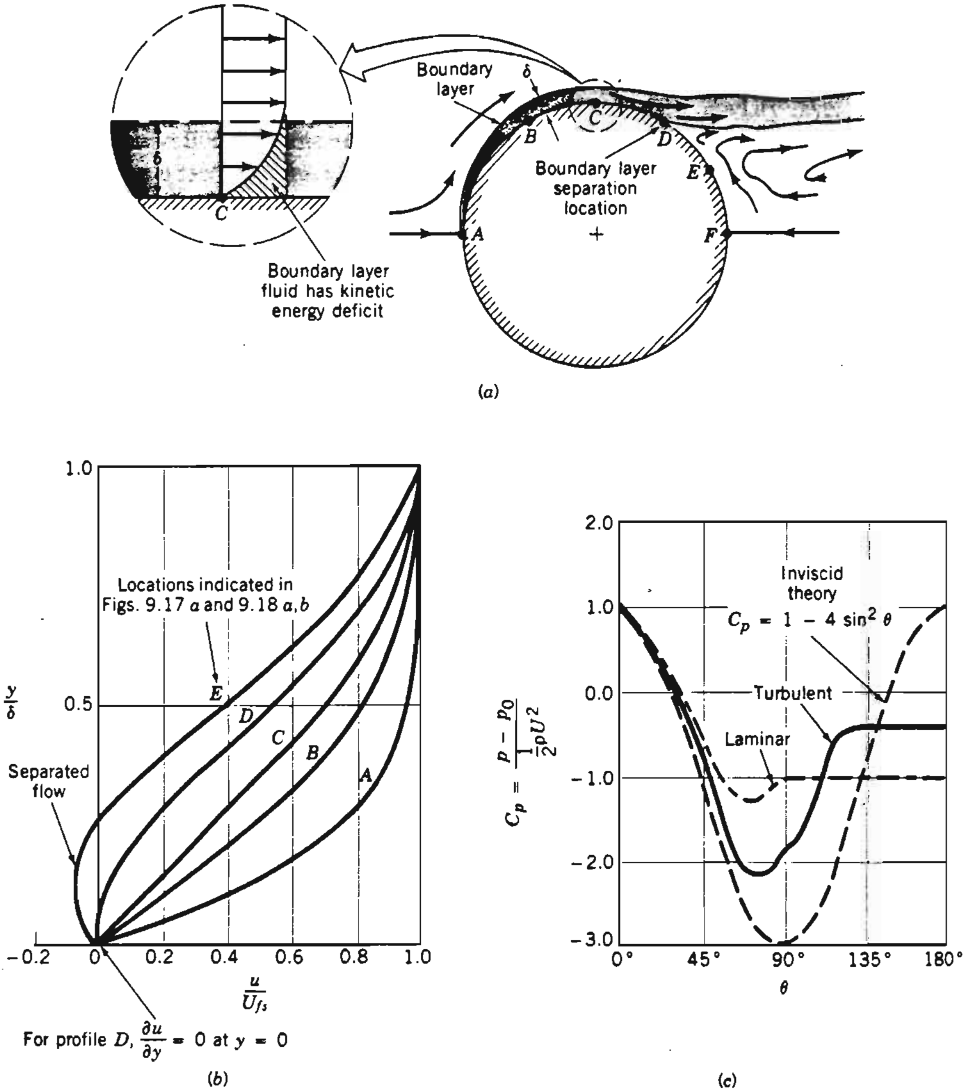
Thus,

$$\omega = (568\ \text{rad/s})(60\ \text{s/min})(1\ \text{rev}/2\pi\ \text{rad}) = 5420\ \text{rpm} \qquad \text{(Ans)}$$

Is it possible to impart this angular velocity to the ball? With larger angular velocities the ball will rise and follow an upward curved path. Similar trajectories can be produced by a well-hit golf ball—rather than falling like a rock, the golf ball trajectory is actually curved up and the spinning ball travels a greater distance than one without spin. However, if top spin is imparted to the ball (as in an improper tee shot) the ball will curve downward more quickly than under the action of gravity alone—the ball is ''topped'' and a negative lift is generated. Similarly, rotation about a vertical axis will cause the ball to hook or slice to one side or the other.



■ **FIGURE 9.16** **Inviscid flow past a circular cylinder: (*a*) streamlines for the flow if there were no viscous effects, (*b*) pressure distribution on the cylinder's surface, (*c*) free stream velocity on the cylinder's surface.**



■ **FIGURE 9.17** **Boundary layer characteristics on a circular cylinder: (*a*) boundary layer separation location, (*b*) typical boundary layer velocity profiles at various locations on the cylinder, (*c*) surface pressure distributions for inviscid flow and boundary layer flow.**

[13-14]

Open Channel Flows

Open Channel Flow [13-14]

$$R_h = \frac{A}{P}; \quad \mathbf{Re} = \frac{VR_h\rho}{\mu}; \quad \mathbf{Fr} = \frac{V}{\sqrt{gy}}; \quad c = \sqrt{gy}$$

Specific Energy: $E = y + \frac{q^2}{2gy^2}$; Specific Momentum: $M = \frac{y^2}{2} + \frac{q^2}{gy}$

Energy Equation: $y_1 + \frac{q_1^2}{2gy_1^2} + z_1 = y_2 + \frac{q_2^2}{2gy_2^2} + z_2 + S_f l \quad \rightarrow \quad E_1 = E_2 + (S_f - S_0)l$

$$E_{min} = \frac{3y_c}{2} \text{ at } \mathbf{Fr} = 1$$

$$\frac{dy}{dx} = \frac{S_f - S_0}{1 - \mathbf{Fr}^2}$$

Uniform Flow: $V = \frac{\kappa}{n} R_h^{\frac{2}{3}} S_0^{\frac{1}{2}} Q = \frac{\kappa}{n} A R_h^{\frac{2}{3}} S_0^{\frac{1}{2}} \kappa = 1(SI) \kappa = 1.49(BGS)$

Hydraulic Jump: $\frac{y_2}{y_1} = \frac{1}{2}(-1 + \sqrt{1 + 8Fr_1^2}) \frac{h_L}{y_1} = 1 - \frac{y_2}{y_1} + \frac{1}{2} Fr_1^2 [1 - (\frac{y_1}{y_2})^2]$

Sharp-Crested Weir: $Q = C_{rectangular} \frac{2}{3} \sqrt{2gh^3} b; Q = C_{triangular} \frac{8}{15} \tan\frac{\theta}{2} \sqrt{2g} H^{5/2}.$

Broad-Crested Weir: $Q = C_{wb} b \sqrt{g} (\frac{2}{3})^{3/2} H^{3/2} : C_{wb} = \frac{0.65}{(1 + H / P_w)^{1/2}}.$

Underflow Gates: $q = C_d a \sqrt{2gy_1}$

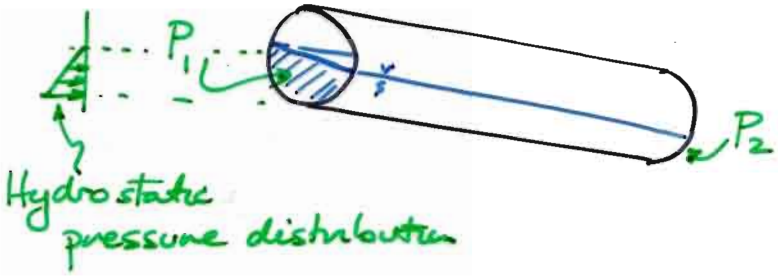
[13:1] Open Channel Flows

Outline

- Flow classifications
  - Hydraulic radius
- Uniform Flows – Chezy/Manning formulae $\frac{dy}{dx} = 0$
- Gradually varying flows $\frac{dy}{dx} << 1$
  - Wave speeds
  - Specific energy

OPEN CHANNEL FLOW

Differences from pipe flow



- ☐ Cannot sustain pressure differential, Δp
- ☐ Remaining driving mechanism $\equiv$ gravity

$\therefore$ Indicies are:

Reynolds' No. $Re = \frac{VD\rho}{\mu} = \frac{\text{inertia}}{\text{viscous}}$

Froude No. $Fr = \frac{V}{\sqrt{gl}} = \frac{\text{inertia}}{\text{gravity}}$

Re for open channels:

$$\underbrace{D_h = \frac{4A}{P}\ ;\ R_h = \frac{A}{P}}_{D_h = 4R_h}$$

$$Re = \frac{V(4R_h)\rho}{\mu} \leq 2000$$

gives turbulent flow.

* Note different definition from pipe flow for Re

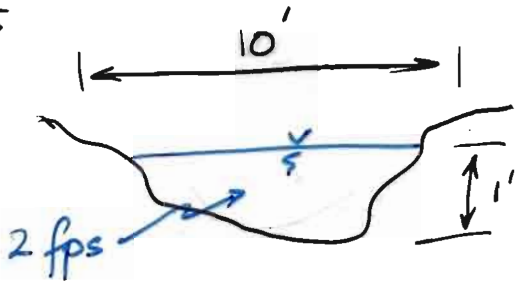
Pipe flow : $Re = \frac{V(D_h)\rho}{\mu}$	<2000	$2000-4000$	>4000
Channel flow : $Re = \frac{V(R_h)\rho}{\mu}$	<500	$500-1000$	>1000
	Laminar	Transitional	Turbulent

But similar limits for turbulent flow.

Open channel flows typically turbulent (water)

$$Re = \frac{V R_h \rho}{\mu} \quad \begin{cases} V \text{ small} & \text{(gravity driven)} \\ R_h \text{ large} & \text{(much larger than pipes)} \end{cases}$$

Example



$$R_h = \frac{A}{P} \simeq \frac{10}{12}$$

$$Re = \frac{(2)(10/12)(1.94)}{2.34 \times 10^{-5} \text{ lb.s/ft}^2}$$

slugs/ft³

$$\simeq 1.4 \times 10^{5} \quad \therefore \text{ turbulent}$$

∴ Turbulent

Wider/larger channel with similar velocity is also more turbulent.

GENERAL DEFINITIONS

- ☐ Uniform $\frac{\partial y}{\partial x} = 0$
  no change in depth
- ☐ Non uniform $\frac{dy}{dx} \neq 0$
  - ☐ GVF $\frac{dy}{dx} \ll 1$
  - ☐ RVF $\frac{dy}{dx} \to ?$

Flow classifications

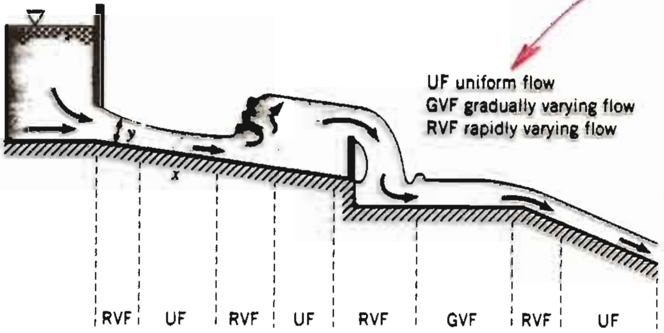


FIGURE 10.1 Classification of open-channel flow.

STEADY -vs- UNSTEADY

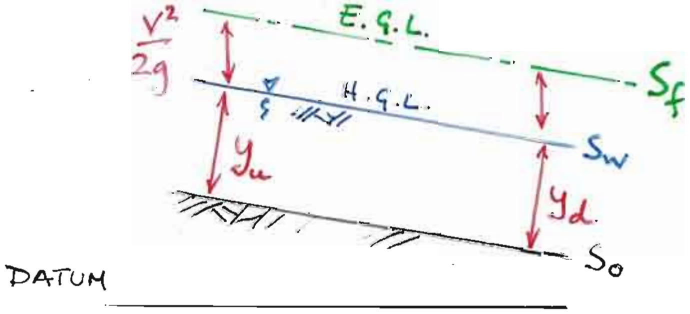
$$\frac{\partial}{\partial t} \to 0 \implies \text{Steady}$$

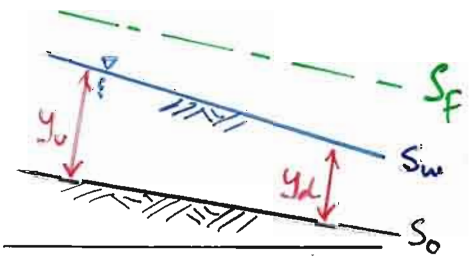
Usually adequate

Nonsteady:
- ☐ Dam release / break
- ☐ Rainfall / flash flood.
- ☐ Tides (quasi-steady)
- ☐ Tsunamis

UNIFORM -vs- NONUNIFORM

$$\frac{\partial}{\partial s} \to 0 \implies \text{Uniform}$$





UNIFORM FLOW

$$y_u = y_d$$

$$S_f = S_w = S_o$$

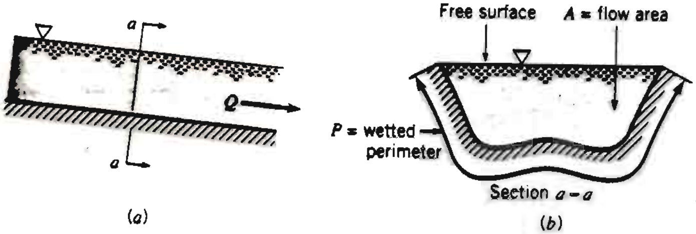
NON-UNIFORM FLOW

$$y_u \neq y_d$$

$$S_f \neq S_w \neq S_o$$

UNIFORM DEPTH FLOWS $\frac{dy}{dx} = 0$ i.e. Adjust base slope, $S_o = S_f$

VISCOUS LOSS EQUALS GRAVITY DRIVE EXACTLY



$$\frac{dy}{dx} = \frac{(S_f - S_o)}{(1 - Fr^2)}$$

■ FIGURE 10.8 Uniform flow in an open channel.

CHEZY & MANNING FORMULAE:

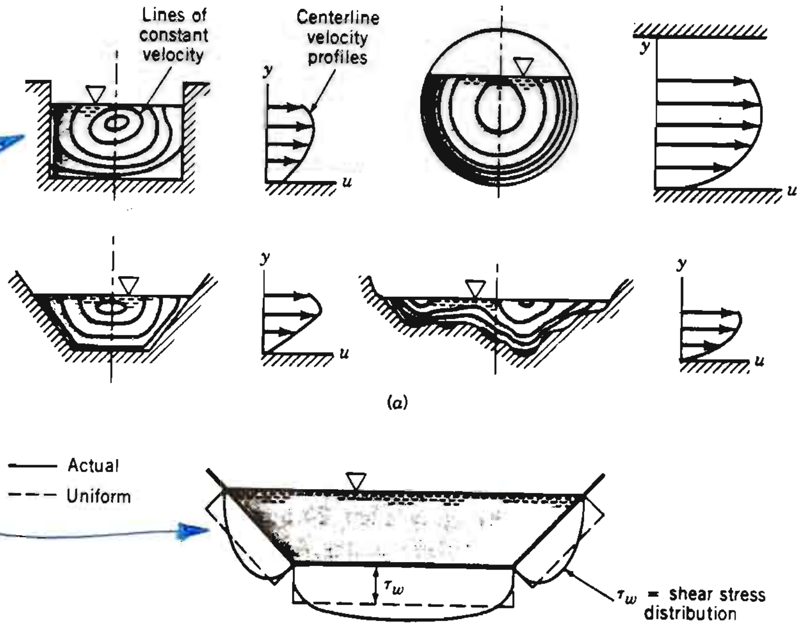
Assume: $\frac{dV}{dz} = 0$

OK assumption

Assume:

τ is uniform

OK



■ FIGURE 10.9 Typical velocity and shear stress distributions in an open channel: (*a*) velocity distribution throughout the cross section, (*b*) shear stress distribution on the wetted perimeter.

DESPITE $\frac{dV}{dz} \neq 0$ and $\tau \neq$ constant

CHEZY & MANNING EQUATIONS ARE OK.

UNIFORM FLOW

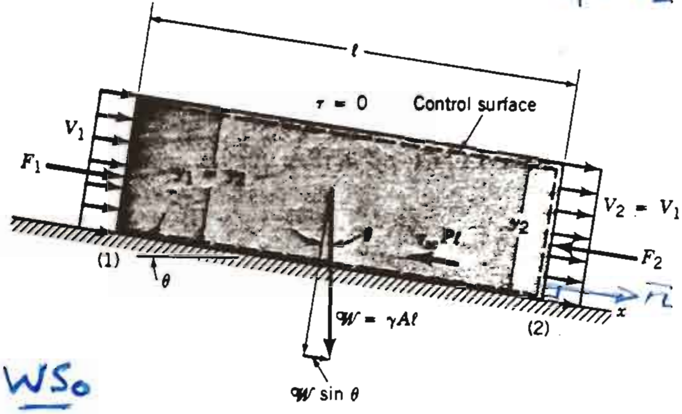
$$\frac{dy}{dx} = 0 \quad \text{i.e. } y_1 = y_2$$

MOMENTUM BALANCE:

$$\Sigma F_x = \rho Q(V_2 - V_1) = 0$$

$$\therefore \Sigma F_x = 0$$

$$\therefore V_1 = V_2$$



FORCE BALANCE:

$$\underbrace{F_1 - F_2}_{0} - \tau_w P\ell + W\sin\theta = 0$$

Rearranging: $\tau_w = \dfrac{W\sin\theta}{P\ell} = \dfrac{WS_o}{P\ell}$

Small θ: $\sin\theta \approx \tan\theta \approx S_o$

$W = A\ell\gamma$

$R_h = \dfrac{A}{P}$

$$\Rightarrow \quad \tau_w = \frac{A}{P}\frac{\ell}{\ell}\gamma S_o = R_h \gamma S_o$$

Turbulent flow $\quad \tau_w = K_L \rho \dfrac{V^2}{2}$ (Darcy-Weisbach formula.)

Combining $\quad V = \sqrt{\dfrac{2\gamma}{K_L\rho}}\sqrt{R_h S_o} = \boxed{C\sqrt{R_h S_o} = V}$

C = Chezy coef.

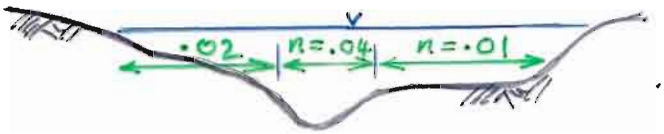
Manning modification $\quad$ SI: $C = \dfrac{R_h^{1/6}}{n}$ $\quad$ English: $C = 1.486\dfrac{R_h^{1/6}}{n}$

$$\boxed{\begin{aligned} V &= \frac{K}{n} R_h^{2/3} S_o^{1/2} \\ Q &= \frac{K}{n} A\, R_h^{2/3} S_o^{1/2} \end{aligned}}$$

K = 1 SI: m/s

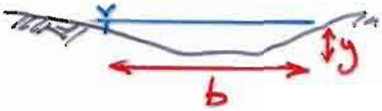
K = 1.49 English: ft/s

AVERAGE 'n' VALUES



WIDE CHANNEL APPROXIMATION

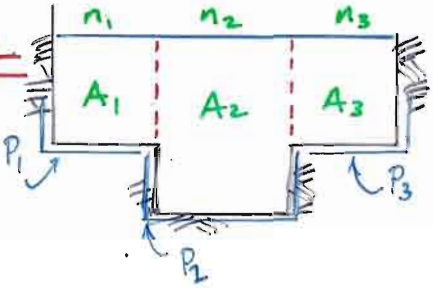
$$R_h = \frac{A}{P} \cong \frac{by}{b+2y}$$



$$\frac{b}{y} > 10 \qquad \boxed{R_h \simeq y}$$

COMPOUND CHANNEL

$$R_h = \frac{A}{P}$$



$$Q = \frac{K}{n_1} A_1 R_1^{2/3} S_o^{1/2} + \frac{K}{n_2} A_2 R_2^{2/3} S_o^{1/2} + \frac{K}{n} A_3 R_3^{2/3} S_o^{1/2}$$

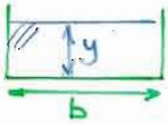
■ TABLE 10.1

Values of the Manning Coefficient, *n* (Ref. 6)

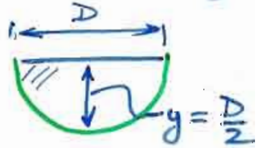
Wetted Perimeter	*n*
A. ***Natural channels***	
Clean and straight	0.030
Sluggish with deep pools	0.040
Major rivers	0.035
B. ***Floodplains***	
Pasture, farmland	0.035
Light brush	0.050
Heavy brush	0.075
Trees	0.15
C. ***Excavated earth channels***	
Clean	0.022
Gravelly	0.025
Weedy	0.030
Stony, cobbles	0.035
D. ***Artificially lined channels***	
Glass	0.010
Brass	0.011
Steel, smooth	0.012
Steel, painted	0.014
Steel, riveted	0.015
Cast iron	0.013
Concrete, finished	0.012
Concrete, unfinished	0.014
Planed wood	0.012
Clay tile	0.014
Brickwork	0.015
Asphalt	0.016
Corrugated metal	0.022
Rubble masonry	0.025

MOST EFFICIENT SECTION

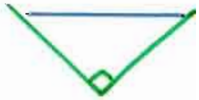
$$F_f = \tau \underset{\uparrow\text{Minimize}}{P} \ell \qquad Q = \frac{K}{n} \underset{\uparrow\text{Maximize}}{A} R_h^{2/3} S_o^{1/2}$$



Rectangle $y = \frac{1}{2} b$



Circle



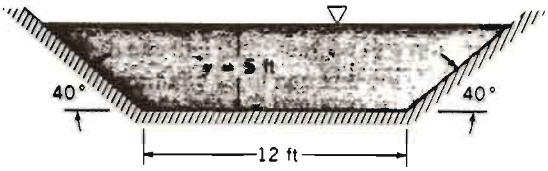
Triangle

$\frac{1}{2}$ Hexagon

EXAMPLE 10.3

FIND Q

Water flows in the canal of trapezoidal cross section shown in Fig. E10.3. The bottom drops 1.4 ft per 1000 ft of length. Determine the flowrate if the canal is lined with new smooth concrete, or if weeds cover the wetted perimeter. Determine the Froude number for each of these flows.



■ **FIGURE E10.3**

1. $R_h = \frac{A}{P}$

2. $S_o = \frac{1.4}{1000}$

3. $n = ?$ use table 10.1.

SOLUTION

From Eq. 10.20,

$$\boxed{Q = \frac{1.49}{n} A R_h^{2/3} S_0^{1/2}} \tag{1}$$

where we have used $\kappa = 1.49$, since the dimensions are given in BG units. For a depth of $y = 5$ ft, the flow area is

$$A = 12 \text{ ft } (5 \text{ ft}) + 5 \text{ ft} \left(\frac{5}{\tan 40°} \text{ ft} \right) = 89.8 \text{ ft}^2$$

so that with a wetted perimeter of $P = 12 \text{ ft} + 2(5/\sin 40° \text{ ft}) = 27.6$ ft, the hydraulic radius is determined to be $R_h = A/P = 3.25$ ft. Note that even though the channel is quite wide (the free-surface width is 23.9 ft), the hydraulic radius is only 3.25 ft, which is less than the depth.

Thus, with $S_0 = 1.4 \text{ ft}/1000 \text{ ft} = 0.0014$, Eq. 1 becomes

$$Q = \frac{1.49}{n}(89.8 \text{ ft}^2)(3.25 \text{ ft})^{2/3}(0.0014)^{1/2} = \frac{10.98}{n}$$

where Q is in ft^3/s.

From Table 10.1, the values of n are estimated to be $n = 0.012$ for the smooth concrete and $n = 0.030$ for the weedy conditions. Thus,

$$Q = \frac{10.98}{0.012} = 915 \text{ cfs} \qquad \text{(SMOOTH)} \qquad \text{(Ans)}$$

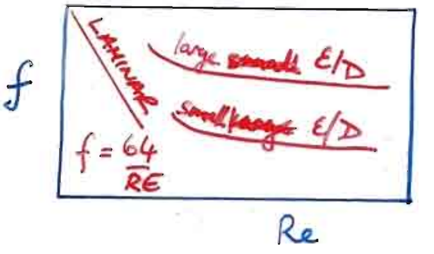
for the new concrete lining and

$$Q = \frac{10.98}{0.030} = 366 \text{ cfs} \qquad \text{(ROUGH)} \qquad \text{(Ans)}$$

for the weedy lining. The corresponding average velocities, $V = Q/A$, are 10.2 ft/s and 4.08 ft/s, respectively. It does not take a very steep slope ($S_0 = 0.0014$ or $\theta = \tan^{-1}(0.0014) = 0.080°$) for this velocity.

V↓ with ↑ ROUGHNESS (ε)

Recall: MOODY for PIPES



∴ f not controlled by Re.

Note that the increased roughness causes a decrease in the flowrate. This is an indication that for the turbulent flows involved, the wall shear stress increases with surface roughness. [For water at 50 °F, the Reynolds number based on the 3.25-ft hydraulic radius of the channel is $\text{Re} = R_h V/\nu = 3.25 \text{ ft } (4.08 \text{ ft/s})/(1.41 \times 10^{-5} \text{ ft}^2/\text{s}) = \underline{9.40 \times 10^5}$, well into the turbulent regime.]

Check $Re = 9.4 \times 10^5$

∴ Turbulent OK.

The Froude numbers based on the maximum depths for the two flows can be determined from $\text{Fr} = V/(gy)^{1/2}$. For the new concrete case,

$$\text{Fr} = \frac{10.2 \text{ ft/s}}{(32.2 \text{ ft/s}^2 \times 5 \text{ ft})^{1/2}} = 0.804 \qquad \text{(Ans)}$$

while for the weedy case

$$\text{Fr} = \frac{4.08 \text{ ft/s}}{(32.2 \text{ ft/s}^2 \times 5 \text{ ft})^{1/2}} = 0.322 \qquad \text{(Ans)}$$

In either case the flow is subcritical.

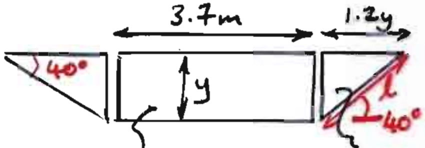
The same results would be obtained for the channel if its size were given in meters. We would use the same value of n but set $\kappa = 1$ for this SI units situation.

NEED Q = PRESCRIBED
DETERMINE y

EXAMPLE 10.4

Water flows in the channel shown in Fig. E10.3 at a rate of $Q = 10.0\ \text{m}^3/\text{s}$. If the canal lining is weedy, determine the depth of the flow.

SOLUTION



$A_1 = 3.7y$ $\quad A_2 = \frac{1}{2}y^2(1.2)$

$A = A_1 + 2A_2$

$l \sin 40 = y \Rightarrow l = \frac{y}{\sin 40}$

Substitute
$\left.\begin{matrix} A = f(y) \\ R_h = f(y) \end{matrix}\right\}$ Chezy.

In this instance neither the flow area nor the hydraulic radius are known, although they can be written in terms of the depth, y, as

$$\boxed{A = 1.19y^2 + 3.66y}$$

where the bottom width is (12 ft)(1 m/3.281 ft) = 3.66 m and A and y are in square meters and meters, respectively. Also, the wetted perimeter is

$$P = 3.66 + 2\left(\frac{y}{\sin 40°}\right) = 3.11y + 3.66$$

so that

$$R_h = \frac{A}{P} = \frac{1.19y^2 + 3.66y}{3.11y + 3.66}$$

where R_h and y are in meters. Thus, with $n = 0.030$ (from Table 10.1), Eq. 10.20 can be written as

$$Q = 10 = \frac{\kappa}{n} A R_h^{2/3} S_0^{1/2}$$

$$= \frac{1.0}{0.030}(1.19y^2 + 3.66y)\left(\frac{1.19y^2 + 3.66y}{3.11y + 3.66}\right)^{2/3}(0.0014)^{1/2}$$

which can be rearranged into the form

$$(1.19y^2 + 3.66y)^5 - 515(3.11y + 3.66)^2 = 0 \qquad \textbf{(1)}$$

where y is in meters. The solution of Eq. 1 can be easily obtained by use of a simple root-finding numerical technique or by trial-and-error methods. The only physically meaningful root of Eq. 1 (i.e., a positive, real number) gives the solution for the normal flow depth at this flowrate as

$$\boxed{y = 1.50\ \text{m}} \qquad \textbf{(Ans)}$$

EXAMPLE 10.6

Water flows in a rectangular channel of width $b = 10$ m that has a Manning coefficient of $n = 0.025$. Plot a graph of flowrate, Q, as a function of slope, S_0, indicating lines of constant depth and lines of constant Froude number.

SOLUTION

For this channel the flow area is $A = by = 10y$, and the hydraulic radius is $R_h = A/P = by/(b + 2y)$, where y is the flow depth. Thus, the Manning equation (Eq. 10.19) becomes

$$V = \frac{\kappa}{n} R_h^{2/3} S_0^{1/2} = \frac{1.0}{0.025}\left(\frac{10y}{10 + 2y}\right)^{2/3} S_0^{1/2} \tag{1}$$

$$V = \sqrt{gy}\,\frac{V}{\sqrt{gy}} = \sqrt{gy}\left(\overset{\substack{\text{Fr}\\ \downarrow}}{\frac{V}{\sqrt{gy}}}\right)$$

where we have set $\kappa = 1$ because we are using SI units. Since we wish to plot Q versus S_0 at constant values of Fr, we use the fact that for a rectangular channel $\text{Fr} = V/(gy)^{1/2}$ and write Eq. 1 as

$$(gy)^{1/2}\text{Fr} = \frac{1}{0.025}\left(\frac{10y}{10 + 2y}\right)^{2/3} S_0^{1/2}$$

which simplifies to

$$S_0 = 0.00613(\text{Fr})^2\, y\left(\frac{5 + y}{5y}\right)^{4/3} \tag{2}$$

$S_o = f[Fr, y]$

and

$Q = f[S_o, Fr, y]$

For a given value of Fr, we pick various values of y, determine the corresponding values of S_0 from Eq. 2, and then calculate $Q = VA$, with V from either Eq. 1 or $V = (gy)^{1/2}$ Fr. The results are indicated in Fig. E10.6.

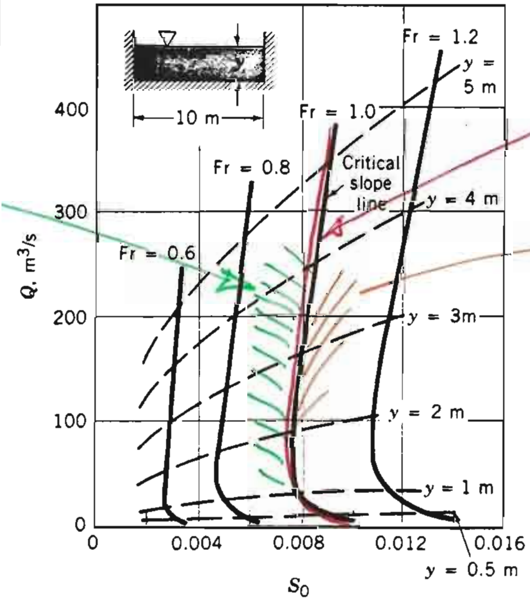


FIGURE E10.6

RESULT

1. Weak dependence of $S_{o_{CRITICAL}}$ on flowrate, Q.

Note that for a given flowrate, there is a specific value of the slope that gives the critical condition, Fr = 1. This slope, denoted S_{0c}, is called the *critical slope*. The critical slope line divides the graph into two regions—one subcritical and the other supercritical. The dependence of S_{0c} on Q is rather weak over a large range of Q. If the slope is such that the flow is critical or nearly critical, it will remain so over a wide range of flowrates (and depths).

Lines of constant depth, y, are also indicated in Fig. E10.6. A figure like this allows one to easily see what effects are to be expected by varying the parameters involved.

EXAMPLE 10.7

COMPOSITE CHANNEL

Water flows along the drainage canal having the properties shown in Fig. E10.7. If the bottom slope is $S_0 = 1 \text{ ft}/500 \text{ ft} = 0.002$, estimate the flowrate.

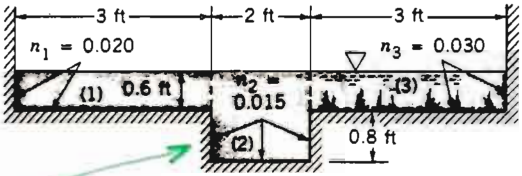


FIGURE E10.7

P= wetted 'static' perimeter only.

SOLUTION

We divide the cross section into three subsections as is indicated in Fig. E10.7 and write the flowrate as $Q = Q_1 + Q_2 + Q_3$, where for each section

$$Q_i = \frac{1.49}{n_i} A_i R_{h_i}^{2/3} S_0^{1/2}$$

The appropriate values of A_i, P_i, R_{hi}, and n_i are listed in Table E10.7. Note that the imaginary portions of the perimeters between sections (denoted by the dashed lines in Fig. E10.7) are not included in the P_i. That is, for section (2)

TABLE E10.7

i	A_i (ft²)	P_i (ft)	R_{hi} (ft)	n_i
1	1.8	3.6	0.500	0.020
2	2.8	3.6	0.778	0.015
3	1.8	3.6	0.500	0.030

$$A_2 = 2 \text{ ft } (0.8 + 0.6) \text{ ft} = 2.8 \text{ ft}^2$$

and

$$P_2 = 2 \text{ ft} + 2(0.8 \text{ ft}) = 3.6 \text{ ft}$$

so that

$$R_{h_2} = \frac{A_2}{P_2} = \frac{2.8 \text{ ft}^2}{3.6 \text{ ft}} = 0.778 \text{ ft}$$

Thus, the total flowrate is

$$Q = Q_1 + Q_2 + Q_3 = 1.49(0.002)^{1/2}$$
$$\times \left[\frac{(1.8 \text{ ft}^2)(0.500 \text{ ft})^{2/3}}{0.020} + \frac{(2.8 \text{ ft}^2)(0.778 \text{ ft})^{2/3}}{0.015} + \frac{(1.8 \text{ ft}^2)(0.500 \text{ ft})^{2/3}}{0.030} \right]$$

or

$$Q = 16.8 \text{ ft}^3/\text{s} \qquad \text{(Ans)}$$

ALTERNATIVE:
May determine 'effective' manning coefficient, n_{eff}.

$n_{max} > n_{effective} > n_{min}$

If the entire channel cross section were considered as one flow area, then $A = A_1 + A_2 + A_3 = 6.4 \text{ ft}^2$ and $P = P_1 + P_2 + P_3 = 10.8$ ft, or $R_h = A/P = 6.4 \text{ ft}^2/10.8 \text{ ft} = 0.593$ ft. The flowrate is given by Eq. 10.20, which can be written as

$$Q = \frac{1.49}{n_{\text{eff}}} A R_h^{2/3} S_0^{1/2}$$

where n_{eff} is the effective value of n for this channel. With $Q = 16.8 \text{ ft}^3/\text{s}$ as determined above, the value of n_{eff} is found to be

$$n_{\text{eff}} = \frac{1.49 A R_h^{2/3} S_0^{1/2}}{Q}$$
$$= \frac{1.49(6.4)(0.593)^{2/3}(0.002)^{1/2}}{16.8} = 0.0179$$

As expected, the effective roughness (Manning n) is between the minimum ($n_2 = 0.015$) and maximum ($n_3 = 0.030$) values for the individual subsections.

$$Q = A \frac{K}{n} R_h^{2/3} S_o^{1/2}$$

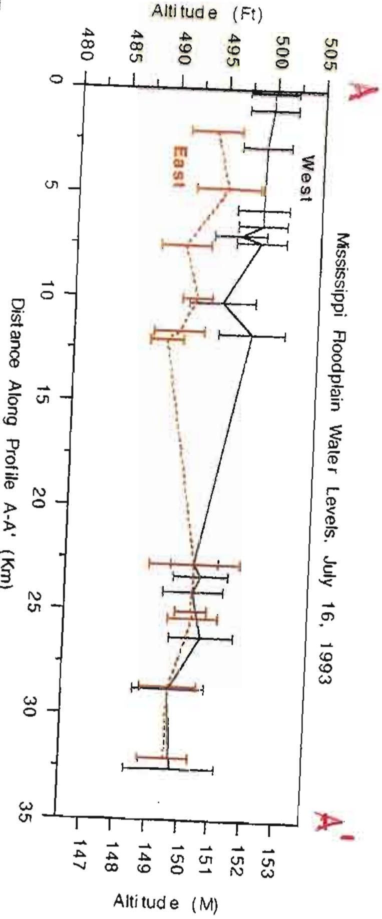


Fig. 4. Longitudinal profiles of the flood stage level on the two opposing sides of the valley reach shown in Figures 1, 2, and 3. Ninety percent confidence intervals are given. Flood stages along the west side of the valley above lock and dam 20 were significantly higher than those along the east side.

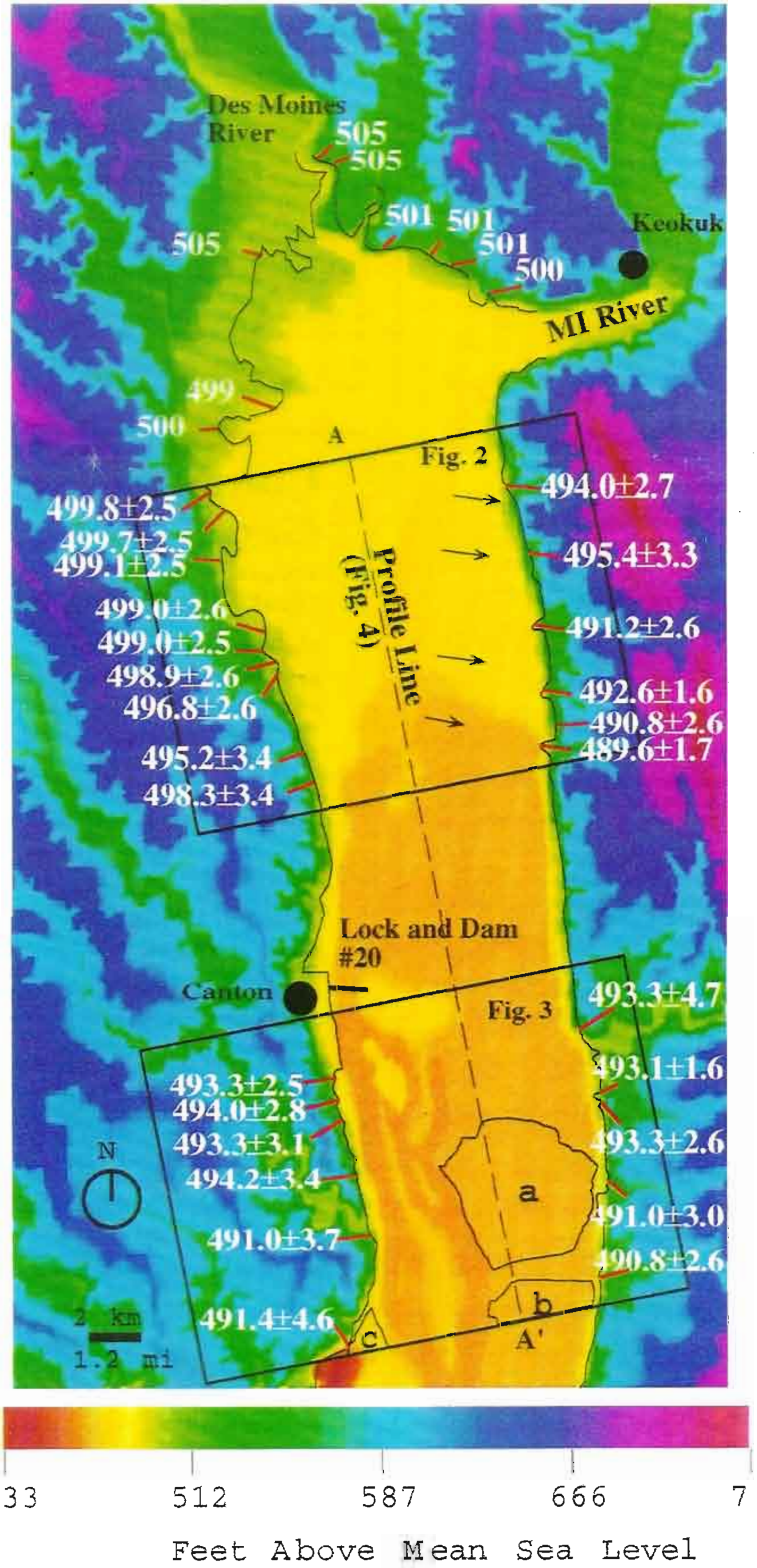


Fig. 1. Topography of the study reach of the upper Mississippi Valley and the overbank flood (black line) on the evening of July 16, 1993. Digital topography is from the U.S. Geological vey. Flood limits are interpreted from an ERS-1 georeferenced fine resolution scene (see Fig 2 and 3). The flood stage measurements illustrated were obtained by comparison of the floc limits to U.S. Geological Survey 1:24,000 topographic maps. Arrows mark zones of inferred rent-induced rough water; lettered areas are plumes of turbid water imaged by the Landsat T matic Mapper sensor on July 25 (see text and Figure 3).

[13:2] Open Channel Flows

Recap

Flow classifications

- Hydraulic radius

Uniform Flows – Chezy/Manning formulae $\frac{dy}{dx} = 0$

Outline

Gradually varying flows $\frac{dy}{dx} << 1$

- Wave speeds
- Specific energy

GENERAL DEFINITIONS

- ☐ Uniform $\frac{\partial y}{\partial x} = 0$
  no change in depth
- ☐ Non uniform $\frac{dy}{dx} \neq 0$
  - ☐ GVF $\frac{dy}{dx} \ll 1$
  - ☐ RVF $\frac{dy}{dx} \to ?$

Flow classifications

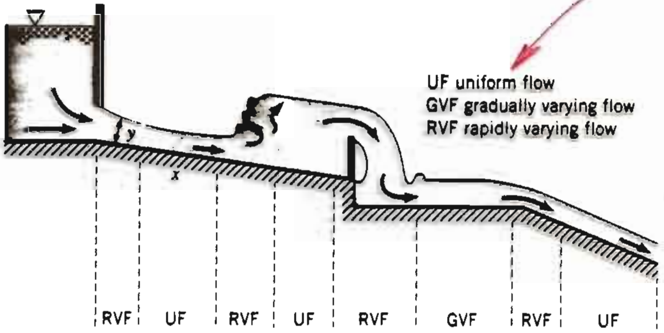


FIGURE 10.1 Classification of open-channel flow.

STEADY -VS- UNSTEADY

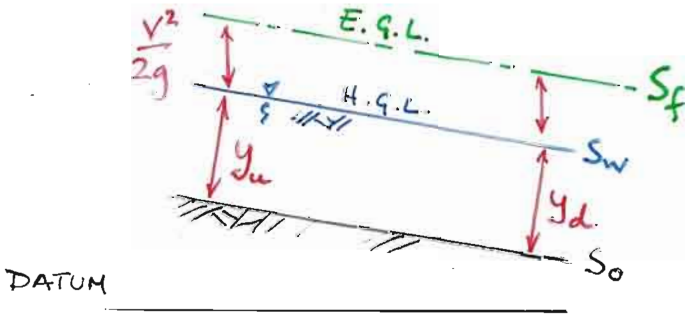
$\frac{\partial}{\partial t} \to 0 \Rightarrow$ Steady

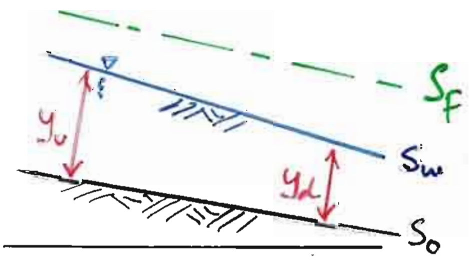
Usually adequate

Nonsteady:
- ☐ Dam release / break
- ☐ Rainfall / flash flood.
- ☐ Tides (quasi-steady)
- ☐ Tsunamis

UNIFORM -VS- NONUNIFORM

$\frac{\partial}{\partial s} \to 0 \Rightarrow$ Uniform





UNIFORM FLOW

$$y_u = y_d$$
$$S_f = S_w = S_o$$

NON-UNIFORM FLOW

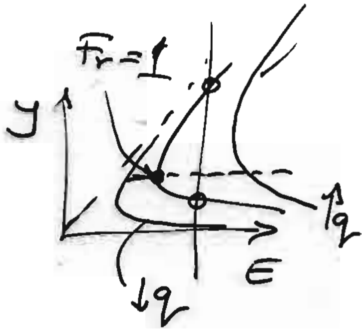
$$y_u \neq y_d$$
$$S_f \neq S_w \neq S_o$$

OPEN CHANNEL HYDRAULICS

Energy Eqn:

$$E = y + \frac{q^2}{2gy^2}$$

$$E_1 + z_1 = E_2 + z_2 + h_L$$



Uniform Flow (Chezy/Manning)

$$\boxed{\frac{dy}{dx} = 0}$$

$$V = \frac{K}{n} R_h^{2/3} S_o^{1/2}$$

$$Q = A\frac{K}{n} R_h^{2/3} S_o^{1/2}$$

GVF

$$\boxed{\frac{dy}{dx} \ll 1}$$

Still to do !

$$\frac{dy}{dx} = \frac{(S_f - S_o)}{(1 - F_r^2)}$$

RVF

$$\boxed{\frac{dy}{dx} \sim 1}$$

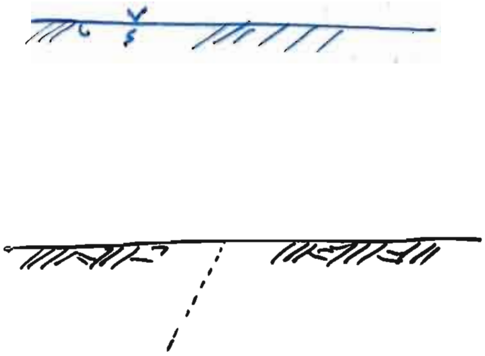
Still to do !

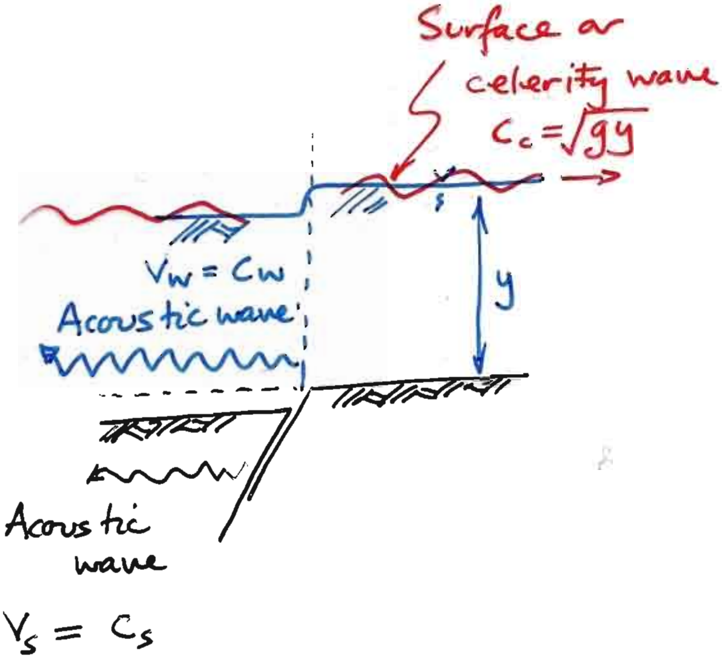
Hydraulic jump

Too complex
for 1-d
representation

SURFACE WAVES

eg. Earthquake

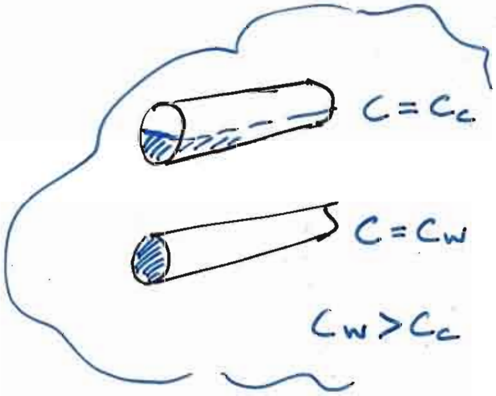




Speed of sound: (Acoustic velocity / Seismic velocity)

Ideal gas: $C_g = \sqrt{kRT}$ Isentropic

Solid $C_s = \sqrt{\frac{E_v}{\rho}}$



Surface wave:

$$C_c = \sqrt{gy}$$

Acoustic waves: $C_{water} \simeq 1500$ m/s

$C_{granite} \simeq 2500$ m/s — 5000 m/s

$C_{air} \simeq 400$ m/s

Celerity wave: $C_c \simeq \sqrt{9.81 \times 1000\text{m}} \simeq 100$ m/s

Tsunami speed?

SURFACE WAVES — WHAT IS WAVE SPEED, c_c?

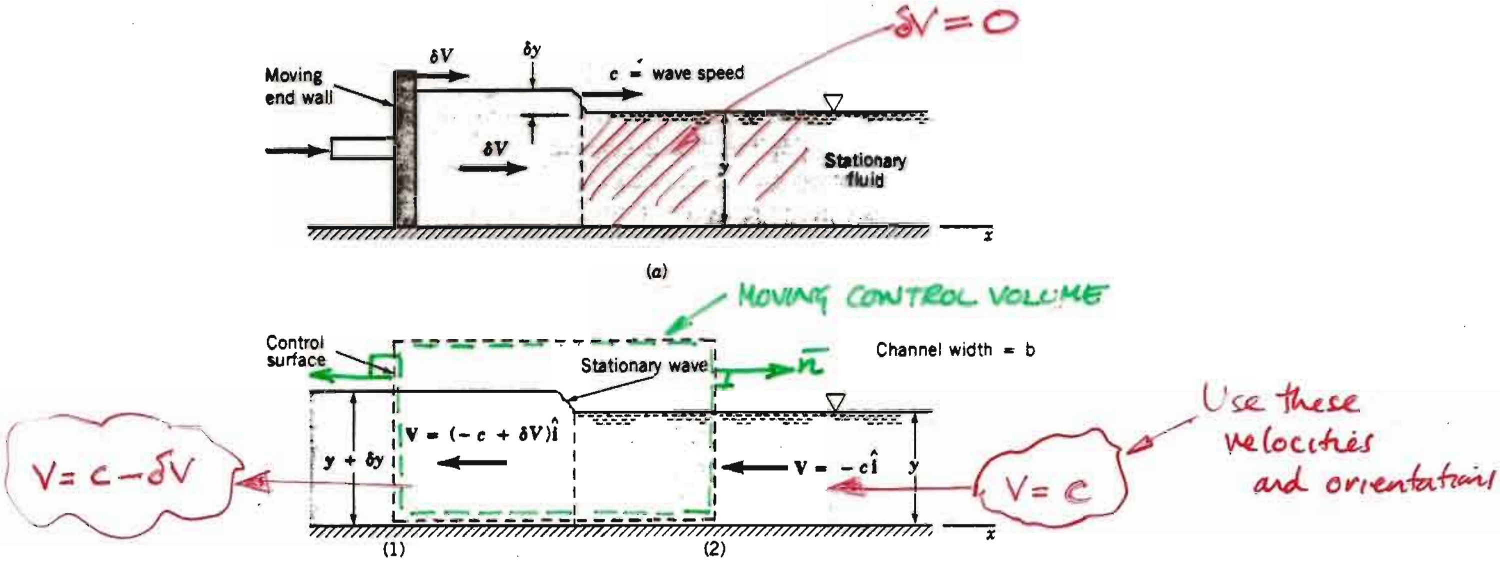


FIGURE 10.2 *(a)* Production of a single elementary wave in a channel as seen by a stationary observer. *(b)* Wave as seen by an observer moving with a speed equal to the wave speed.

CHOOSE MOVING CONTROL VOLUME ☐

CONTINUITY:

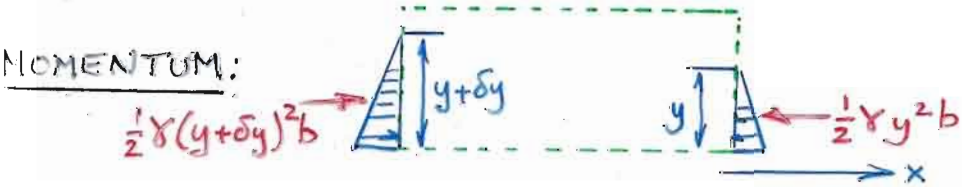
$$-\cancel{\rho} c y \cancel{b} + \cancel{\rho}(+c - \delta V)(y + \delta y)\cancel{b} = 0$$

Expanding: $-cy + c(y+\delta y) = +\delta V(y+\delta y)$

$$c = \frac{\delta V(y+\delta y)}{\delta y}$$

In the limit as $\delta y \ll y \Rightarrow$ $\boxed{c = y\frac{\delta V}{\delta y}}$ ①

MOMENTUM:



$$\frac{1}{2}\gamma(y+\delta y)^2 b - \frac{1}{2}\gamma y^2 b = \dot{m}(W\cdot n)$$

since in −ve x direction

$$\left\{\begin{array}{l}\dot{m} = -\rho c\, by \\ W\cdot n = -c + (c - \delta V) = -\delta V\end{array}\right.$$

$$\frac{1}{2}\gamma\left\{\left[y^2 + \underbrace{\delta y^2 + 2y\delta y}_{\text{Assume } \delta y^2 \ll y\delta y}\right] - y^2\right\} b = \rho c b y\, \delta V$$

Rearranging: $\frac{1}{2}\cancel{\rho} g \cancel{2} y \delta y b = \cancel{\rho} c b y \delta V$

$$\boxed{\frac{g}{c} = \frac{\delta V}{\delta y}} \quad ②$$

Equating ① and ② $\frac{c}{y} = \frac{g}{c} \Rightarrow \boxed{c = \sqrt{gy}}$

<u>Note</u>:

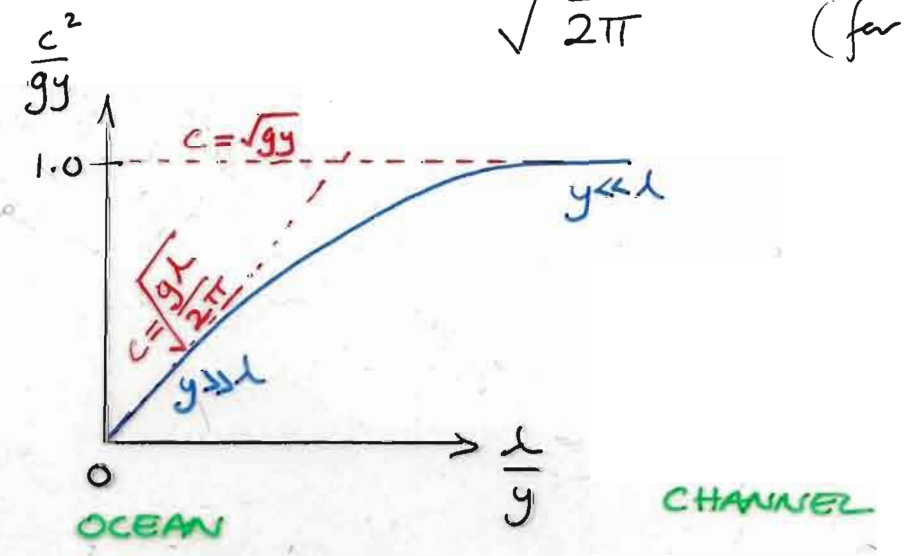
- □ Independent of amplitude, δy.
- □ Proportional to depth, y
- □ Smaller than acoustic wave velocity
- □ Independent of wavelength (λ).

<u>For shallow waves ($\delta y / y \ll 1$)</u>

$$c = \sqrt{gy}$$

<u>For deep (non-solitary i.e. wavetrain)</u>

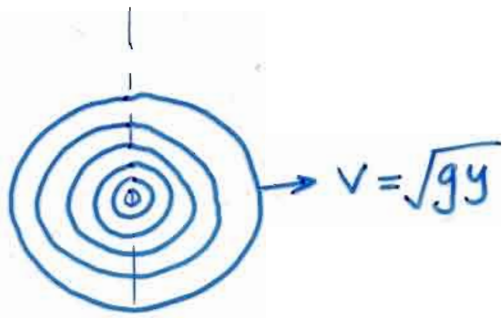
$$c = \sqrt{\frac{g\lambda}{2\pi}} \quad (\text{for } y \gg \lambda)$$



SURFACE WAVES & FROUDE NO.

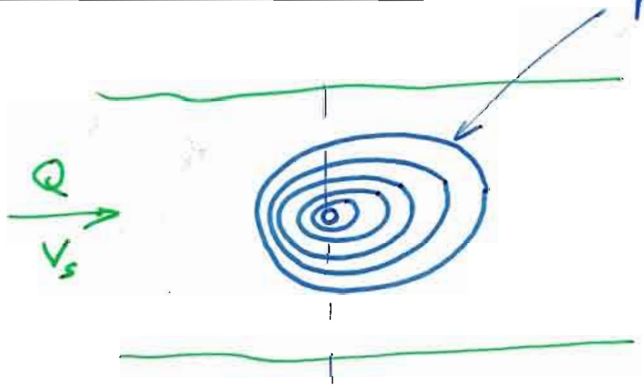
$$\boxed{\begin{array}{c} Fr = \dfrac{V}{\sqrt{gy}} \\ c_c = \sqrt{gy} \end{array}} = \frac{V}{c_c}$$

STATIC



$Fr = 1$

SUBCRITICAL STREAM



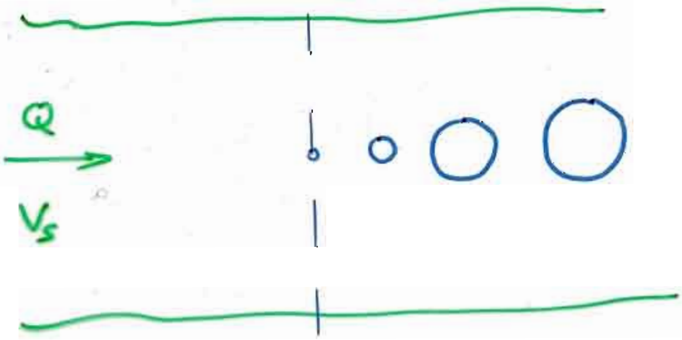
Non-concentric circles

$V_s < \sqrt{gy}$

So $Fr = \dfrac{V_s}{\sqrt{gy}}$ $\boxed{Fr < 1}$

gravity force > inertial force

SUPERCRITICAL STREAM



$V_s > \sqrt{gy}$

$Fr = \dfrac{V_s}{\sqrt{gy}}$ $\boxed{Fr > 1}$

gravity force < inertial force

$Fr = \dfrac{\text{inertial force}}{\text{gravity force}}$

Downstream disturbance has no communication with upstream !!

CONSIDERATION OF:

☐ Steady flows

☐ Homogeneous flows

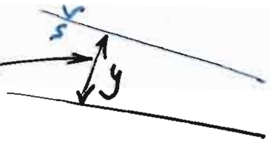
↓

Indexing parameters are:

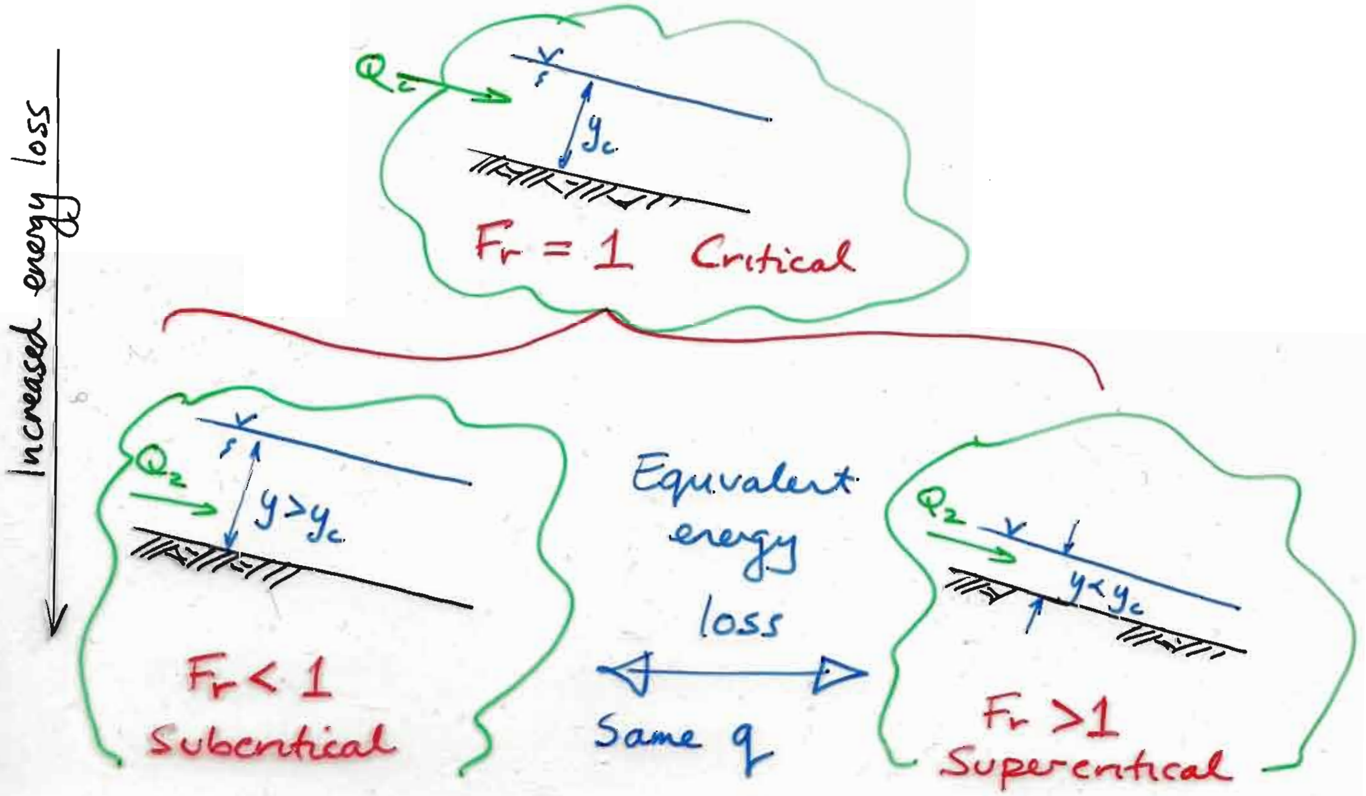
Reynolds No. $Re = \frac{VR_h\rho}{\mu}$

but always turbulent

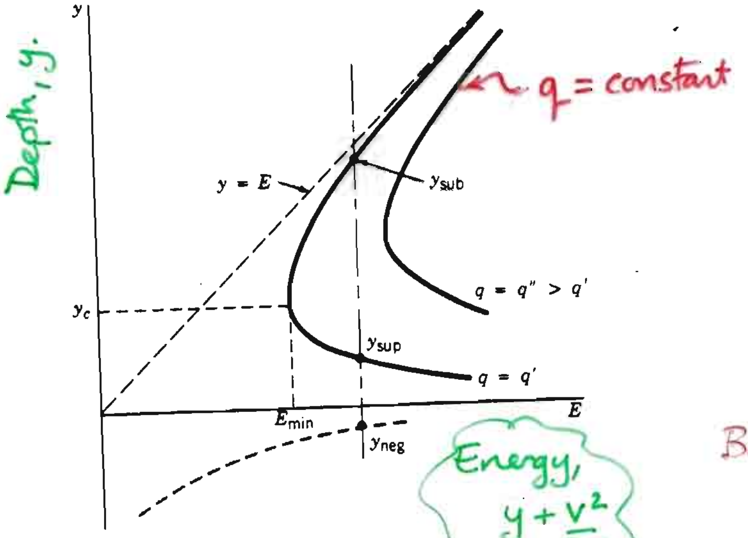
Froude No. $Fr = \frac{V}{\sqrt{(gy)}}$

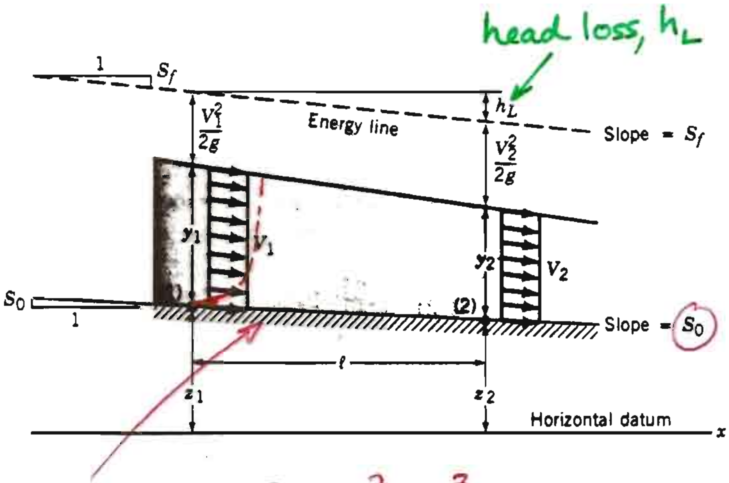


Froude No. indexes the flow



ENERGY EQUATION





Energy, $y + \frac{V^2}{2g}$

BASAL SLOPE, $S_0 = \frac{z_1 - z_2}{\ell}$

Energy Equation: $$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

Substitute $S_0\ell = z_1 - z_2$; $y_1 = \frac{P_1}{\gamma}$; $y_2 = \frac{P_2}{\gamma}$

$$y_1 + \frac{V_1^2}{2g} + S_0\ell = y_2 + \frac{V_2^2}{2g} + h_L$$

Slope of energy line (E.G.L). $S_f = \frac{h_L}{\ell}$

$$\boxed{y_1 - y_2 = \frac{(V_2^2 - V_1^2)}{2g} + (S_f - S_0)\ell}$$

Where viscous losses ($S_f\ell = h_L$) directly balance energy driving the system ($S_0\ell$ = gravity)

$$\boxed{y_1 - y_2 = \frac{V_2^2 - V_1^2}{2g}}$$

Equivalent to Bernoulli since

$S_f = 0$ $\mu \rightarrow 0$

$S_0 = 0$ Horizontal channel.

SPECIFIC ENERGY

Definition, Specific Energy, E: $E = y + \frac{V^2}{2g}$

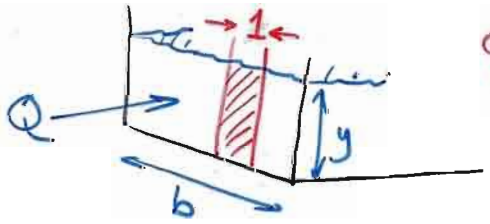
Therefore: $y_1 - y_2 = \frac{V_2^2 - V_1^2}{2g} + (S_f - S_0)\ell \Rightarrow E_1 = E_2 + (S_f - S_0)\ell$

Check if makes sense? Bernoulli, $S_f \to 0$ then

$$E_1 = E_2 + (-S_0)\ell$$

$$E_1 = E_2 - z_1 + z_2$$ QED Bernoulli

Define Specific Energy as a flow rate, q (per unit width)



$$q = \frac{Q}{b} = \frac{Vby}{b} = Vy$$

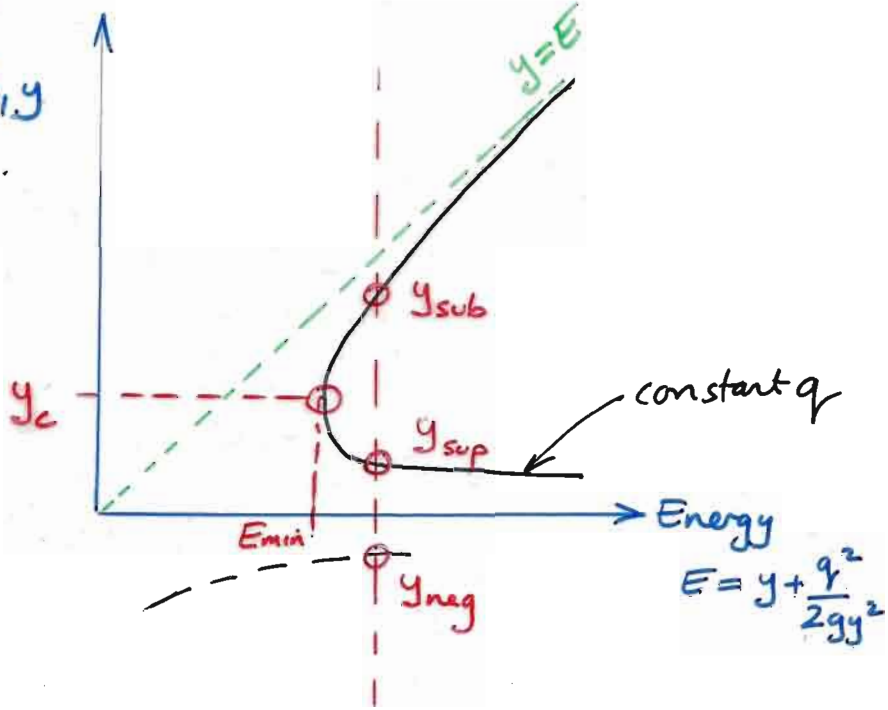
$$E = y + \frac{V^2}{2g} = y + \frac{V^2}{2g}\frac{y^2}{y^2}$$

$$\boxed{E = y + \frac{q^2}{2gy^2}}$$

Energy is the most convenient parameter to enable flow criticality (Fr) to be determined.

SPECIFIC ENERGY DIAGRAM

Plot Energy -vs- depth. =>



$$E = y + \frac{q^2}{2gy^2}$$

$$\Downarrow$$

$$\boxed{0 = -Ey^2 + y^3 + \frac{q^2}{2g}}$$

Cubic ∴ 3 roots y_{sub} ; y_{sup} ; y_{neg}

For minimum energy: E_{min} @ y_{min} One root

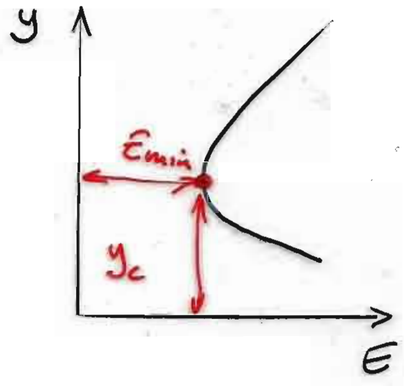
For $E > E_{min}$ y_{sub} = subcritical flow

y_{sup} = supercritical flow

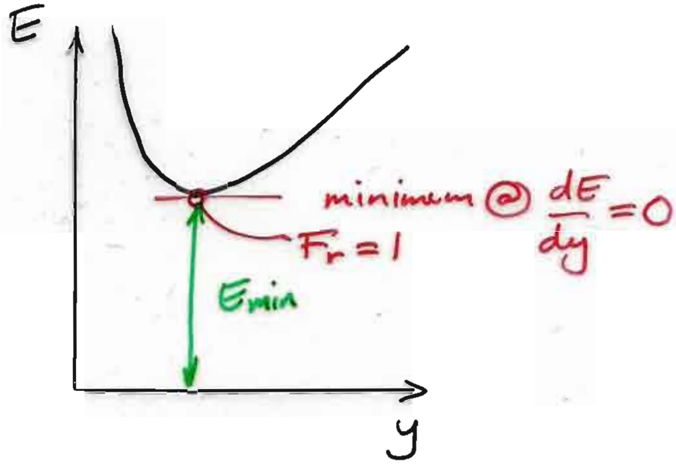
Division between supercritical and subcritical @ y_c

This flow @ y_c is for $\underline{\underline{F_R = 1}}$

DETERMINE F_R @ y_c



Switch axes $\Longrightarrow$



$$\frac{dE}{dy} = \frac{d}{dy}\left[y + \frac{q^2}{2gy^2}\right] = 1 - \frac{q^2}{gy^3} = 0$$

y_c

Rearrange as: $y_c = \left(\frac{q^2}{g}\right)^{1/3}$ * OR $q^2 = y_c^3 g$

Substitute for $\boxed{E_{min} = y_c + \frac{q^2}{2gy_c^2} = \frac{3y_c}{2}}$

Determine velocity, V_c, at minimum energy, E_{min}.

$V_c = \frac{q}{y_c}$ From * we have $\sqrt{y_c^3 g} = q$

$$V_c = \frac{y_c^{3/2} g^{1/2}}{y_c} = \sqrt{g y_c}$$

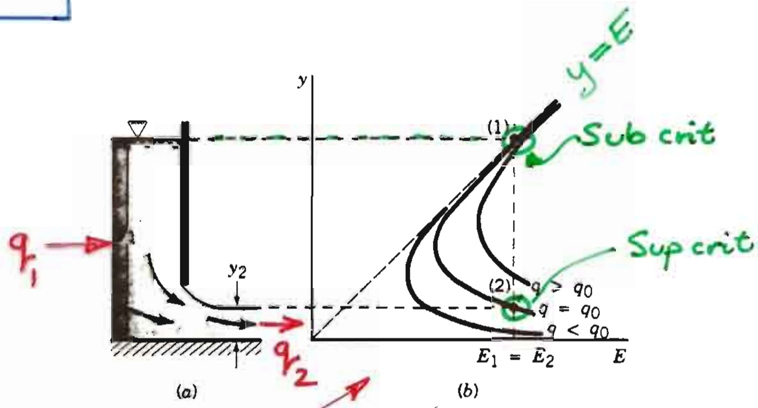
$$F_{R_c} = \frac{V_c}{\sqrt{g y_c}} = 1$$

DEFINE SPECIFIC ENERGY DIAGRAM FOR INVISCID FLOW BENEATH SLUICE

INVISCID → $S_f = 0$

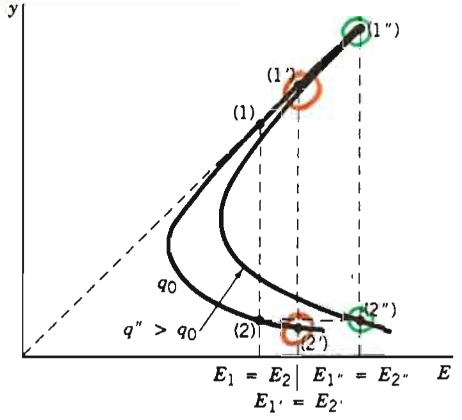
HORIZONTAL BED → $S_o = 0$

$$E_1 = E_2 + (\cancelto{0}{S_f} - \cancelto{0}{S_o})\ell$$



CONTINUITY: $q_1 = q_2$ ∴ same line

ENERGY: $E = y + \frac{q^2}{2gy^2}$.



(c)

Note:

1. Changing q changes y_{sup} most as y_{sub} is constrained by $y = E$ line

Changes in configuration

1. Raise upstream level, $y_{1'}$, but keep q constant by lowering sluice gap, $y_{2'}$. $q = q_0$

2. Raise upstream level, $y_{1''}$, and allow q to increase by not lowering sluice gap, $y_{2''} = y_2$. $q > q_0$

WATER FLOWS UP 0.5ft RAMP

$q = 5.75 \text{ ft}^2/\text{s}$

$y_1 = 2.3 \text{ ft}$

DETERMINE DOWNSTREAM SURFACE ELEVATION?

INVISCID $\therefore h_L = 0$

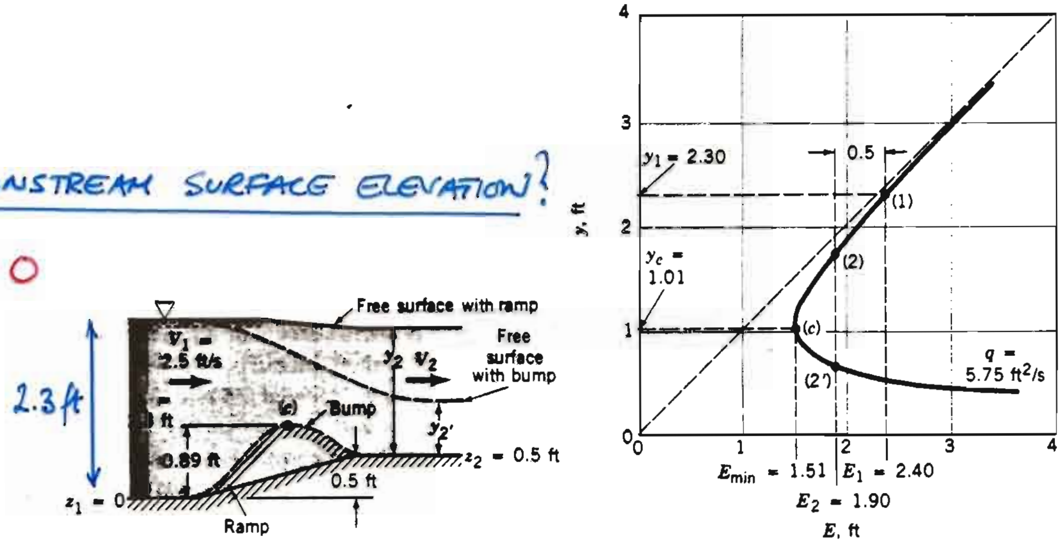
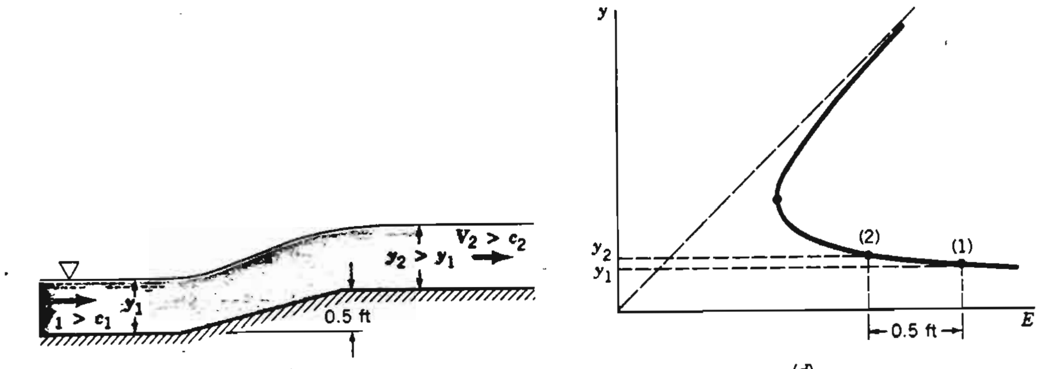


FIGURE E10.2

$$E_1 + z_1 = E_2 + z_2 + h_L$$



ENERGY EQUATION:

$$\underset{2.3}{\underset{\uparrow}{y_1}} + \frac{V_1^2}{2g} + \underset{0}{\underset{\uparrow}{z_1}} = \underset{?}{\underset{\uparrow}{y_2}} + \underset{?}{\underset{\uparrow}{\frac{V_2^2}{2g}}} + \underset{0.5}{\underset{\uparrow}{z_2}} + \cancelto{0}{\underbrace{S_f \ell}_{h_L}}$$

DEFINE $\frac{V_1^2}{2g}$ as $\Rightarrow \frac{q_1^2}{y_1^2 2g} = \frac{(5.75)^2}{(2.3)^2 2(32.4)} = 0.0971$

ONLY REMAINING UNKNOWNS y_2, V_2.

$2.3 + 0.097 - 0.5$

$$\boxed{1.90 = y_2 + \frac{V_2^2}{2g}} \quad ①$$

ENERGY EQUATION

CONTINUITY : $$\underset{q_1 \nearrow}{y_1 V_1} = y_2 V_2$$

$$\boxed{5.75\ ft^2/s = y_2 V_2} \quad ②$$

Substitute ② into ① as $V_2 = \frac{5.75}{y_2}$

to give $$1.90 = y_2 + \frac{(5.75)^2}{y_2^2\, 2g} \Rightarrow 0 = y_2^3 - 1.9y_2^2 + 0.513$$

$$\text{Solutions} \begin{cases} 1.72\ ft & \text{Subcritical} \\ 0.638\ ft & \text{Supercritical} \\ -.466\ ft & \text{False} \end{cases} \Big\} y_2$$

Free surface elevations

$y_2 + z_2 = 1.72 + 0.5 = 2.22\ ft$

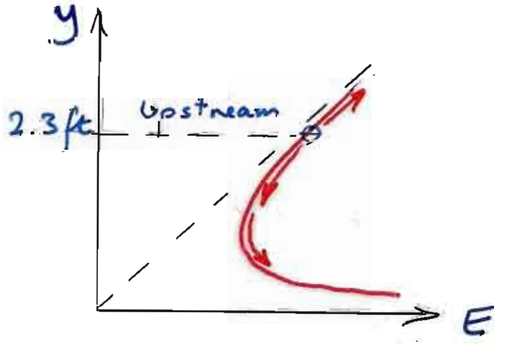
$y_2 + z_2 = 0.64 + 0.5 = 1.14\ ft$

Does <u>sub</u> critical or <u>super</u> critical flow develop?

Use definition of energy, E $$E = y + \frac{q^2}{2gy^2}$$

$$E = y + \frac{0.513}{y^2}$$

DRAW SPECIFIC ENERGY DIAGRAM

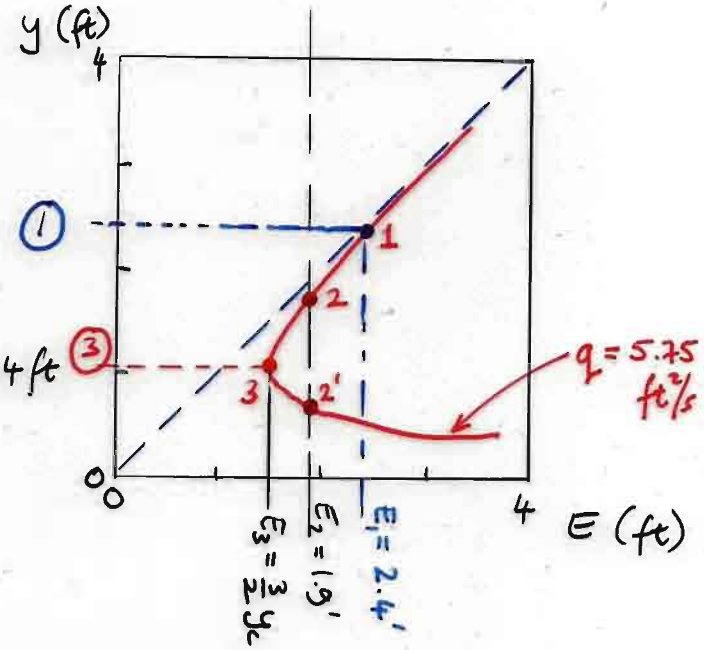


DETERMINING FLOW CONDITIONS

Upstream: $y = 2.3$ ft ①

and $E_1 = 2.4$ ft

∴ Subcritical



Downstream: $E_1 = \overset{2.3}{y} + \frac{0.513}{y^2} = 2.4$ ft

Also $E_1 = E_2 + \underbrace{(z_2 - z_1)}_{0.5\ \text{ft}}$

$$\boxed{E_2 = 1.9 \text{ft}}$$

Subcritical (2) or Supercritical (2')?

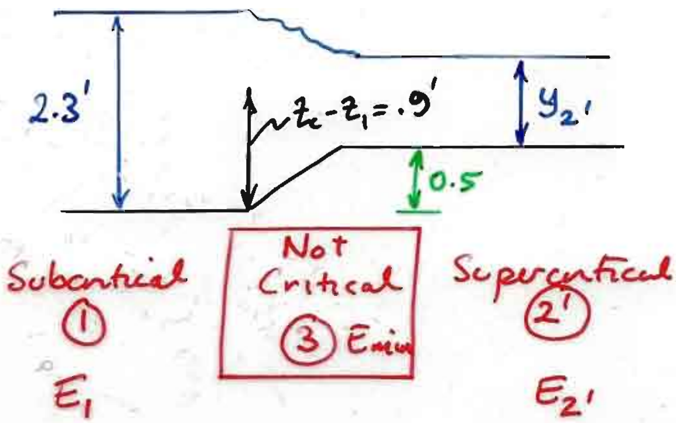
If supercritical then must pass through $F_r = 1$ (Minimum Energy).

@ E_{min}: $y_c = 1.01$ ③ $E_{min} = \frac{3y_c}{2} = 1.51$ ft

$E_1 + z_1 = E_{min} + z_c$

$z_c - z_1 = E_1 - E_{min}$

$z_c - z_1 = 2.4 - 1.51 = 0.89'$



$z_c - z_1 = 0.9' > 0.5'$

∴ this flow is not supercritical

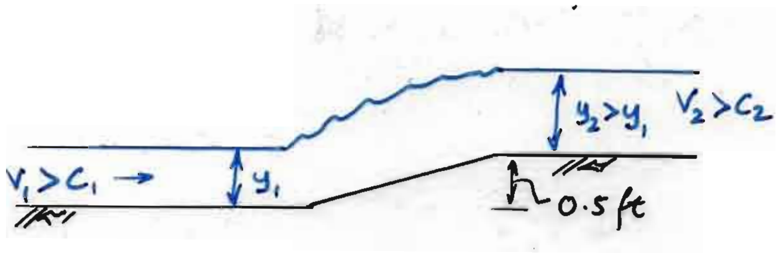
State ②' is inaccessible!!

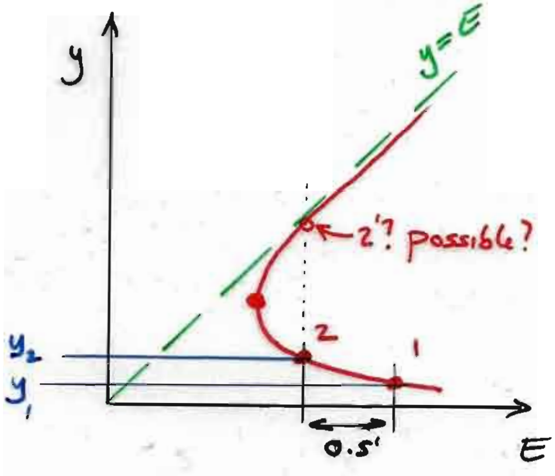
∴ subcritical flow ②.

$$\boxed{y_2 + z_2 = 2.22 \text{ft}}$$

$y = 1.72'$

ALTERNATIVE SCENARIO

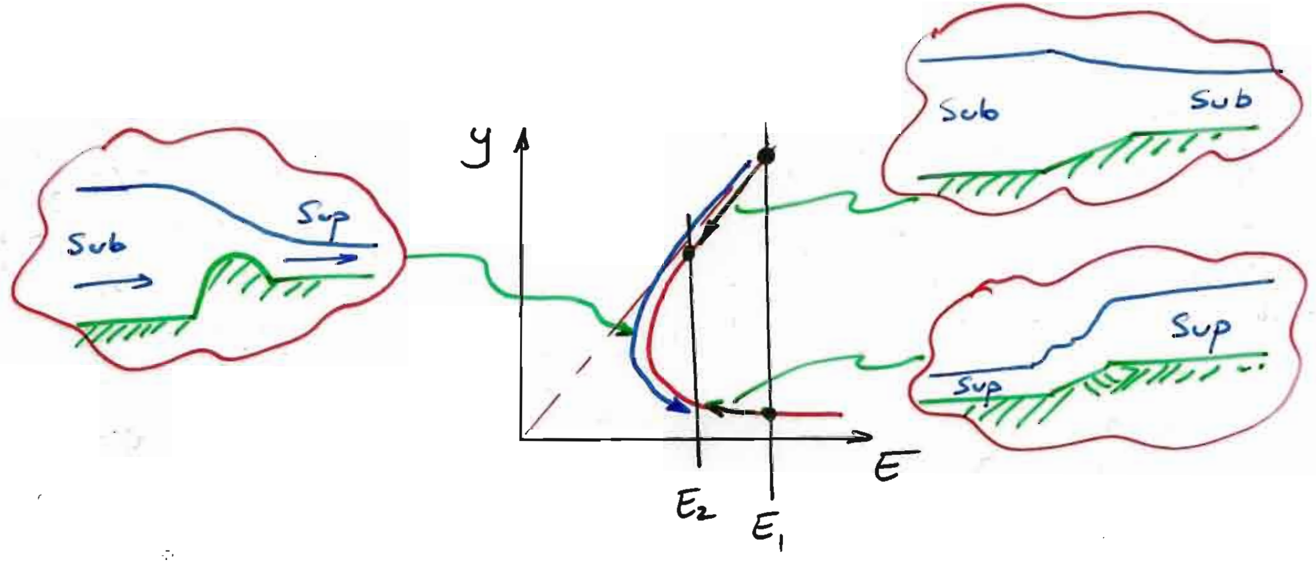




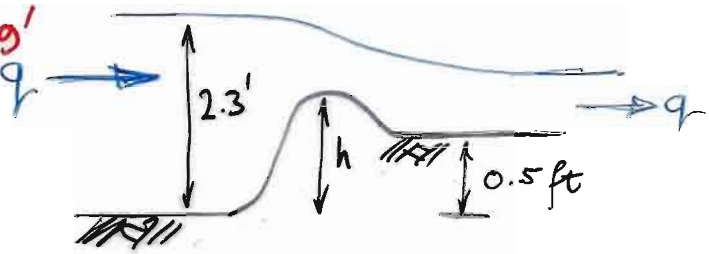
If upstream conditions supercritical.

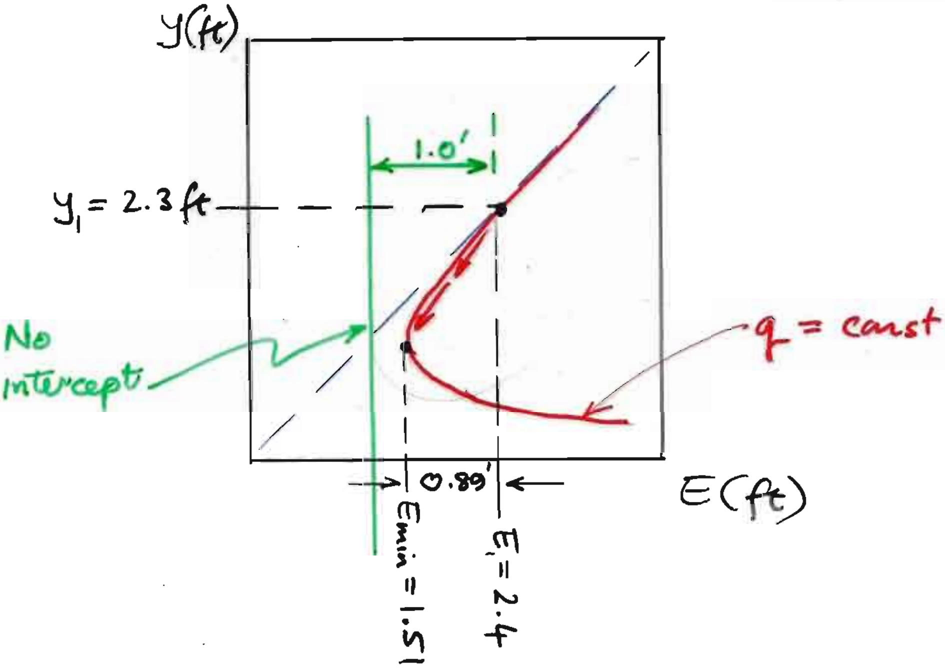
Changing Energy by −0.5 ft ↑ y_2 Flow remains supercritical

It is important to know the initial upstream conditions



LARGE 'BUMP' PRESENT? $h > 0.89'$





What if downstream water elevation (bump) higher than $\underline{0.89'}$?

say $h = 1.0$ ft.

$$\underset{2.4'}{E_1} + \underset{0'}{z_1} = E_2 + \underset{h=1.0\,ft}{z_2} \Rightarrow \begin{aligned} 1.4 &= E_2 \\ \underbrace{1.4 = y_2 + \frac{(5.75)^2}{2g\,y_2^2}} \end{aligned}$$

No (+ve) solution – see graph

∴ Flow is "choked" – cannot flow @ $q = 5.75\ ft^2/s$ in this configuration.

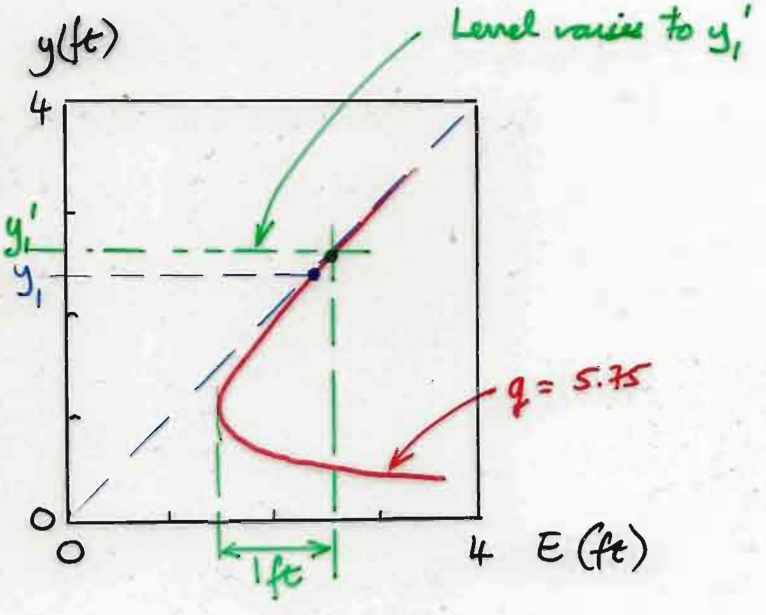
TWO POSSIBLE RESULTS

1. **We are controlling flowrate:**

q fixed @ 5.75 ft^2/s

∴ same specific energy curve

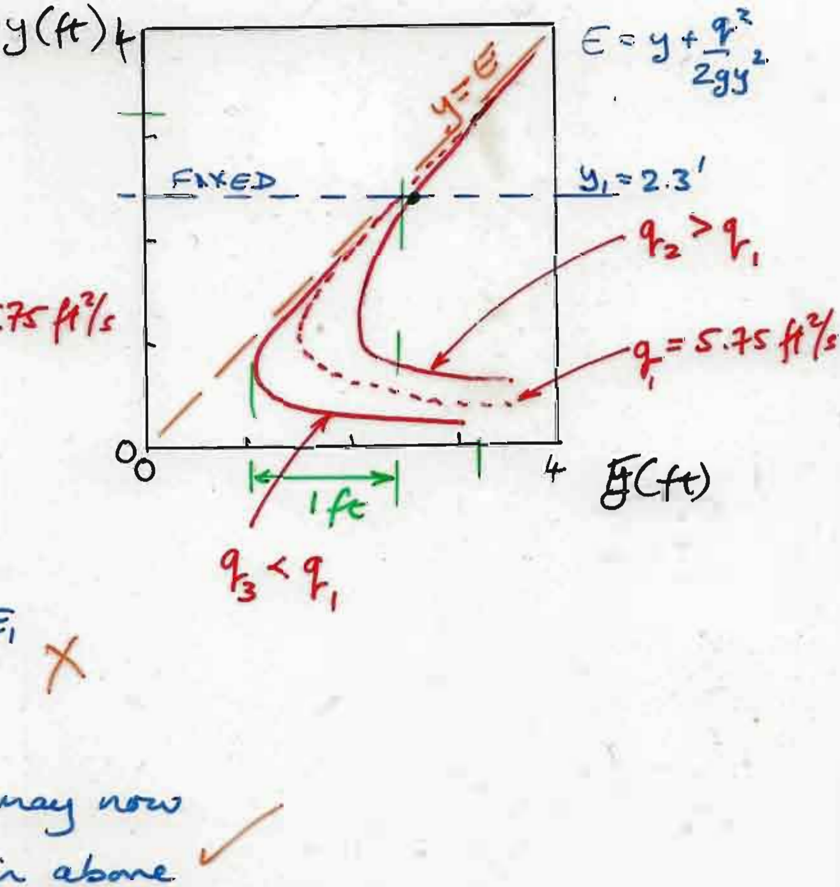
∴ upstream water level rises to overcome obstacle and stem 'choking' of flow.



2. **We control upstream depth:**

y_1 fixed @ 2.3 ft

∴ Migrate from specific energy curve for $q_1 = 5.75$ ft^2/s to another.

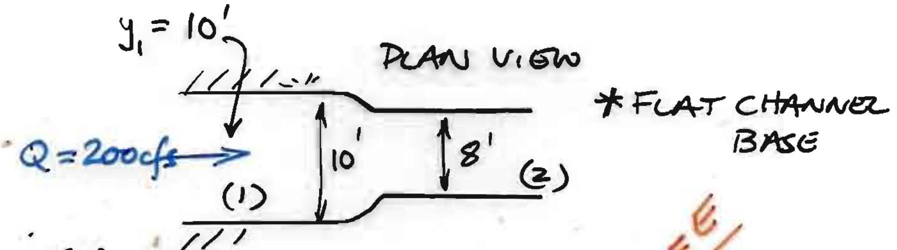


Two possible curves

$q > q_1$ – NO. Since need $\xleftrightarrow{1ft}$ from y_1, E_1 to y_2, E_2. ✗

$q < q_1$ – OK. Since we may now fit $\xleftrightarrow{1ft}$ in above E_{min} @ y_{crit}. ✓

FLOW WIDTH CONTRACTION

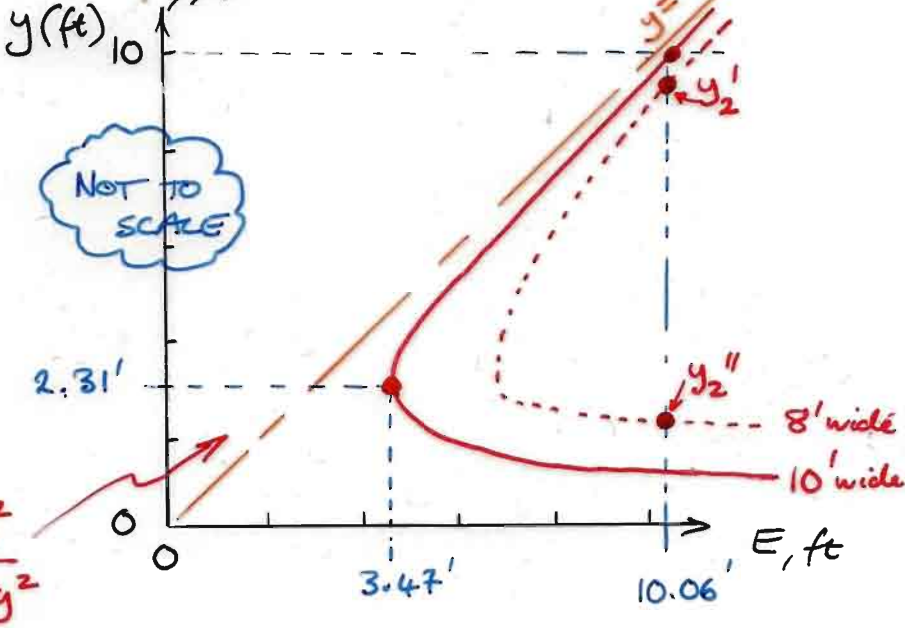


$Q = 200$ cfs

$\underline{q} = \frac{Q}{b} = 20 \ ft^2/s$

If $y_1 = 10'$ $V_1 = 2 \ ft/s$

$E_1 = 10 + \frac{4}{64.4} = \underline{10.06'}$



Plot Spec. E. dia $E = y + \frac{q^2}{2gy^2}$

$y_c = \underline{2.31'}$; $E_{min} = \underline{3.47'}$ (i.e. @ Fr = 1 $y_c = \sqrt[3]{\frac{q^2}{g}} = \sqrt[3]{\frac{20^2}{32.2}}$)

CONTRACT TO 8' WIDTH

If INVISCID Then $\left.\begin{array}{l} h_L = S_f l = 0 \\ \& \ z_1 = z_2 \end{array}\right\}$ ∴ $E_2 = E_1 = 10.06'$

But $b = 8'$ ∴ $q = \frac{Q}{b} = 25 \ ft^2/s$ ⇒ New curve

25 ft²/s

Plot new Spec. E dia. $E = y + \frac{q^2}{2gy^2}$

Choose between y_2' and y_2''

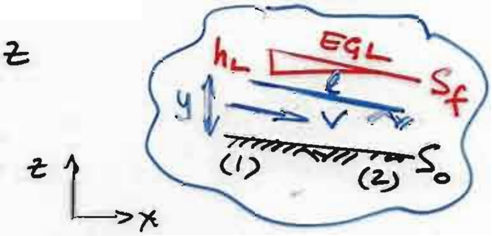
Equivalent to solving $10.06 = y_2 + \frac{25^2}{2gy^2}$

2 roots!!

CHANNEL DEPTH VARIATIONS

Gradually varying flows $\frac{dy}{dx} \ll 1$

One - dimensional flow: $H = \frac{V^2}{2g} + y + z$



$$H_1 = H_2 + h_L$$

EGL Slope : $\frac{dH}{dx} = \frac{dh_L}{dx} = S_f$ BASE SLOPE: $\frac{dz}{dx} = S_o$

$$\frac{dH}{dx} = \frac{d}{dx}\left(\frac{V^2}{2g} + y + z\right) = \underbrace{\frac{V}{g}\frac{dV}{dx}}_{\frac{d(\,)}{dV}\frac{dV}{dx}} + \frac{dy}{dx} + \frac{dz}{dx}$$

RESUBSTITUTING:

$$\frac{dh_L}{dx} = S_f = \frac{V}{g}\frac{dV}{dx} + \frac{dy}{dx} + S_o \qquad ①$$

USING FLOWRATE / UNIT WIDTH: $q = Vy$

$$\frac{dV}{dx} = \frac{d}{dx}\left(\frac{q}{y}\right) = -\frac{q}{y^2}\frac{dy}{dx} = -\frac{V}{y}\frac{dy}{dx}$$

DETERMINING THE TERM $\frac{V}{g}\frac{dV}{dx}$ in ①

$$\frac{V}{g}\frac{dV}{dx} = -\frac{V^2}{gy}\frac{dy}{dx} = -F_r^2\frac{dy}{dx}$$

FROM ①:

$$\frac{V}{g}\frac{dV}{dx} + \frac{dy}{dx} = S_f - S_o$$

$$(1 - F_r^2)\frac{dy}{dx} = S_f - S_o \Rightarrow \boxed{\frac{dy}{dx} = \frac{(S_f - S_o)}{(1 - F_r^2)}}$$

IMPORTANCE OF THIS FACTOR?

$$\boxed{\frac{dy}{dx} = \frac{(S_f - S_o)}{(1 - Fr^{*2})}}$$

* Fr is local Froude No.

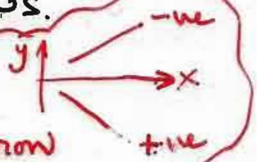
Depth change, $\frac{dy}{dx} = \phi(S_f, S_o, Fr)$

Denominator $= (1 - Fr^2)$ Changes sign +ve $Fr < 1$

−ve $Fr > 1$

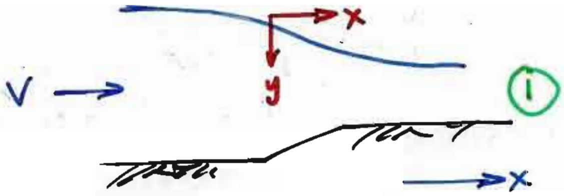
$\frac{dy}{dx}$ behaviour opposite for subcritical or supercritical flows.

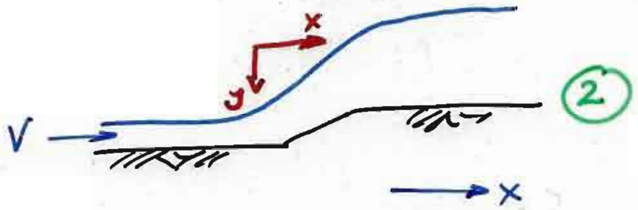
NOTE: SLOPE CONVENTION



EXAMPLE:

$S_f > S_o$





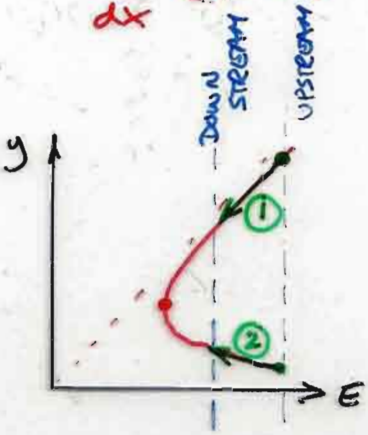
SUBCRITICAL

$\frac{dy}{dx}$ (+ve)

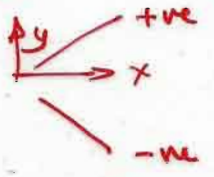
SUPERCRITICAL

$\frac{dy}{dx}$ (−ve)

i.e.



* Note slope convention not same as



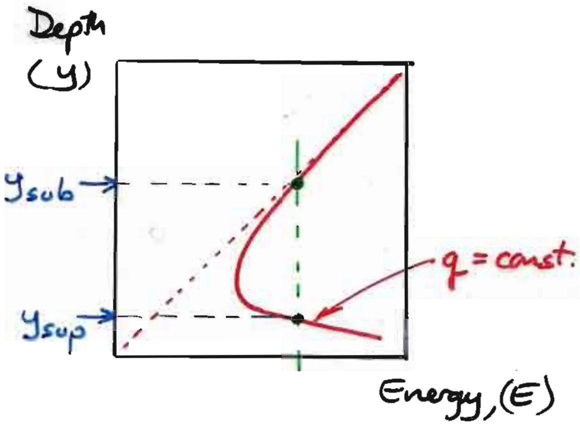
OPEN CHANNEL FLOW CONCEPTS

I GENERIC CONCEPTS

Froude No. $F_R = \frac{V}{\sqrt{gy}}$

Energy. $E = y + \frac{q^2}{2gy^2}$

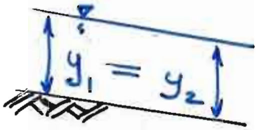
Energy Eqn. $E_1 + z_1 = E_2 + z_2 + h_L$



$$\boxed{\frac{dy}{dx} = \frac{(S_f - S_o)}{(1 - Fr^2)}}$$

II SPECIFIC CONCEPTS

(i) UNIFORM FLOW $\frac{dy}{dx} = 0$



$S_o = S_f$

(ii) GRADUALLY VARIED FLOW $\frac{dy}{dx} \ll 1$

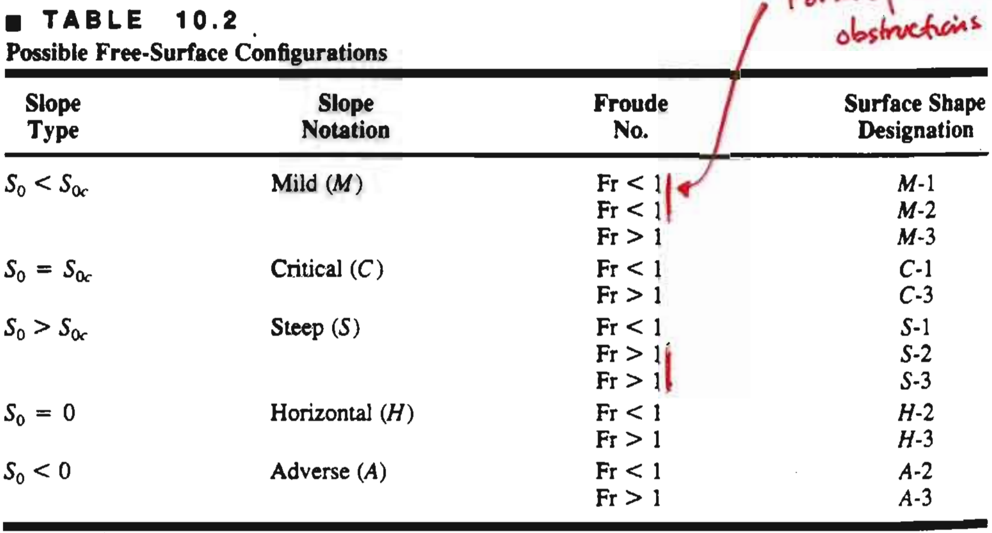
(iii) RAPIDLY VARIED FLOW $\frac{dy}{dx} \sim 1$

GRADUALLY VARIED FLOW

12 configurations of surface shapes:

Note: $S_o = S_f$ @ critical flow. i.e. $dy/dx = 0$

Form of downstream obstructions



■ **TABLE 10.2**

Possible Free-Surface Configurations

Slope Type	Slope Notation	Froude No.	Surface Shape Designation
$S_0 < S_{0c}$	Mild (*M*)	Fr < 1	*M*-1
		Fr < 1	*M*-2
		Fr > 1	*M*-3
$S_0 = S_{0c}$	Critical (*C*)	Fr < 1	*C*-1
		Fr > 1	*C*-3
$S_0 > S_{0c}$	Steep (*S*)	Fr < 1	*S*-1
		Fr > 1	*S*-2
		Fr > 1	*S*-3
$S_0 = 0$	Horizontal (*H*)	Fr < 1	*H*-2
		Fr > 1	*H*-3
$S_0 < 0$	Adverse (*A*)	Fr < 1	*A*-2
		Fr > 1	*A*-3

Surface shapes indexed: $f[S_o ; Fr]$

i.e.
$$\boxed{\frac{dy}{dx} = \frac{(S_f - S_o)}{(1 - Fr^2)}}$$

☐ 5 potential channel slopes: 3 relative to critical, S_{oc}
2 relative to horizontal

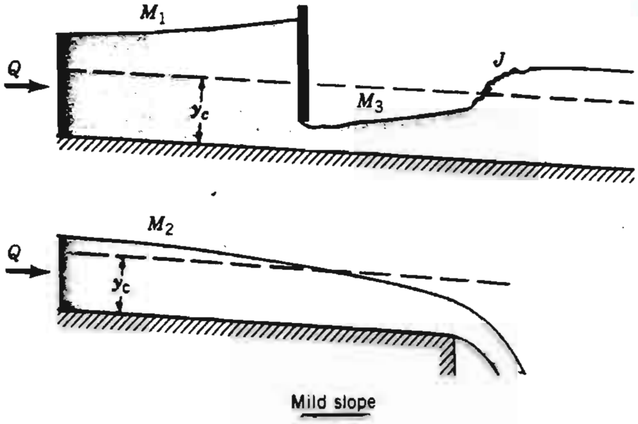
☐ Froude No. designation $Fr > 1$
$Fr < 1$

Terminology
1 upstream of obstruction
2 no-obstruction
3 downstream of obstruction

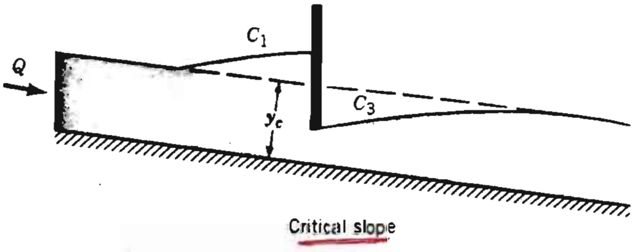
GRADUALLY VARIED

Note $\frac{dy}{dx} << 1$

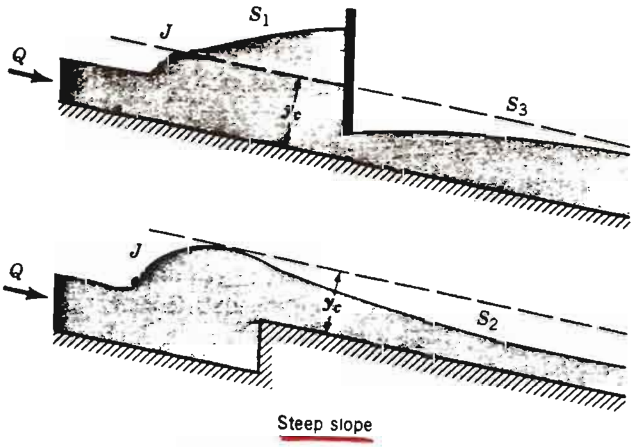
∴ scales exagerated



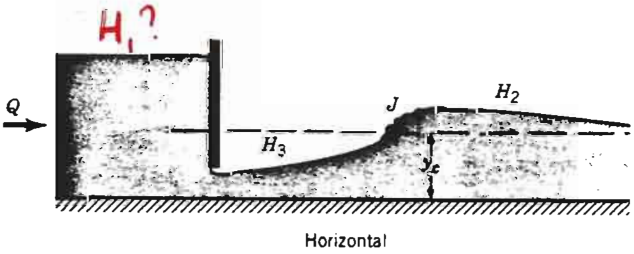
■ FIGURE 10.11 Typical surface configurations for nonuniform depth flow with a mild slope, $S_0 < S_{0c}$.



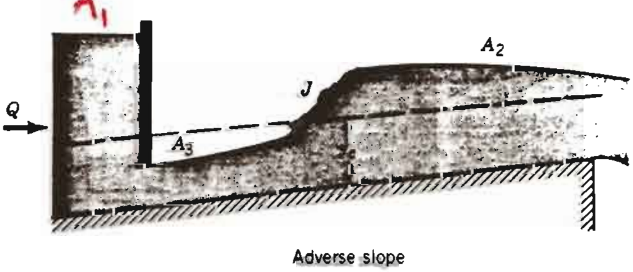
■ FIGURE 10.12 Typical surface configurations for nonuniform depth flow with a critical slope, $S_0 = S_{0c}$.



■ FIGURE 10.13 Typical surface configurations for nonuniform depth flow with a steep slope, $S_0 > S_{0c}$.



■ FIGURE 10.14 Typical surface configurations for nonuniform depth flow with a horizontal slope, $S_0 = 0$.



■ FIGURE 10.15 Typical surface configurations for nonuniform depth flow with an adverse slope, $S_0 < 0$.

[13:3] Open Channel Flows

Recap

Flow classifications

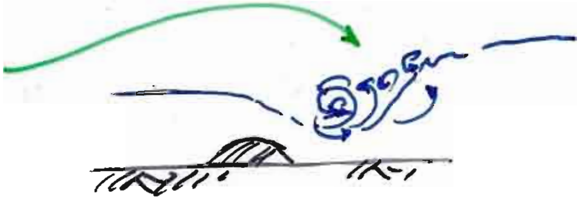
- Uniform Flows – Chezy/Manning formulae $\frac{dy}{dx} = 0$
- Gradually Varying Flows $\frac{dy}{dx} << 1$

Outline

Rapidly varying flows $\frac{dy}{dx} \sim 1$

- Momentum and energy concepts
- Hydraulic jumps

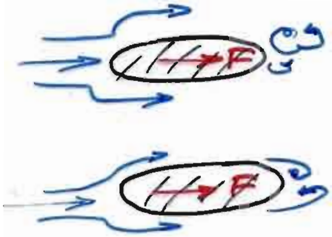
RAPIDLY VARIED FLOWS $\frac{dy}{dx} \sim 1$

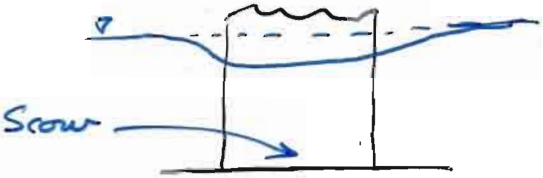


- Behavior loses its 1-D nature
  ∴ 1-D solutions only approximate.

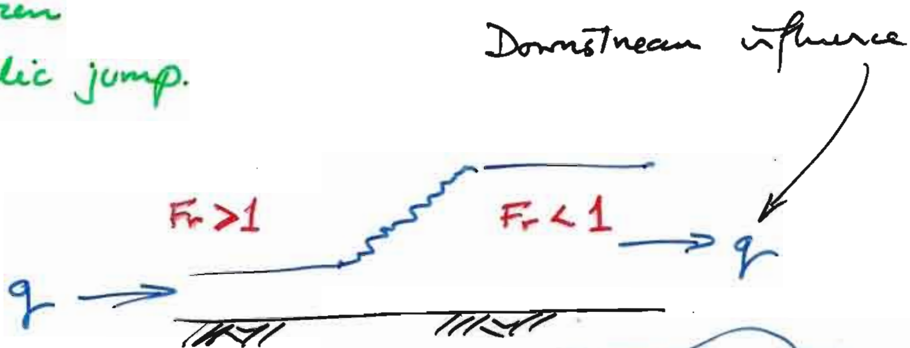
- Usually the 3-d flow complexity that is desired i.e.

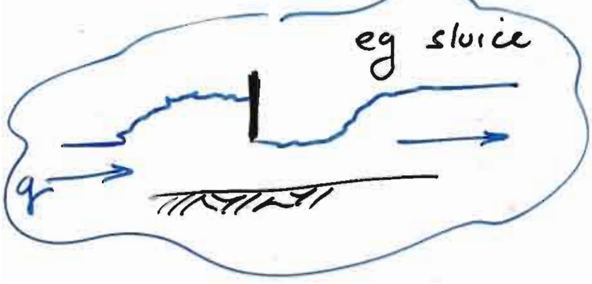
  Bridge pier — ☐ Scour
  ☐ Forces on pier





- Some result from no apparent flow obstruction
  eg. Hydraulic jump.

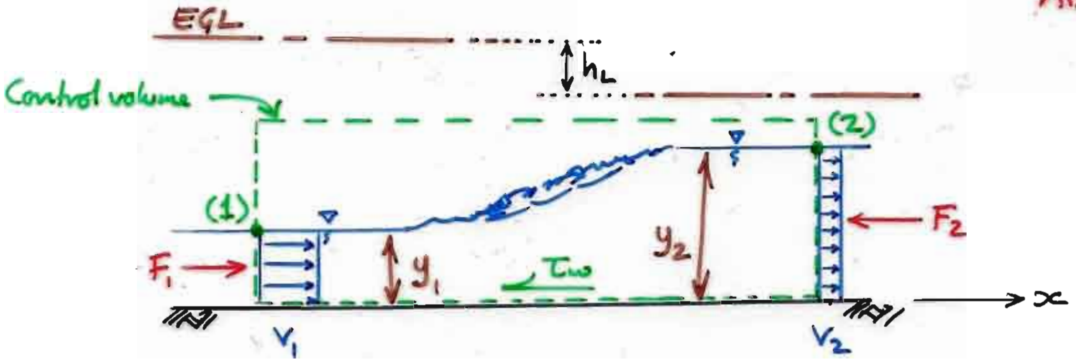




HYDRAULIC JUMP

$\frac{dy}{dx} = \infty$ – Discontinuity

Also discontinuity in $\frac{dE}{dx}$.



Neglect τ_w

MOMENTUM : $F_1 - F_2 = \rho Q (V_2 - V_1)$

$$\begin{cases} F_1 = \frac{1}{2}\gamma y_1^2 b \\ F_2 = \frac{1}{2}\gamma y_2^2 b \end{cases}$$

$Q = qb$; $V = \frac{q}{y}$

SUBSTITUTING : $\frac{1}{2}\gamma b [\, y_1^2 - y_2^2 \,] = \frac{\gamma}{g} q b \left(\frac{q}{y_2} - \frac{q}{y_1} \right)$

$$\boxed{\frac{y_1^2}{2} + \frac{q^2}{g y_1} = \frac{y_2^2}{2} + \frac{q^2}{g y_2}}$$

MOMENTUM : $\boxed{M = \frac{y^2}{2} + \frac{q^2}{g y}}$

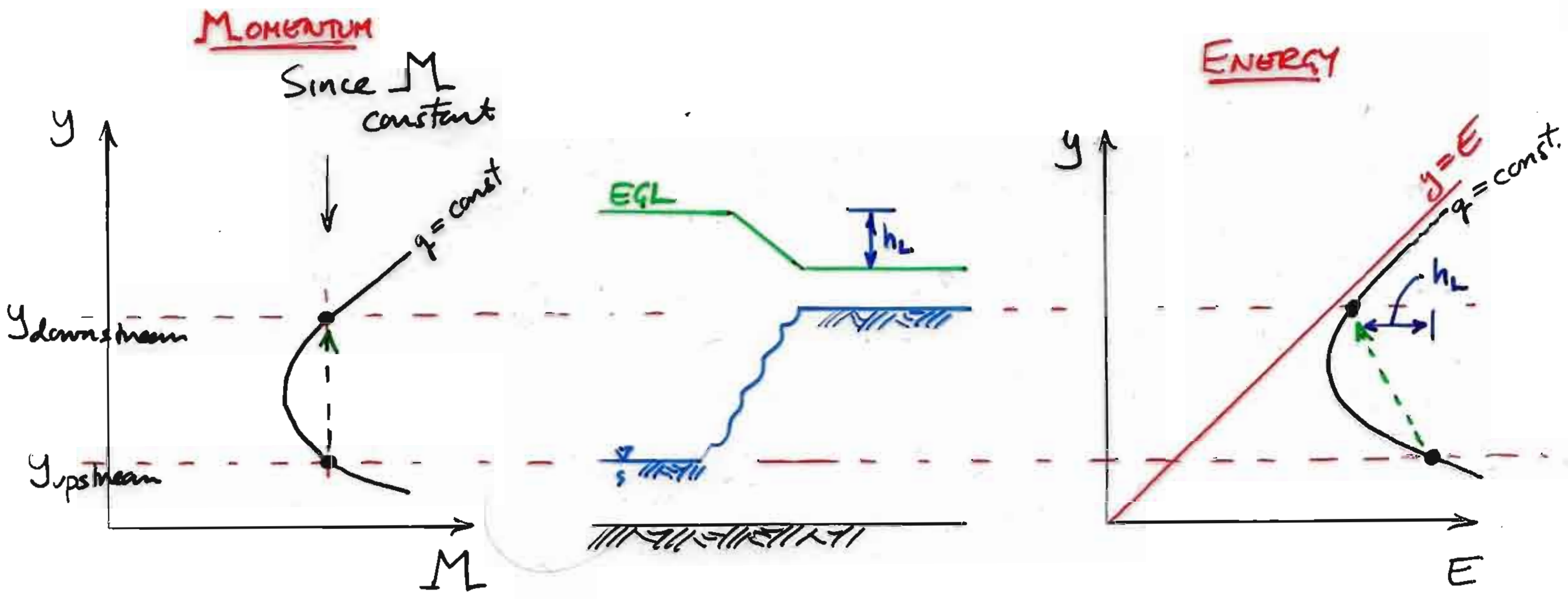
MOMENTUM = const.

M = constant through jump.

CONTINUITY : $\boxed{y_1 V_1 b = y_2 V_2 b}$

ENERGY EQN : $\boxed{y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + h_L}$

MOMENTUM & ENERGY DIAGRAMS



$$\mathcal{M} = \frac{y^2}{2} + \frac{q^2}{gy}$$

$$E = y + \frac{q^2}{2gy^2}$$

☐ Plot $\mathcal{M}$ for a given q, variable with flow depth, y.

☐ If upstream $\mathcal{M}$ is known then know that $\mathcal{M}_{\text{downstream}}$ is the same

☐ Project up on $\mathcal{M}$ diagram to determine downstream depth, $y_{\text{downstream}}$.

SOLVE FOR DEPTH CHANGE

$$F_r = \frac{V}{\sqrt{gy}} \; ; \quad V = \frac{q}{y} \quad \Rightarrow \quad F_r = \frac{q}{y\sqrt{gy}} \quad \Rightarrow \quad F_r^2 = \frac{q^2}{g y_1^3} \leftarrow *$$

MOMENTUM EQUATION:

$$\frac{1}{2}\left(y_1^2 - y_2^2\right) = \frac{1}{g}\left(\frac{q^2}{y_2} - \frac{q^2}{y_1}\right)$$

$$\frac{1}{2}(y_1 + y_2)(y_1 - y_2) = \frac{1}{g}\left(\frac{y_1^3 q^2}{y_1^3 y_2} - \frac{y_1^2 q^2}{y_1^2 y_1}\right)$$

$$= \left(y_1^2 \frac{y_1}{y_2} F_r^2 - y_1^2 F_r^2\right) = y_1^2 F_r^2\left(\frac{y_1}{y_2} - 1\right)$$

$$\frac{1}{2}(y_1 + y_2)(y_1 - y_2) = \frac{y_1^2}{y_2} F_r^2 (y_1 - y_2)$$

Trivial solution $y_1 = y_2$ ∴ no jump.

Divide by y_2 → $$\frac{1}{2}\left(\frac{y_1}{y_2} + 1\right) = \left(\frac{y_1}{y_2}\right)^2 F_r^2$$

Solve quadratic → $$\frac{1}{2} + \frac{1}{2}\left(\frac{y_1}{y_2}\right) - F_r^2\left(\frac{y_1}{y_2}\right)^2 = 0$$

$$\boxed{\frac{y_2}{y_1} = \frac{1}{2}\left(-1 \pm \sqrt{1 + 8F_r^2}\right)}$$

+ only since $\frac{y_2}{y_1} < 0$ not possible.

$F_r^2 = \frac{q^2}{g y_1^2}$ (upstream Fr)

ENERGY EQUATION:

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + h_L \Rightarrow \boxed{\frac{h_L}{y_1} = 1 - \frac{y_2}{y_1} + \frac{F_r^2}{2}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right]}$$

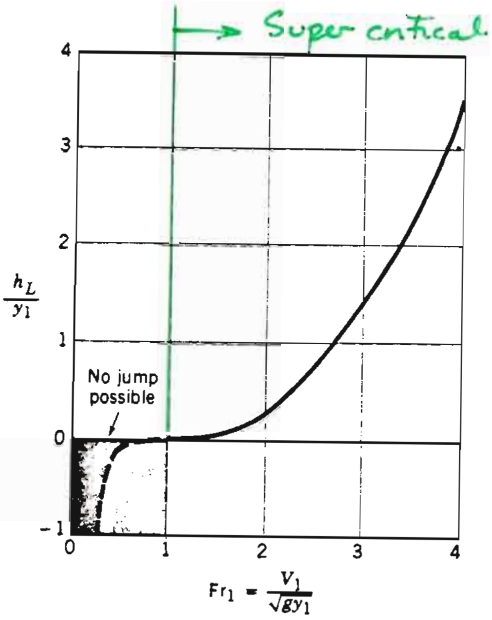
HYDRAULIC JUMP

HEAD LOSS:

$$\frac{h_L}{y_1} = 1 - \frac{y_2}{y_1} + \frac{Fr^2}{2}\left[1 - \overset{*}{\left(\frac{y_1}{y_2}\right)^2}\right] \quad (2)$$

$\rightarrow h_L \Rightarrow$ Thermal energy ΔT.

* Note (1) substituted into (2) for a given Fr.



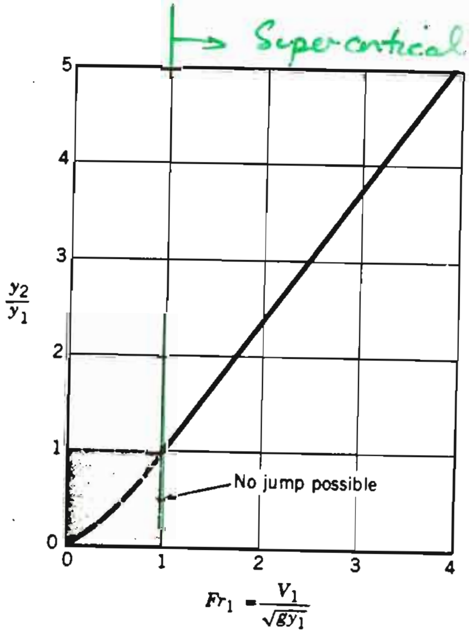
■ FIGURE 10.22 Dimensionless head loss across a hydraulic jump as a function of upstream Froude number.

Take advantage of h_L to dissipate (intentionally) the flow energy.

- ☐ Spillways
- ☐ Balance with erosion cavitation

UPSTREAM DEPTH RATIO:

$$\frac{y_2}{y_1} = \frac{1}{2}\left(-1 + \sqrt{1 + 8Fr^2}\right) \quad (1)$$



■ FIGURE 10.21 Depth ratio across a hydraulic jump as a function of upstream Froude number.

EXAMPLE 10.9

Water on the horizontal apron of the 100-ft-wide spillway shown in Fig. E10.9*a* has a depth of 0.60 ft and a velocity of 18 ft/s. Determine the depth, y_2, after the jump, the Froude numbers before and after the jump, Fr_1 and Fr_2, and the power dissipated, $\mathcal{P}_d$, within the jump.

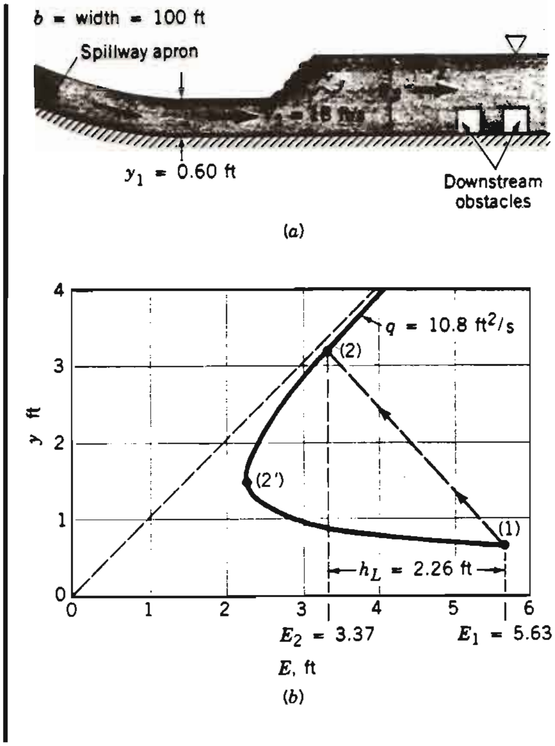


FIGURE E10.9

$$Fr_1 = \frac{V_1}{\sqrt{gy_1}} = 4.10$$

ok $Fr > 1$ ✓

$$\frac{y_2}{y_1} = \frac{1}{2}\left(-1 + \sqrt{1+8Fr^2}\right) = 5.32$$

$$\therefore y_2 = 3.19 ft$$

$$Fr_2 = \frac{V_2}{\sqrt{gy_2}} = 0.334$$

ok $Fr < 1$

∴ changes from super → sub. ✓

Plot $E = y + \dfrac{q^2}{2gy^2}$

gives:

$$E_1 = 5.63' \text{ @ } y_1 = 0.6 ft$$

$$E_1 + z_1 = E_2 + z_2 + h_L$$

$$E_2 = E_1 - h_L \cong 3.33$$

Note path of travel
- due to discontinuity
- i.e. 1-d representation is not very good for path of behavior.

$$\frac{h_L}{y_1} = 1 - \frac{y_2}{y_1} + \frac{Fr_1^2}{2}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right] \Rightarrow h_L = 2.3'$$

NOTE, use $\boxed{Fr_1}$

POWER: $P = \gamma Q h_L$

$$P = (62.4 lb/ft^3)(100 ft)(0.6 ft)(18 ft/s)(2.3')$$

$$P = 1.52 \times 10^5 ft.lb/s \Rightarrow$$

$$P = \frac{1.52 \times 10^5 ft.lb/s}{550 (ft.lb./s)/hp} = 277 hp.$$

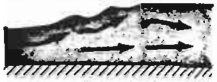
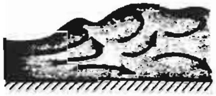
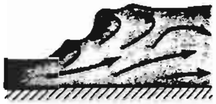
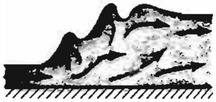
$\longrightarrow P \Rightarrow \Delta T$

May also use specific E concepts

HYDRAULIC JUMPS

■ **TABLE 10.3**
Classification of Hydraulic Jumps (Ref. 12)

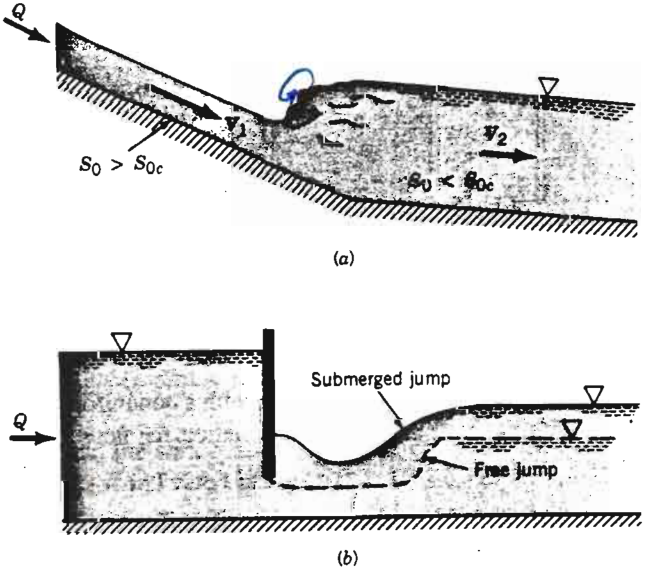
Upstream →

Fr_1	y_2/y_1	Classification	Sketch
<1	1	Jump impossible	
1 to 1.7	1 to 2.0	Standing wave or undulant jump	
1.7 to 2.5	2.0 to 3.1	Weak jump	
2.5 to 4.5	3.1 to 5.9	Oscillating jump	
4.5 to 9.0	5.9 to 12	Stable, well-balanced steady jump; insensitive to downstream conditions	
>9.0	>12	Rough, somewhat intermittent strong jump	

Length of jump = 7 downstream depths.

Also develop in:
☐ Sloping channels

☐ Obstructed flows. give SUBMERGED jump.



(a)

(b)

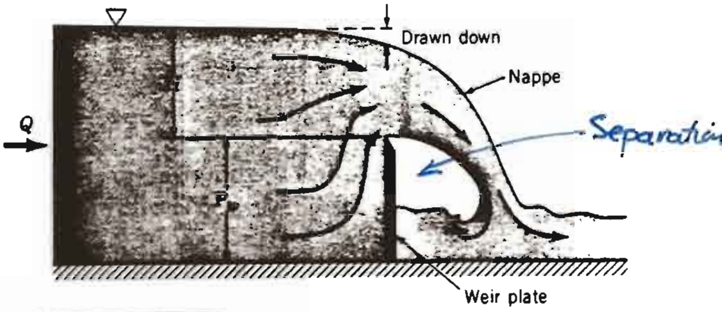
■ **FIGURE 10.23**
Hydraulic jump variations: (*a*) jump caused by a change in channel slope, (*b*) submerged jump.

SHARP CRESTED WEIR

- ☐ Sharp crested ⇒ Separation occurs
- ☐ Gravity drives flow (obviously).

Assumptions

- ☐ 1-d horizontal flow
- ☐ Atmospheric pressure through Nappe.



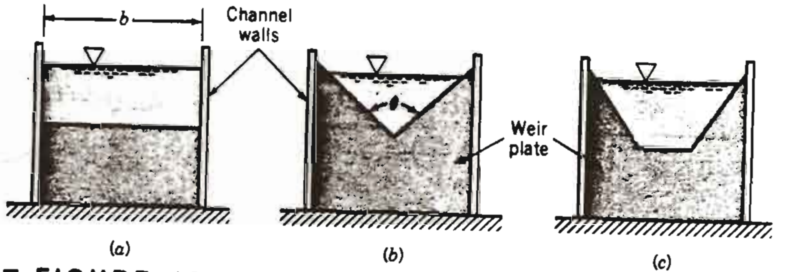
■ **FIGURE 10.24**
Sharp-crested weir geometry.

Apply similarity to FREE JET

$$Q = C_{wr}\,\frac{2}{3}\sqrt{2gH^3}\;b$$

$$\uparrow \qquad = C_{wr}\,\frac{2}{3}\sqrt{2gH}\,(Hb)$$

Correction factor (rectangular)



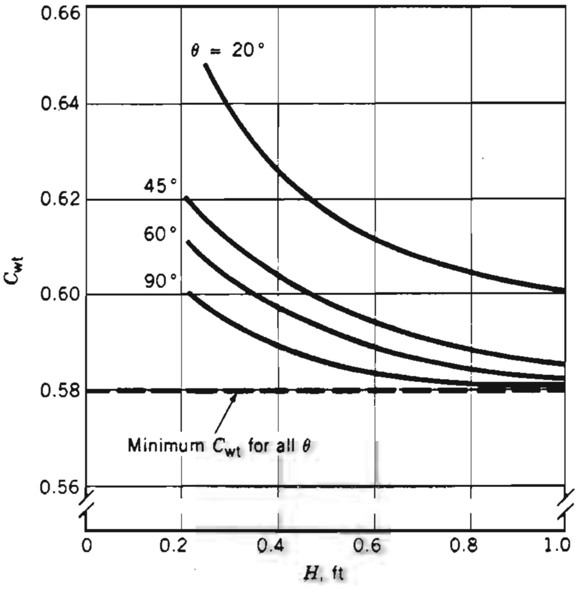
■ **FIGURE 10.25** **Sharp-crested weir plate geometry: (*a*) rectangular, (*b*) triangular, (*c*) trapezoidal.**

$$C_{wr} = 0.611 + 0.075\left(\frac{H}{P_w}\right)$$

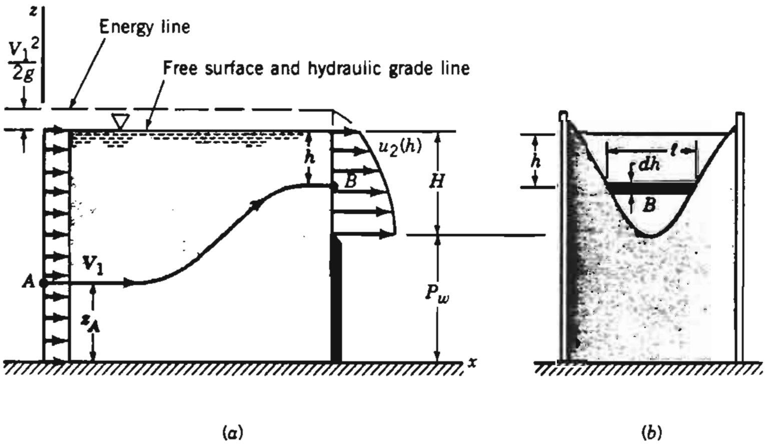
TRIANGULAR WEIR

$$Q = C_{wt}\,\frac{8}{15}\,\tan\left(\frac{\theta}{2}\right)\sqrt{2g}\;H^{5/2}$$

Sharp-crested weir → Requires 'ventilation' of Nappe.

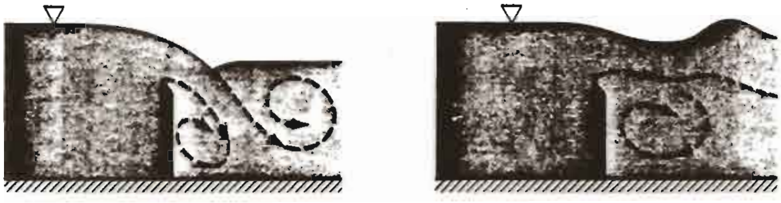


■ **FIGURE 10.27** **Weir coefficient for triangular sharp-crested weirs (Ref. 10).**



■ **FIGURE 10.26** **Assumed flow structure over a weir.**

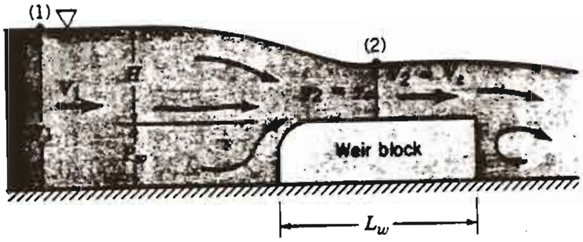
BROAD-CRESTED WEIR – (Unventilated Nappe)


(a) (b)

■ **FIGURE 10.28** Flow conditions over a weir without a free nappe: (*a*) plunging nappe, (*b*) submerged nappe.

← See previous for sharp crest. Without ventilation, the previous relations cannot be applied.

BROAD-CRESTED WEIR



■ **FIGURE 10.29** Broad-crested weir geometry.

Flow accelerates above weir and becomes critical at

$y_2 = y_c$ $\qquad H_w / L_w > 0.08$

Use Bernoulli with $V_2 = V_c = \sqrt{g y_c}$ and $h_L = 0$

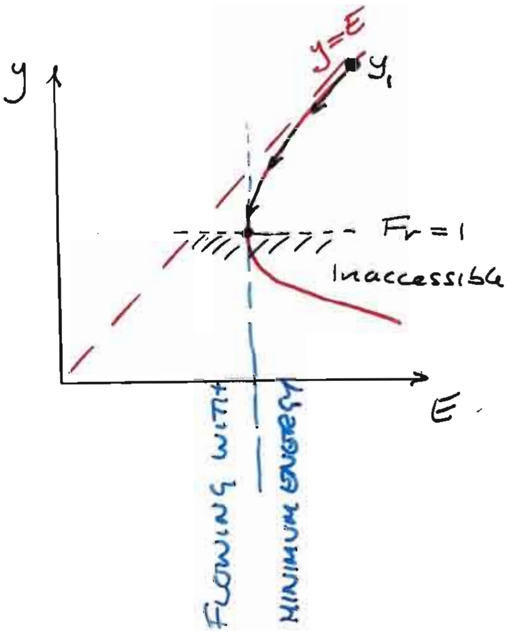
[illegible]

$$Q = C_{wb}\, b \sqrt{g} \left(\frac{2}{3}\right)^{3/2} H^{3/2}$$

$$C_{wb} = \frac{0.65}{\left(1 + H/P_w\right)^{1/2}}$$

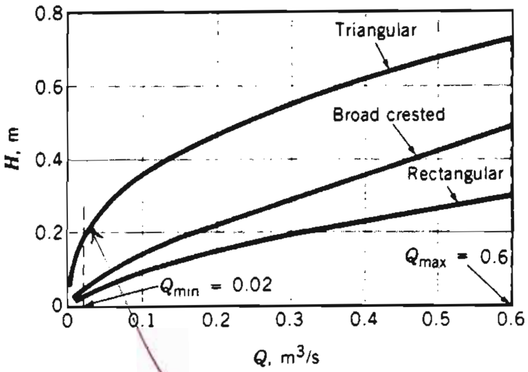
☐ Flow accelerates over weir block and attempts to become supercritical → →

☐ Flow @ (2) cannot be supercritical since downstream flow cannot communicate the end of the weir block upstream (ie. since $V = V_c$ above block).



EXAMPLE 10.10

Water flows in a rectangular channel of width $b = 2$ m with flowrates between $Q_{\min} = 0.02$ m^3/s and $Q_{\max} = 0.60$ m^3/s. This flowrate is to be measured by using either (a) a rectangular sharp-crested weir, (b) a triangular sharp-crested weir with $\theta = 90°$, or (c) a broad-crested weir. In all cases the bottom of the flow area over the weir is a distance $P_w = 1$ m above the channel bottom. Plot a graph of $Q = Q(H)$ for each weir and comment on which weir would be best for this application.

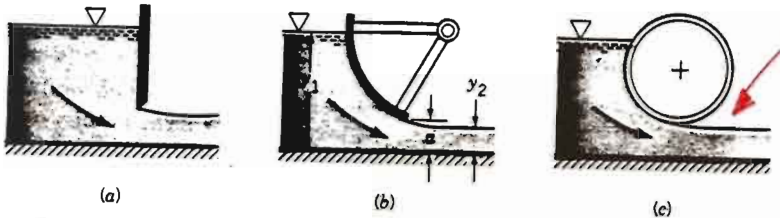


USE WEIR EQUATIONS

$Q = \phi[b, H, P_w, \theta]$

Δr gives best resolution at lowest flowrates.

UNDERFLOW GATES



Induces supercritical flow = free outflow

■ **FIGURE 10.30** **Three variations of underflow gates: (*a*) vertical gate, (*b*) radial gate, (*c*) drum gate.**

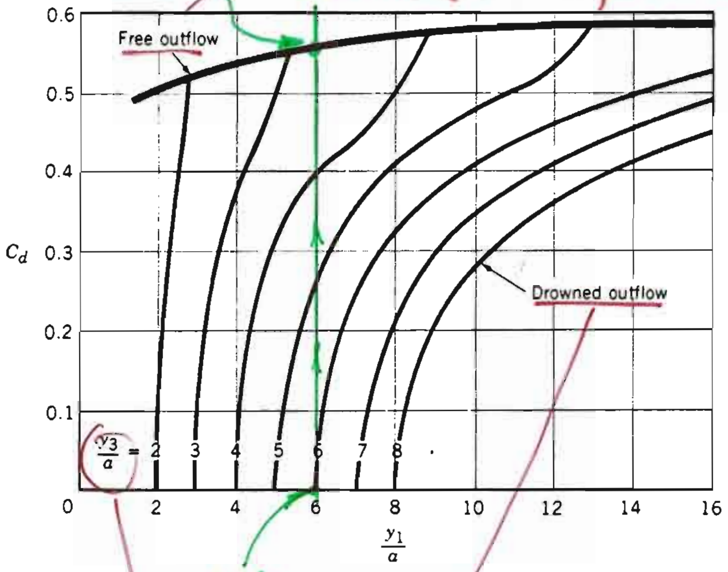
$$q = C_d a \sqrt{2 g y_1}$$

a = separation bed to gate base

y_1 = upstream depth.

q = max since Cd = max

Free outflow = supercritical downstream



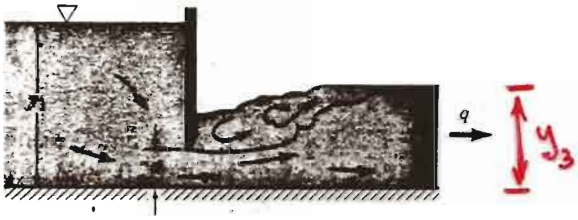
No flow q=0 since Cd=0

■ **FIGURE 10.31** **Typical discharge coefficients for underflow gates (Ref. 3).**

Two conditions:

1. Free outflow Cd = 0.5 – 0.6
2. Drowned outflow Cd = 0 – 0.6

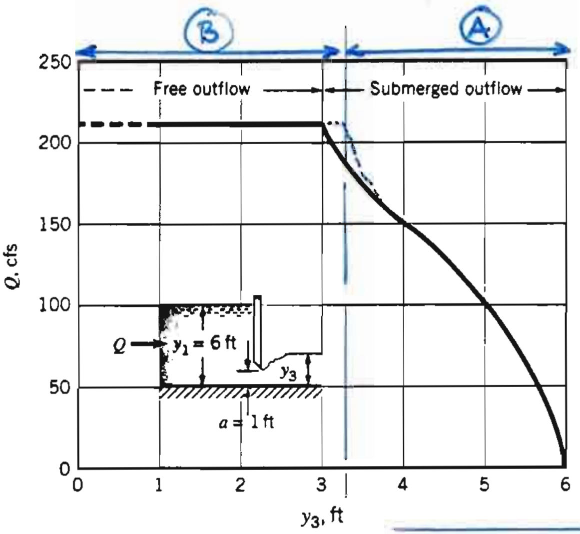
Drowned outflow.



■ **FIGURE 10.32** **Drowned outflow from a sluice gate.**

EXAMPLE 10.11

Water flows under the sluice gate shown in Fig. E10.11. The channel width is $b = 20$ ft, the upstream depth is $y_1 = 6$ ft, and the gate is $a = 1.0$ ft off the channel bottom. Plot a graph of flowrate, Q, as a function of y_3.



$q = C_d a \sqrt{2gy_1}$

∴ evaluate $Q = qb$

→ $Q = 393 C_d$ cfs.

$\frac{y_1}{a} = \frac{6}{1} = 6$.

∴ Use Figure 10.31.

Ⓐ For $\frac{y_1}{a} = 6$ $\begin{cases} \frac{y_3}{a} = 3.2 \\ \frac{y_3}{a} = 6 \end{cases}$

y_3 3.2

$y_3 = 6$

↑

Submerged range

Ⓑ For $\frac{y_3}{a} \lesssim 3.2$ C_d = constant = 0.56

∴ Q = constant