Extended Finite Element Method

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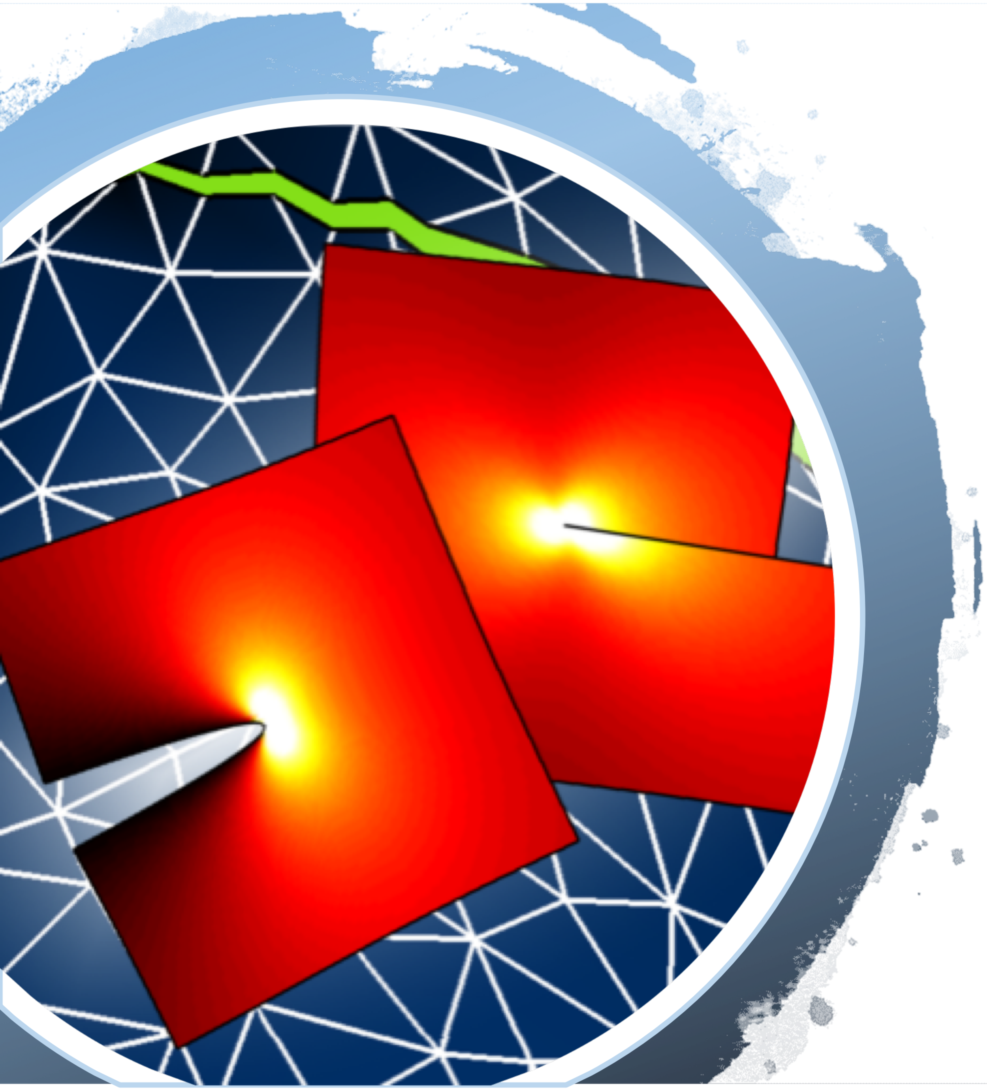
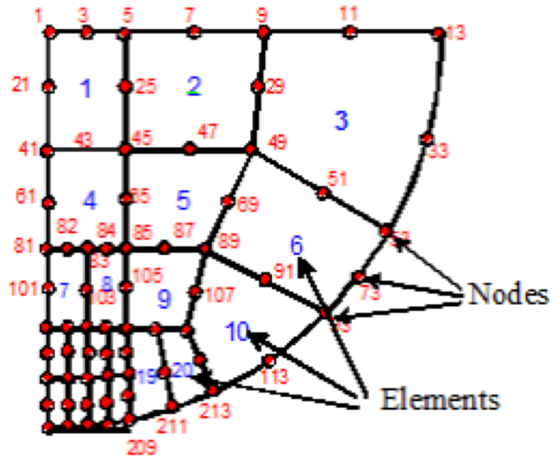


Introduction

Introduction

➢ **Finite Element Method (FEM)**

- A numerical tool to obtain approximate solutions of PDEs.
- Steps:
  - Discretization of the solution region;
  - Derivation of equations;
  - Assemble all the elements;
  - Solving equations.
- Requirements:
  - the mesh has to conform to the geometry;
  - Remeshing at each step.



Mesh in FEM

Allan F. Bower, "Applied Mechanic of Solids".

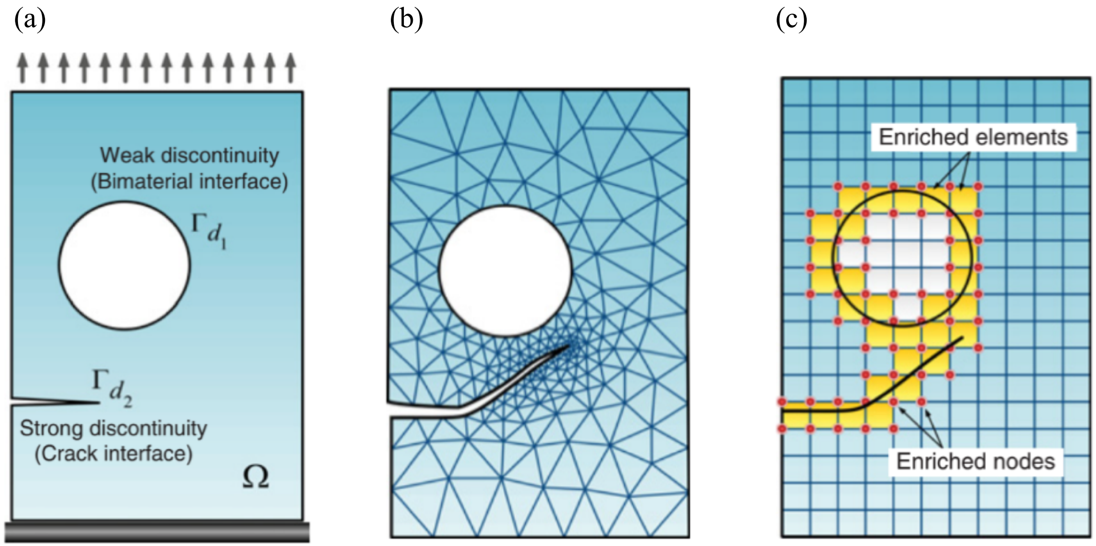
Introduction

➢ **Extended Finite Element Method (XFEM)**

- A powerful tool for discontinuous problems.
- Enables accurate capture of non-smooth features.
- Avoid using a mesh that conforms to cracks with overcoming remeshing difficulties as in FEM.
- Enrich the elements near the crack tip and along the crack faces.

Introduction

➢ **FEM vs. XFEM Performance**



Modeling of discontinuities in FEM and XFEM. (a) Crack propagation in a plate with a hole; (b) FEM using adaptive mesh refinement; (c) XFEM with enrichment of the elements.

Extended Finite Element Method: Theory and Applications, First Edition. Amir R. Khoei.

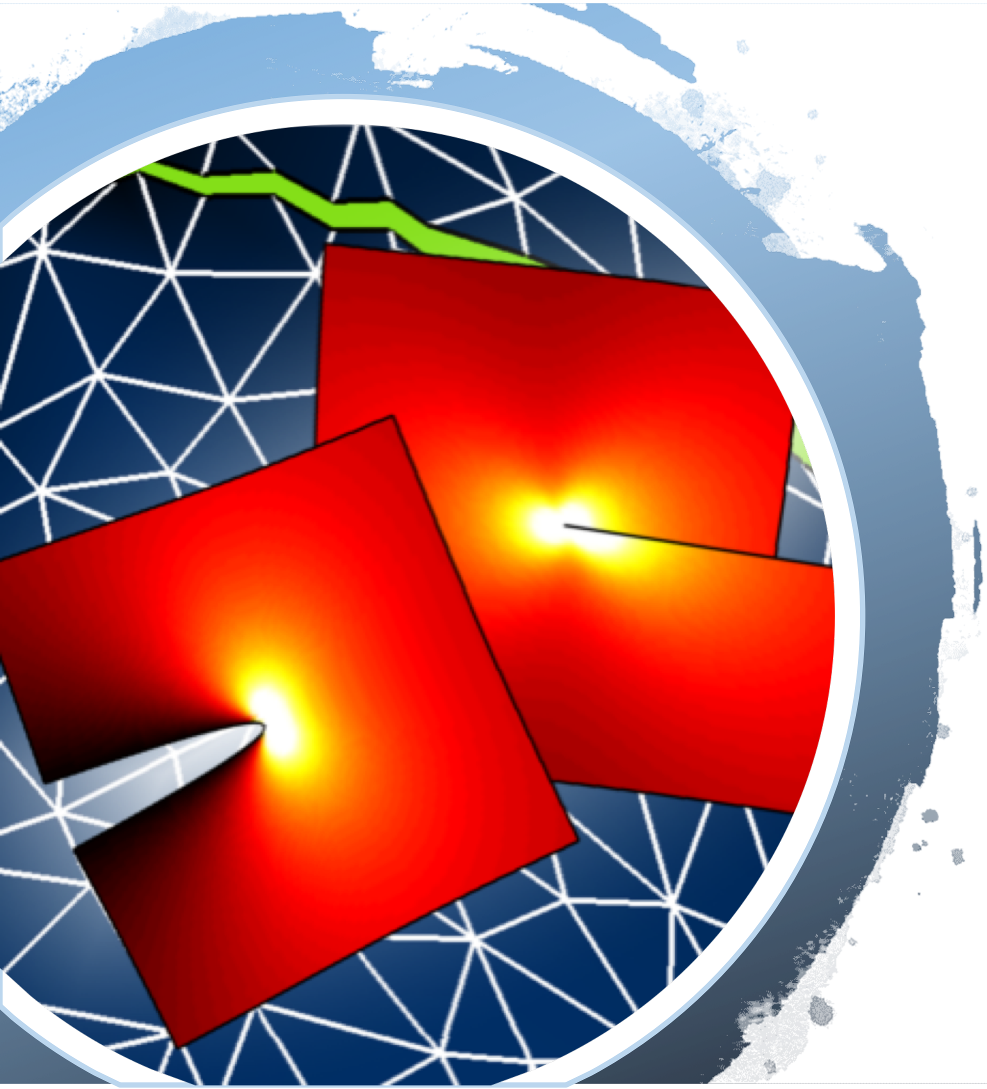
Introduction

➢ Comparisons

- XFEM vs. FEM
  - XFEM : useful for discontinuities problems since enrichment functions are added to FEM; cracks can propagate along a natural arbitrary path.
  - FEM: poor for arbitrary discontinuities problems; cracks only propagate along the element edge.
- Boundary Element Method (BEM) vs. XFEM
  - It is applicable for multi-material/ phase problems.

Introduction

- **Advantages of XFEM**
  - Cracks with complex geometry can be modeled
  - No need of remeshing
  - Less expensive
- **Shortcomings of XFEM**
  - Hard to localize the initial fracture
  - Only for linear elastic fracture mechanics



Historical Perspective

Historical Perspective

- **Ted Belytschko and Tom Black (1999)**
  - A improved technique for finite elements based on a partition-of-unity which involves minimal remeshing.
  - Not yet applicable for long cracks and 3D.

- **Nicolas Moës, John Dolbow, and Ted Belytschko (1999)**
  - Incorporating a discontinuous field across the crack faces away from the crack tip.
  - Independent of remeshing

Historical Perspective

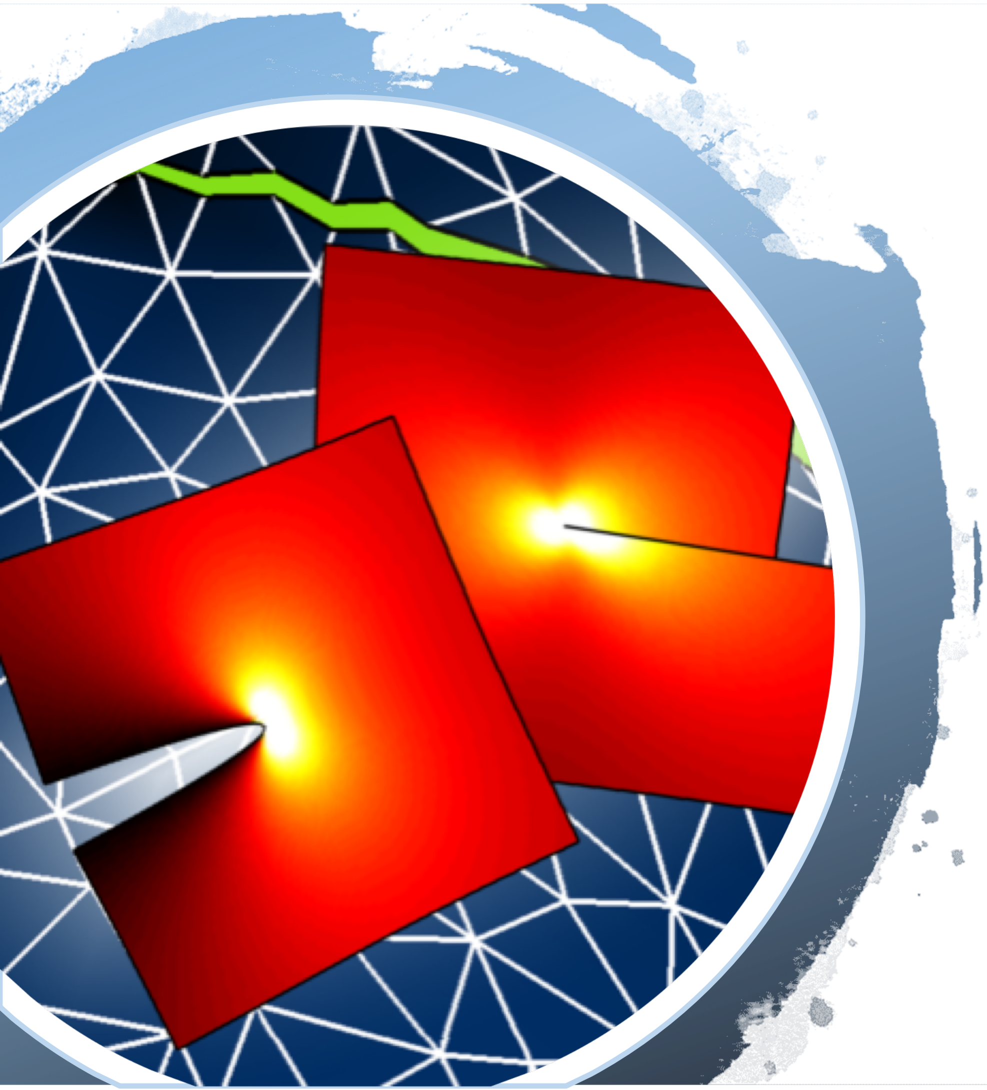
- **Stolarska et al. (2001)**
  - Apply level set method within the framework of X-FEM
  - The level set method is used to represent the crack location, including the location of crack tips
- **Sukumar et al. (2001)**
  - Proposed enrichment function for holes and inclusion

Historical Perspective

- **Réthoré et al (2005)**
  - Discuss the mathematical properties of X-FEM
  - Prove the stability of the numerical scheme in the linear case
- **Chessa and Belytschko (2004-2005)**
  - Extent the enriching shape function in time axis
  - space-time extended finite element method
- **Belytschko (2006)**
  - mesh-free method
  - no additional unknowns are introduced at the nodes whose supports are crossed by discontinuities

Historical Perspective

- **Gravouil(2007)**
  - Model of frictional contact along crack faces via X-FEM
- **Fang and Jin(2007)**
  - X-FEM algorithm was coupled with commercial software ABAQUS



General Principles

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Enrichment Function

- **Goal**
  - Increase the accuracy of the approximation by including information of the analytical solution.
- **Ways of enrichment**
  - Intrinsic enrichment: Enrich the basic vector
  - Extrinsic enrichment: Enrich the approximation

Enrichment Function

➢ **Intrinsic Enrichment**

- Enhance the approximation space u(x) by including the new basis functions.

- $\boldsymbol{u}(\boldsymbol{x}) = \sum_{i=1}^{p} \widehat{\boldsymbol{N}}_i(\boldsymbol{x})\overline{\boldsymbol{a}}_i$ $\qquad \widehat{\boldsymbol{N}}(\boldsymbol{x}) = \langle \boldsymbol{N}^{std}(\boldsymbol{x}), \boldsymbol{N}^{enr}(\boldsymbol{x}) \rangle$

- $\boldsymbol{N}^{std}(\boldsymbol{x})$ - Standard polynomial functions $\mathrm{N_i}(x)$
- $\boldsymbol{N}^{enr}(\boldsymbol{x})$ - Enriched shape functions obtained from $\mathrm{N_i}(x)p_j(x)$
- $\overline{\boldsymbol{a}}$ - A vector of coefficients obtained from one of the least-squares techniques

No additional unknowns

Enrichment Function

➢ **Extrinsic Enrichment**

- Enrich the approximation by adding the enrichment functions to the standard approximation.
- *Local* extrinsic enrichment, instead of enriching whole domain of the solution.
- Shows a systematical error in partially enriched elements.

Enrichment Function

➢ **Extrinsic Enrichment**

- The general enhanced solution field in the X-FEM:

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{i=1}^{N} \boldsymbol{N}_i(\boldsymbol{x})\overline{\boldsymbol{u}}_i + \underbrace{\sum_{k=1}^{p}\sum_{j=1}^{M_k} \overline{\boldsymbol{N}}_j(\boldsymbol{x})\boldsymbol{\psi}_k(\boldsymbol{x})\overline{\boldsymbol{a}}_{kj}}_{\text{Enrichment}}$$

Additional unknowns

Enrichment

- $\boldsymbol{M_k} \subseteq N$ - sets of nodal points, enriched by functions $\boldsymbol{\psi_k}(\boldsymbol{x})$
- $\boldsymbol{\psi_k}(\boldsymbol{x})$ - Enrichment functions

Enrichment Function

➢ **Different techniques used for enrichment function**

- Signed distance function
- Level set function
- Heaviside jump function
- …

➢ **Dependency on the conditions of problem**

- Ex. Discontinuity
  - Different types of material properties ⟶ Level Set Function
  - Different displacement fields on either sides of the discontinuity ⟶ Heaviside Function

Enrichment Function

➢ Level Set Method

- Definition: A numerical technique for tracking moving interfaces.

- The interface is represented as the zero level set of a function

  - One dimension (time) higher than the dimension of the interface
  - Evolved by solving the hyperbolic conservation laws
  - Independent of element mesh

- Most common function: signed distance function

Enrichment Function

- **Signed Distance Function**
  - $\boldsymbol{\varphi}(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{x}^*\| \boldsymbol{sign}(\boldsymbol{n}_{\boldsymbol{\Gamma}_{\boldsymbol{d}}}(\boldsymbol{x} - \boldsymbol{x}^*))$
  - $\boldsymbol{x}^*$ - the closest point projection of x onto the discontinuity $\boldsymbol{\Gamma}_{\boldsymbol{d}}$
  - $\boldsymbol{n}_{\boldsymbol{\Gamma}_{\boldsymbol{d}}}$ - the normal vector to the interface at point $\boldsymbol{x}^*$
  - $\|\boldsymbol{x} - \boldsymbol{x}^*\|$ - specifies the distance of point x to discontinuity $\boldsymbol{\Gamma}_{\boldsymbol{d}}$

Enrichment Function

- **Heaviside Function**

  - $H(x) = \begin{cases} 0, & if\ \varphi(x) < 0 \\ 1, & if\ \varphi(x) > 0 \end{cases}$ $\quad\quad H(x) = \begin{cases} -1, & if\ \varphi(x) < 0 \\ +1, & if\ \varphi(x) > 0 \end{cases}$

    Signed distance function

  - The approximation field can be written as

  - $u(x) = \sum_{i=1}^{N} N_i(x)\bar{u}_i + \sum_{j=1}^{M} N_j(x)H(x)\bar{a}_j$

  - Basis of definition of the kinematics of the strong discontinuity (a jump in the displacement field).

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Enriched Shape Function

➢ **Describing a Strong Discontinuity Surface**

- $u(x) = \sum_{i=1}^{N} N_i(x) u_i + \sum_{j=1}^{M} N_j(x) H(f(x)) a_j(t) + \sum_{k=1}^{S} N_k(x) \phi(x) b_k(t)$

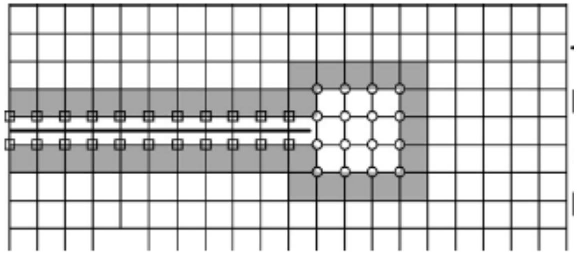
  Crack-crossed — Crack-embedded

- a_j , b_k - Enriched DOFs at the element node

- $H(x) = \begin{cases} -1, & if\ x < 0 \\ +1, & if\ x \geq 0 \end{cases}$

- $f(x) = min\|x - x^*\| sign(n_{\Gamma_d}(x - x^*))$

- $\phi(x) = \left[\sqrt{r}\, sin\frac{\theta}{2}, \sqrt{r}\, sin\frac{\theta}{2} sin\theta, \sqrt{r}\, cos\frac{\theta}{2}, \sqrt{r}\, cos\frac{\theta}{2} sin\theta\right]$



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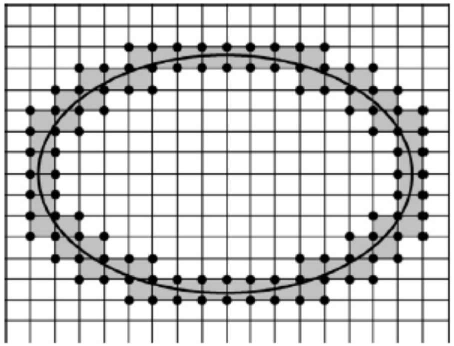
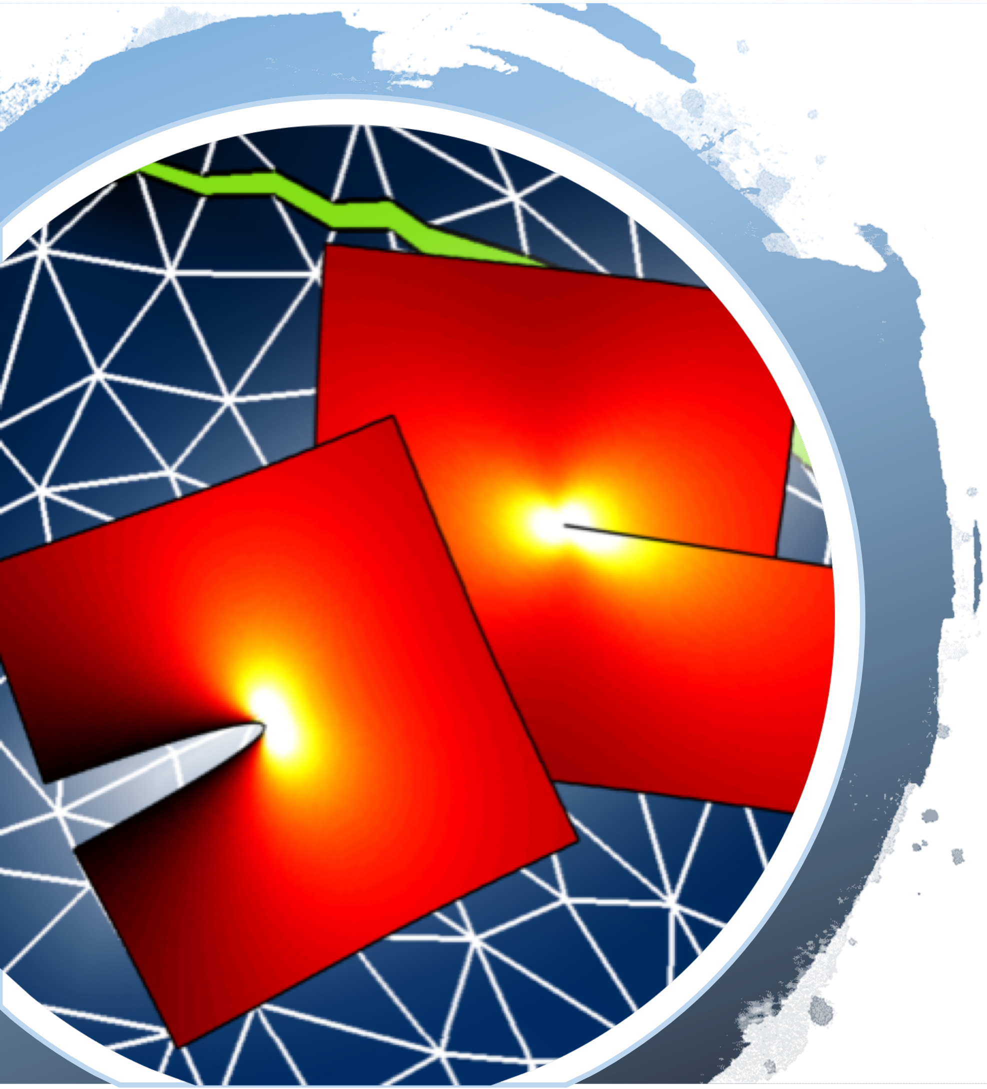
Enriched Shape Function

- **Describing a Weak Discontinuity Surface**
  - Displacement field continues at the interface
  - Derivative of displacement field (strain field) is discontinues
    - Material difference
  - $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) = \sum_{i=1}^{N} \boldsymbol{N}_i(\boldsymbol{x})\boldsymbol{u}_i(\boldsymbol{t}) + \sum_{j=1}^{M} \boldsymbol{N}_j(\boldsymbol{x})\boldsymbol{\phi}(\boldsymbol{x})\boldsymbol{q}_j$

    Interface
  - $\boldsymbol{\phi}(\boldsymbol{x}) = \sum_i |\boldsymbol{f}(\boldsymbol{x}_i)|\, \boldsymbol{N}_i(\boldsymbol{x}) - \left|\sum_i |\boldsymbol{f}(\boldsymbol{x}_i)|\, \boldsymbol{N}_i(\boldsymbol{x})\right|$
  - $\boldsymbol{q}_j$ - the new added DOF at the node





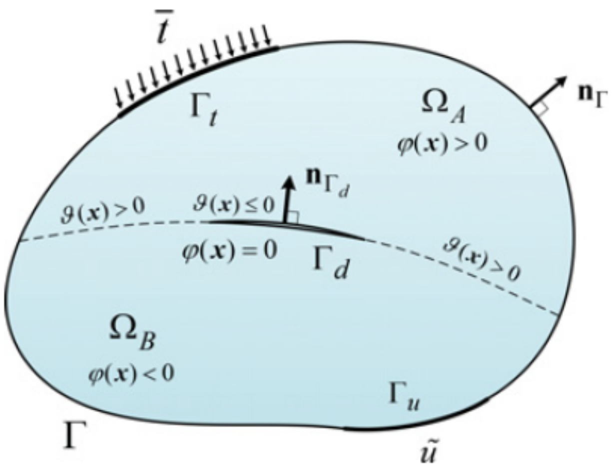
Governing Equations

Governing Equation

- **Strong Form of the Equilibrium Equation:**

$$\nabla \cdot \sigma + b = 0 \qquad \text{in } \Omega$$

- ***Boundary Conditions*:**
  - Displacement (Dirichlet) B.C.: $u = \tilde{u} \; on \; \Gamma_u$
  - Traction (Neumann) B.C.: $\sigma \cdot n_\Gamma = \tilde{t} \; on \; \Gamma_t$
  - Internal B.C.: $\sigma \cdot n_{\Gamma_d} = \bar{t}_d \; on \; \Gamma_d$



- For domain with strong discontinuity, $\sigma \cdot n_{\Gamma_d} = 0$

Governing Equation

➢ **Weak Form formulation of the Equilibrium Equation:**

$\nabla \cdot \sigma + b = 0$ Multiplying by the test functions, $\delta u(x,t)$, and integrate over the domain Ω

$$\int_{\Omega} \delta\mathbf{u}(\boldsymbol{x},t)\,(\nabla\cdot\boldsymbol{\sigma}+\mathbf{b})\,d\Omega = 0$$

Divergence theorem

$$\int_{\Omega} \nabla\delta\mathbf{u} : \boldsymbol{\sigma}\,d\Omega + \cancel{\int_{\Gamma_d} [\delta\mathbf{u}\cdot\boldsymbol{\sigma}]\mathbf{n}_{\Gamma_d}\,d\Gamma} - \int_{\Gamma_t} \delta\mathbf{u}\cdot\bar{\mathbf{t}}\,d\Gamma - \int_{\Omega} \delta\mathbf{u}\cdot\mathbf{b}\,d\Omega = 0$$

Weak Discontinuities

$$\int_{\Gamma_d} [\delta\mathbf{u}\cdot\boldsymbol{\sigma}]\mathbf{n}_{\Gamma_d}\,d\Gamma = \int_{\Gamma_d} [\delta\mathbf{u}\cdot\bar{\mathbf{t}}_d]\,d\Gamma = \int_{\Gamma_d} (\delta\mathbf{u}^{+}\bar{\mathbf{t}}_d^{+} - \delta\mathbf{u}^{-}\bar{\mathbf{t}}_d^{-})\,d\Gamma = 0$$

Strong Discontinuities

$$\int_{\Gamma_d} [\delta\mathbf{u}\cdot\boldsymbol{\sigma}]\mathbf{n}_{\Gamma_d}\,d\Gamma = \int_{\Gamma_d} [\delta\mathbf{u}\cdot\bar{\mathbf{t}}_d]\,d\Gamma = \int_{\Gamma_d} (\delta\mathbf{u}^{+} - \delta\mathbf{u}^{-})\bar{\mathbf{t}}_d\,d\Gamma = 0$$

$$\int_{\Omega} \nabla\delta\mathbf{u} : \boldsymbol{\sigma}\,d\Omega - \int_{\Gamma_t} \delta\mathbf{u}\cdot\bar{\mathbf{t}}\,d\Gamma - \int_{\Omega} \delta\mathbf{u}\cdot\mathbf{b}\,d\Omega = 0$$

Governing Equation

$$\int_{\Omega} \nabla \delta \mathbf{u} : \boldsymbol{\sigma} \, d\Omega - \int_{\Gamma_t} \delta \mathbf{u} \cdot \bar{\mathbf{t}} \, d\Gamma - \int_{\Omega} \delta \mathbf{u} \cdot \mathbf{b} \, d\Omega = 0$$

⇩

$$K\overline{U} - F = 0$$

- $\boldsymbol{K}$, total stiffness matrix
- $\boldsymbol{F}$, external force vector
- $\overline{\boldsymbol{U}}$, vector of degrees of nodal freedom (classical and enriched)

Governing Equation

$$K\bar{U} - F = 0 \quad \Rightarrow \quad \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{ua} \\ \mathbf{K}_{au} & \mathbf{K}_{aa} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{a}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ \mathbf{F}_a \end{Bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \int_\Omega (\mathbf{B}^{std})^T \mathbf{D}\, \mathbf{B}^{std} d\Omega & \int_\Omega (\mathbf{B}^{std})^T \mathbf{D}\, \mathbf{B}^{enr} d\Omega \\ \int_\Omega (\mathbf{B}^{enr})^T \mathbf{D}\, \mathbf{B}^{std} d\Omega & \int_\Omega (\mathbf{B}^{enr})^T \mathbf{D}\, \mathbf{B}^{enr} d\Omega \end{bmatrix}$$

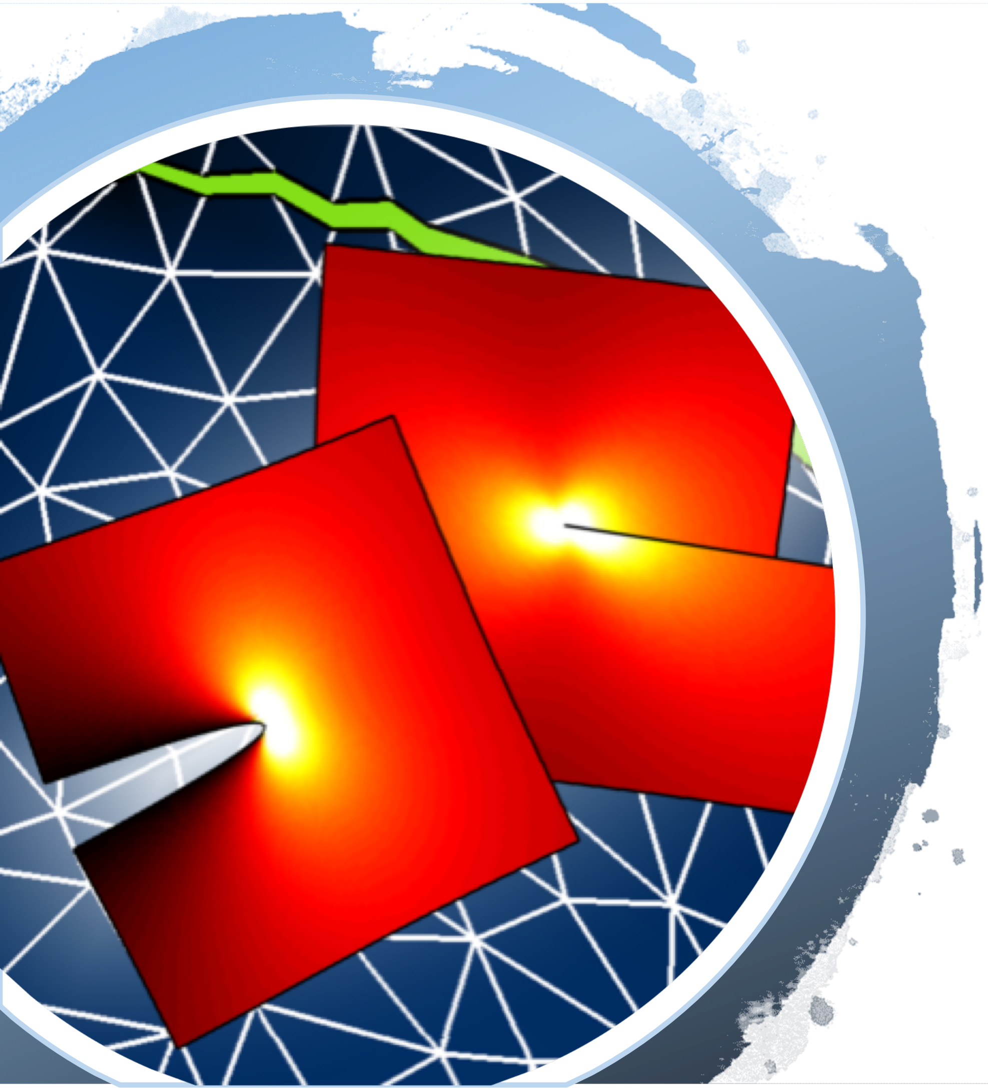
$$\mathbf{F} = \begin{Bmatrix} \int_{\Gamma_t} (\mathbf{N}^{std})^T \bar{\mathbf{t}}\, d\Gamma + \int_\Omega (\mathbf{N}^{std})^T \mathbf{b}\, d\Omega \\ \int_{\Gamma_t} (\mathbf{N}^{enr})^T \bar{\mathbf{t}}\, d\Gamma + \int_\Omega (\mathbf{N}^{enr})^T \mathbf{b}\, d\Omega \end{Bmatrix}$$

$$\mathbf{B}_i^{std} = \begin{bmatrix} \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y \\ \partial N_i/\partial y & \partial N_i/\partial x \end{bmatrix}$$

$$\mathbf{N}_i^{std} = \begin{bmatrix} N_i(\boldsymbol{x}) & 0 \\ 0 & N_i(\boldsymbol{x}) \end{bmatrix}$$

$$\mathbf{B}_j^{enr} = \begin{bmatrix} \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial x & 0 \\ 0 & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial y \\ \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial y & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial x \end{bmatrix}$$

$$\mathbf{N}_j^{enr} = \begin{bmatrix} N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right) & 0 \\ 0 & N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right) \end{bmatrix}$$



Hand-Calculation Examples

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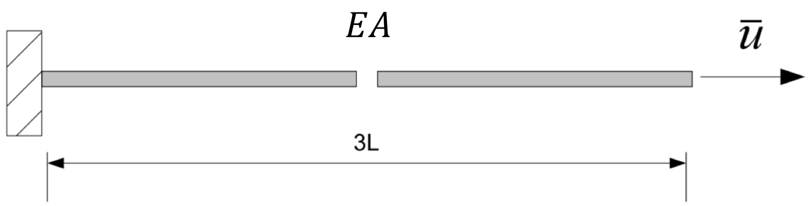
Hand-Calculation

Consider a 1D bar of length 3L.

Let E be the elastic moduli and A be the cross-sectional area of the bar.

The bar is subjected to a prescribed displacement at the end while the other end of the bar is fixed.

The bar is cracked at its mid length, L=1.5L.

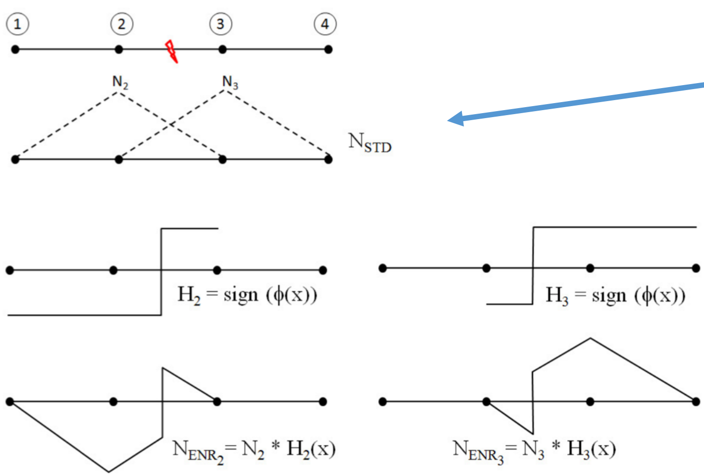


Hand-Calculation

- **XFEM Solution with Non-aligned Mesh**

$$u_{XFEM} = \sum_{i=1}^{N} N_i u_i + \sum_{j=1}^{M} N_j H(x) a_j$$

Enriched basis function for a strong discontinuity in 1D



$$[N] = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

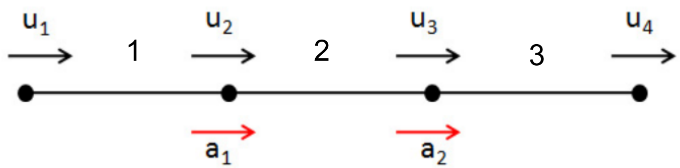
$$H(x) = \begin{cases} +1 & x - x_0 > 0 \\ 0 & x = x_0 \\ -1 & x - x_0 < 0 \end{cases}$$

$$\boldsymbol{u(x) = \sum_{i=1}^{N} N_i(x) u_i + \sum_{j=1}^{M} N_j(x) H(f(x)) a_j(t) + \sum_{k=1}^{S} N_k(x) \phi(x) b_k(t)}$$

Hand-Calculation

➢ XFEM Solution with Non-aligned Mesh

Element No.1



The step function $H(x) = -1$

$$N_{std}^{u} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \qquad N_{enr}^{a} = H\begin{bmatrix} \frac{x}{L} \end{bmatrix} = \begin{bmatrix} -\frac{x}{L} \end{bmatrix}$$

$$B_{std}^{u} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \qquad B_{enr}^{a} = H\begin{bmatrix} \frac{1}{L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} \end{bmatrix}$$

$$K_{uu} = EA\int_0^L (B_{std}^{u})^T B_{std}^{u}\, dx = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{ua} = EA\int_0^L (B_{std}^{u})^T B_{enr}^{a}\, dx = \frac{EA}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{au} = {K_{ua}}^T$$

$$K_{aa} = EA\int_0^L (B_{enr}^{a})^T B_{enr}^{a}\, dx = \frac{EA}{L}$$

$$K_{e^1} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

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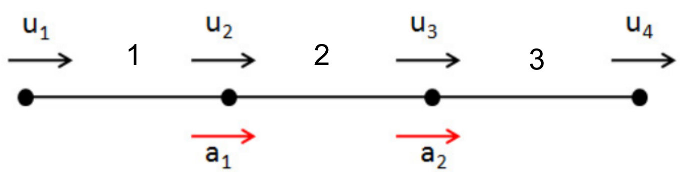
Hand-Calculation

➢ XFEM Solution with Non-aligned Mesh

Element No.3



The step function $H(x) = +1$

$$N_{std}^{u} = \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} \qquad N_{enr}^{a} = H\begin{bmatrix} 1-\frac{x}{L} \end{bmatrix} = \begin{bmatrix} 1-\frac{x}{L} \end{bmatrix}$$

$$B_{std}^{u} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \qquad B_{enr}^{a} = H\begin{bmatrix} -\frac{1}{L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} \end{bmatrix}$$

$$K_{uu} = EA\int_0^L (B_{std}^{u})^T B_{std}^{u}\, dx = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{ua} = EA\int_0^L (B_{std}^{u})^T B_{enr}^{a}\, dx = \frac{EA}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{au} = {K_{ua}}^{T}$$

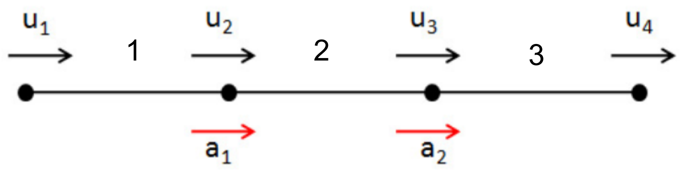
$$K_{aa} = EA\int_0^L (B_{enr}^{a})^T B_{enr}^{a}\, dx = \frac{EA}{L}$$

$$K_{e^3} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Hand-Calculation

▯ XFEM Solution with Non-aligned Mesh

Element No.2



The enrichment function $H(x) = \begin{cases} +1 & x \to {x_0}^+ \\ -1 & x \to {x_0}^- \end{cases}$

$$N_{std}^{u} = \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix}$$

$$N_{enr}^{a} = H\begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} = \begin{cases} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} & x \to {x_0}^+ \\ \begin{bmatrix} \frac{x}{L} & 1-\frac{x}{L} \end{bmatrix} & x \to {x_0}^- \end{cases}$$

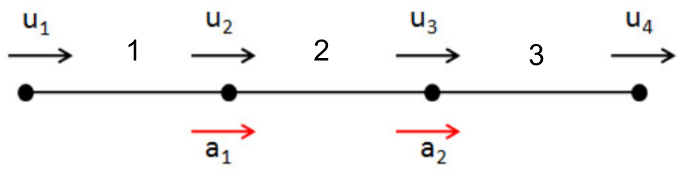
$$B_{std}^{u} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

$$B_{enr}^{a} = H\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} = \begin{cases} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} & x \to {x_0}^+ \\ \begin{bmatrix} \frac{1}{L} & -\frac{x}{L} \end{bmatrix} & x \to {x_0}^- \end{cases}$$

Hand-Calculation

- **XFEM Solution with Non-aligned Mesh**

Element No.2



$$K_{uu} = EA\int_0^L (B^u_{std})^T B^u_{std}\,dx = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{ua} = K^+_{ua} + K^-_{ua}$$

$$K_{au} = K^+_{au} + K^-_{au}$$

$$K_{aa} = K^+_{aa} + K^-_{aa}$$

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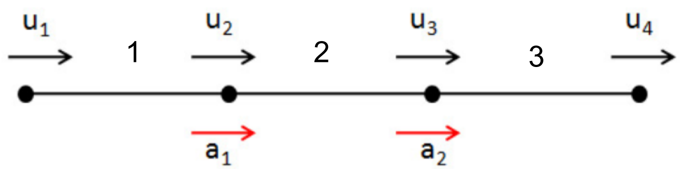
Hand-Calculation

- **XFEM Solution with Non-aligned Mesh**

Element No.2



Integrating on +, the enrichment function $H(x) = +1$

$$K_{ua}^{+} = EA\int_{0}^{\frac{L}{2}} (B_{std}^{u})^T B_{enr}^{a}\, dx = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{aa}^{+} = EA\int_{0}^{\frac{L}{2}} (B_{enr}^{a})^T B_{enr}^{a}\, dx = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Integrating on -, the enrichment function $H(x) = -1$

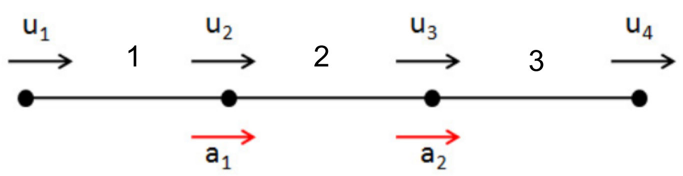
$$K_{ua}^{-} = EA\int_{\frac{L}{2}}^{L} (B_{std}^{u})^T B_{enr}^{a}\, dx = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{aa}^{-} = EA\int_{\frac{L}{2}}^{L} (B_{enr}^{a})^T B_{enr}^{a}\, dx = \frac{EA}{2L}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Hand-Calculation

- **XFEM Solution with Non-aligned Mesh**

Element No.2



$$K_{ua} = K_{ua}^{+} + K_{ua}^{-} = \frac{EA}{2L}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

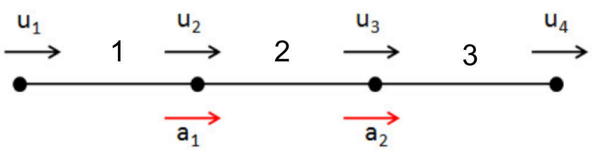
$$K_{aa} = K_{aa}^{+} + K_{aa}^{-} = \frac{EA}{L}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$K_{au} = K_{ua}{}^{T}$$

$$K_{e^2} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Hand-Calculation

- **XFEM Solution with Non-aligned Mesh**



Not a nodal interpolant.

$$K = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 & -1 & 2 \end{bmatrix}$$

$K\bar{u} - F = 0$

$u_1 = 0$
$u_4 = \bar{u}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\bar{u}}{2} \\ \frac{\bar{u}}{2} \\ \bar{u} \\ -\frac{\bar{u}}{2} \\ -\frac{\bar{u}}{2} \end{bmatrix}$$

$$u(X) = N_i u_i + H N_j a_j$$

$$\begin{aligned} u(x_1) &= u_1 = 0 \\ u(x_2) &= u_2 + H(x_2)a_1 = \frac{\bar{u}}{2} - \frac{\bar{u}}{2} = 0 \\ u(x_3) &= u_3 + H(x_3)a_2 = \frac{\bar{u}}{2} + (-1)\left(-\frac{\bar{u}}{2}\right) = \bar{u} \\ u(x_4) &= u_4 = \bar{u} \end{aligned}$$

Enriched displacement approximation

Hand-Calculation

- **Standard Finite Element Method**

The finite element mesh has to be aligned with the crack.

$$K\bar{u} - F = 0$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

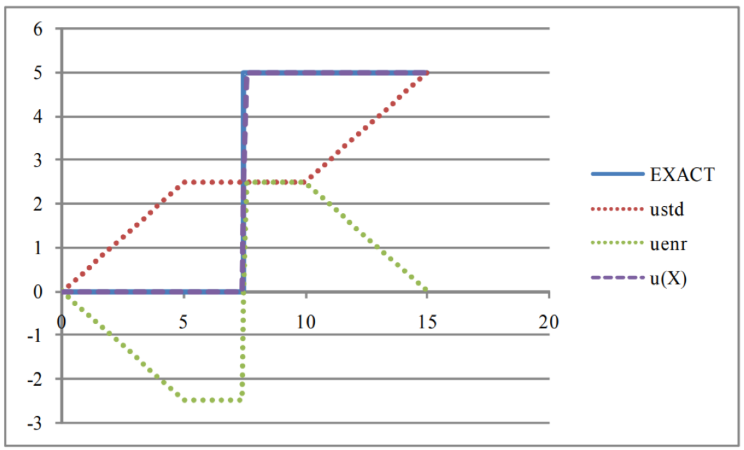
$u_1 = 0 \; u_4 = \bar{\bar{u}}$

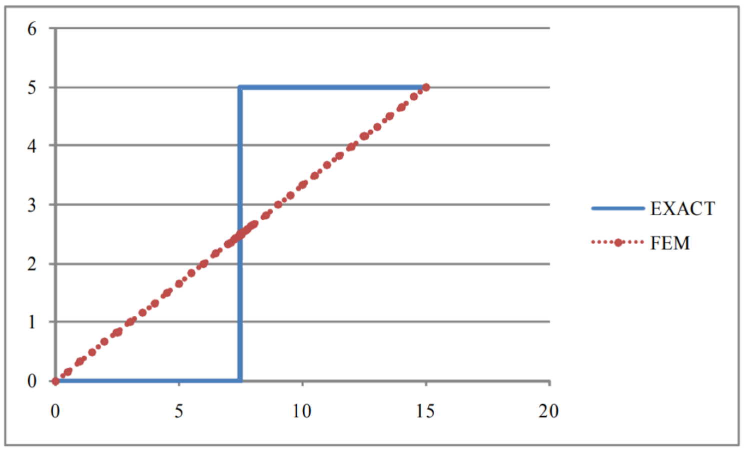
$$u_1 = 0, \qquad u_2 = \frac{\bar{u}}{3}, \qquad u_3 = \frac{2\bar{u}}{3}, \qquad u_4 = \bar{u}$$

Hand-Calculation

- **Compare XFEM Solution with Standard Finite Element Method**

Numerical solution of displacement field using XFEM and FEM





Hand-Calculation

$$u^h(x) = \sum_{i \in I} u_i N_i(x) + \sum_{i \in I} a_i N_i(x) H(x) + \sum_{i \in K_1} N_i(x) \left(\sum_{l=1}^{4} b_{i,1}^l F_1^l(x) \right) + \sum_{i \in K_2} N_i(x) \left(\sum_{l=1}^{4} b_{i,1}^l F_2^l(x) \right)$$

- I is the set of nodes in the mesh.
- u_i is the classical (vectorial) degree of freedom at node i.
- N_i is the scalar shape function associated to node i.
- $L \subset I$ is the subset of nodes enriched by the Heaviside function. The corresponding (vectorial) DOF are denoted a_i.
- $K_1 \subset I$ and $K_2 \subset I$ are the set of nodes to enrich to model crack tips numbered 1 and 2, respectively. The corresponding degrees of freedom are $b_{i,1}^l$ and $b_{i,2}^l, l = 1, \ldots, 4$.
- Functions $F_1^l(x), l = 1, \ldots, 4$ modeling the crack tip are given in elasticity by :

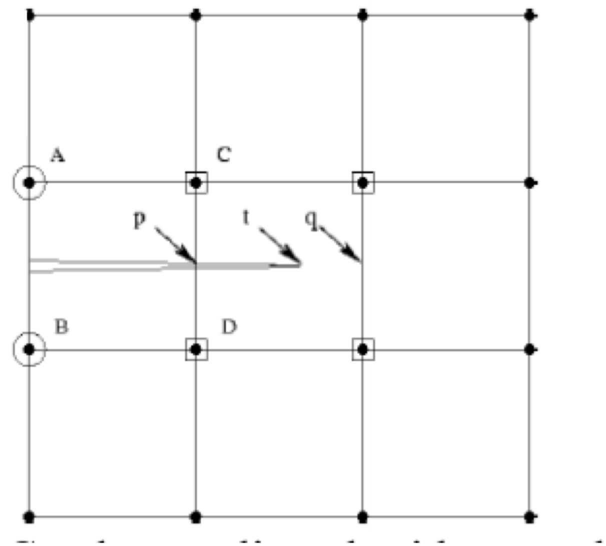
$$\{F_1^l(x)\} = \{\sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)\}$$

<u>Vector Enrichment:</u>

$$\underline{u} = \sum u_i N_i + \sum K_{1i} N_i G_1(r,\theta) + \sum K_{2i} N_i G_2(r,\theta)$$

<u>Scalar Enrichment:</u>

$$\underline{u} = \sum u_i N_i + \sum a_i N_i F_1(r,\theta) + \sum b_i N_i F_2(r,\theta) + \sum c_i N_i F_3(r,\theta) + \sum d_i N_i F_4(r,\theta$$



Source: Dibakar Datta

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Hand-Calculation

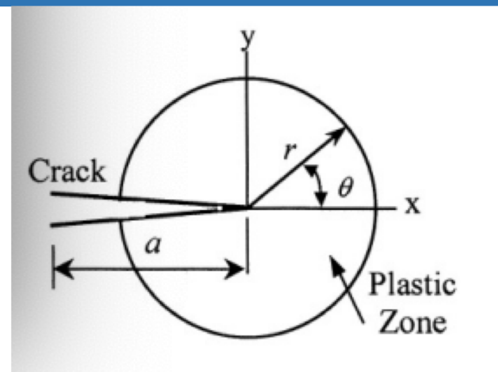


Fig 2.11: Crack tip circular region

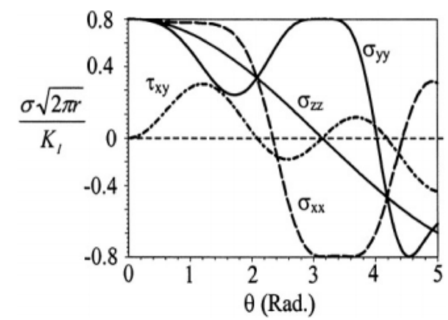


Fig 2.9: Normalized Stress Distribution for Mode 1.

Solution for Stress Field:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\begin{bmatrix}1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\\ 1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\\ \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\end{bmatrix}$$

Solution for Displacement Field:

$$\begin{bmatrix}\mu_x\\ \mu_y\\ \mu_z\end{bmatrix}=\frac{2K_I}{\sqrt{2\pi E}}\begin{bmatrix}\sqrt{r}\cos\frac{\theta}{2}\left[(1-\upsilon)+(1+\upsilon)\sin^2\frac{\theta}{2}\right]\\ \sqrt{r}\sin\frac{\theta}{2}\left[2-(1+\upsilon)\right]\cos^2\frac{\theta}{2}\\ -\frac{\upsilon B}{\sqrt{r}}\cos\frac{\theta}{2}\end{bmatrix}$$

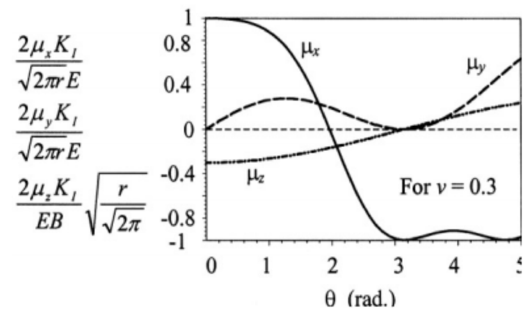


Fig 2.10: Normalized Displacement Distribution for Mode 1.

Source: Dibakar Datta

Hand-Calculation

Fig 2.11:Comparison of error for different scalar type of enrichment radius for polynomial degree 1

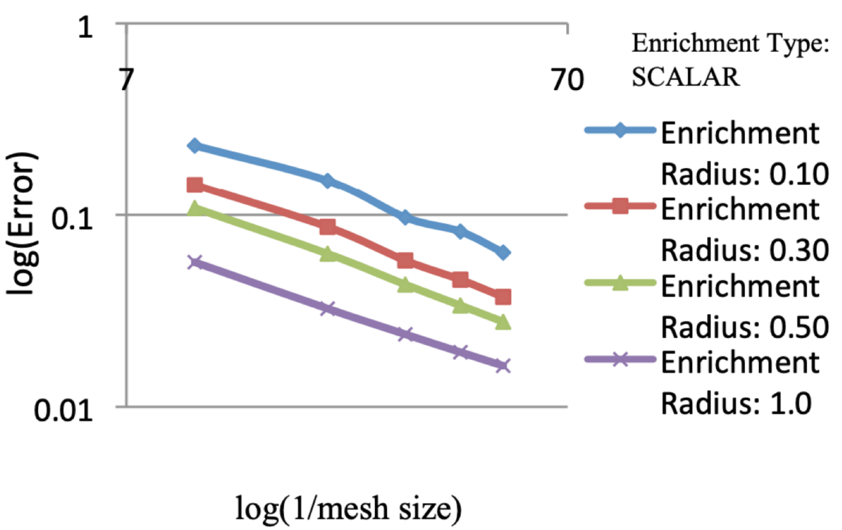
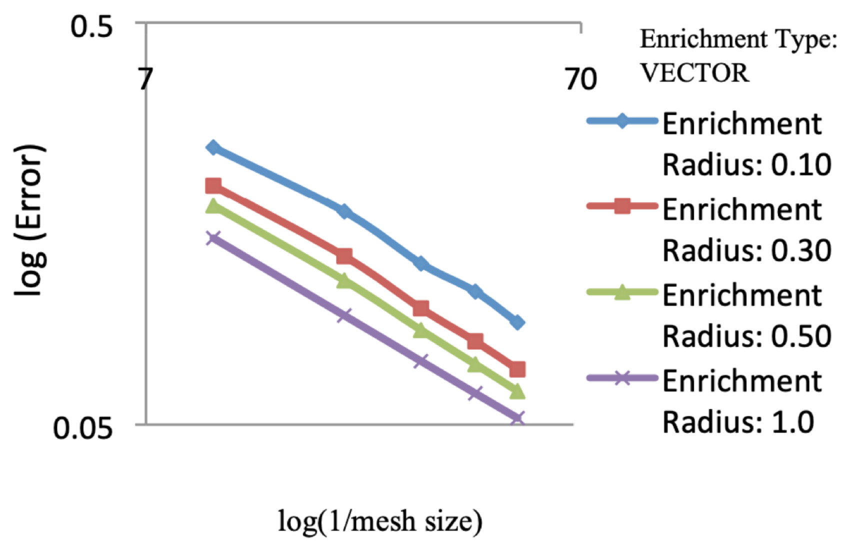
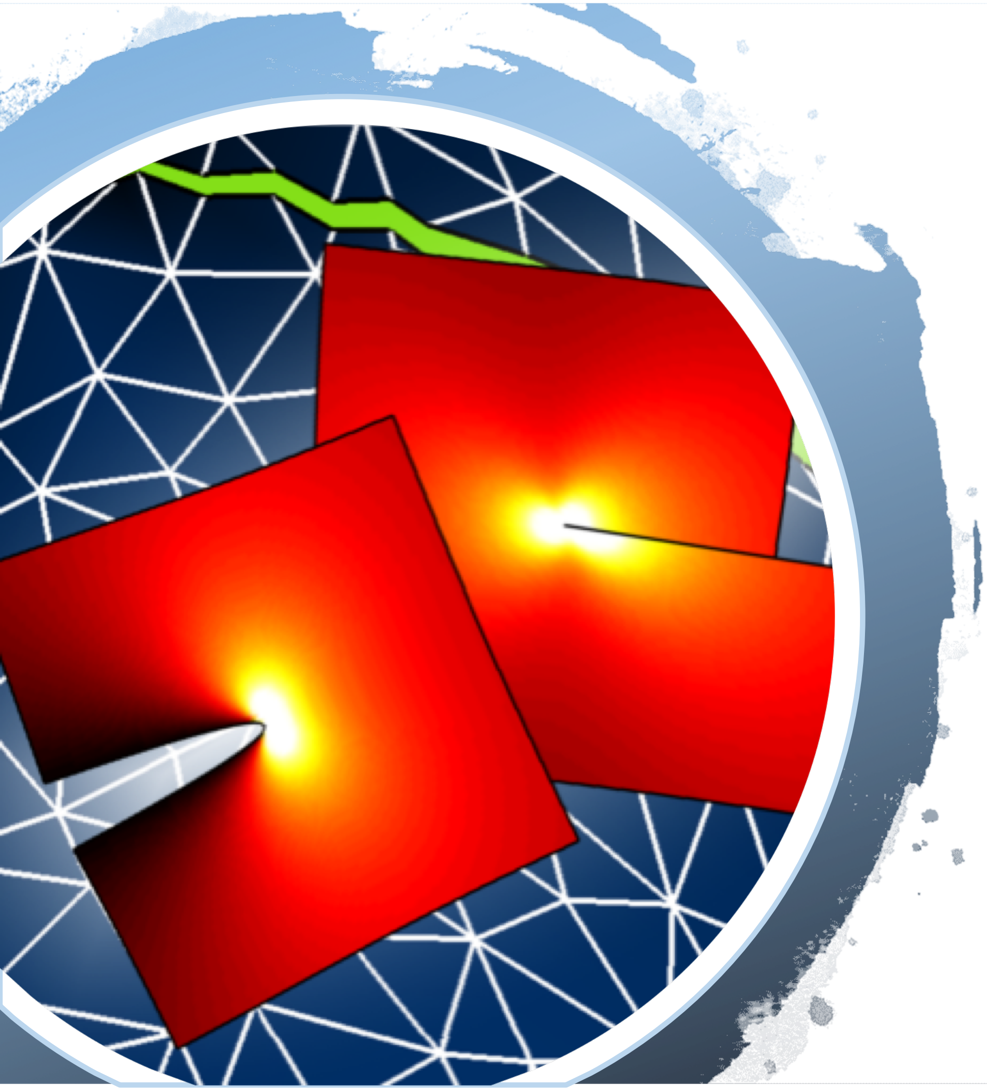


Fig 2.12: Comparison of error for different vector type of enrichment radius for polynomial degree 1

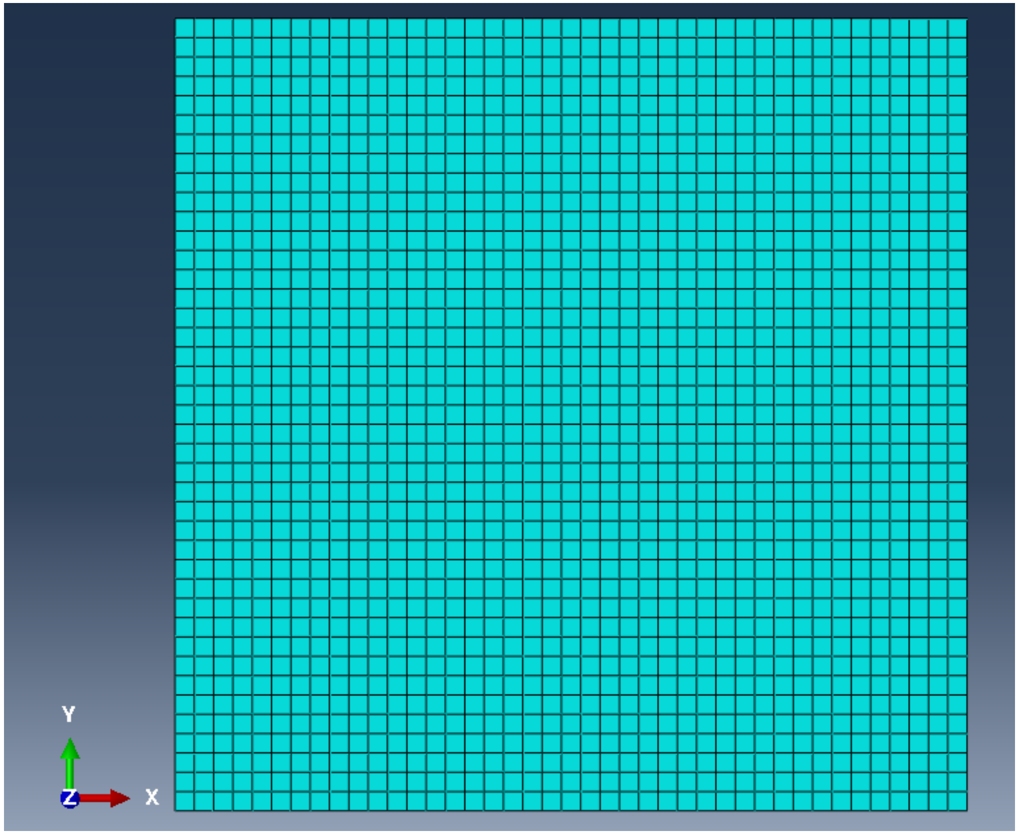


Source: Dibakar Datta



Numerical Example

Numerical Example



2D Static Edge Crack

Un-cracked Domain:

- 2D Planar,
- $\begin{bmatrix} (-2,2) & (2,2) \\ (-2,-2) & (2,-2) \end{bmatrix}$
- 41x41
- Aluminum
- E = 70 Gpa
- $\upsilon = 0.33$
- Max. Principle Stress: 500 Mpa

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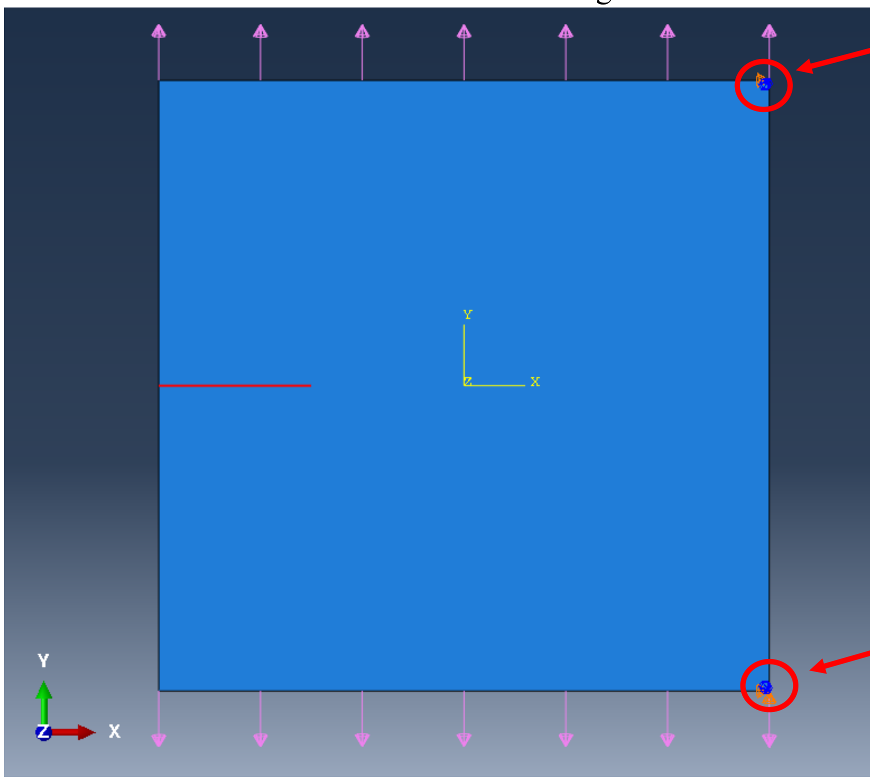
Numerical Example

Uniform Pressure loading



Uniform Pressure loading

Roller B.C.:
$u_1 = u_3 = 0$

2D Static Edge Crack

Cracked Domain:

- 2D Planar with wire
- $[(-2,0) \quad (-1,0)]$ for the crack

Fixed B.C.:
$u_1 = u_2 = u_3 = 0$

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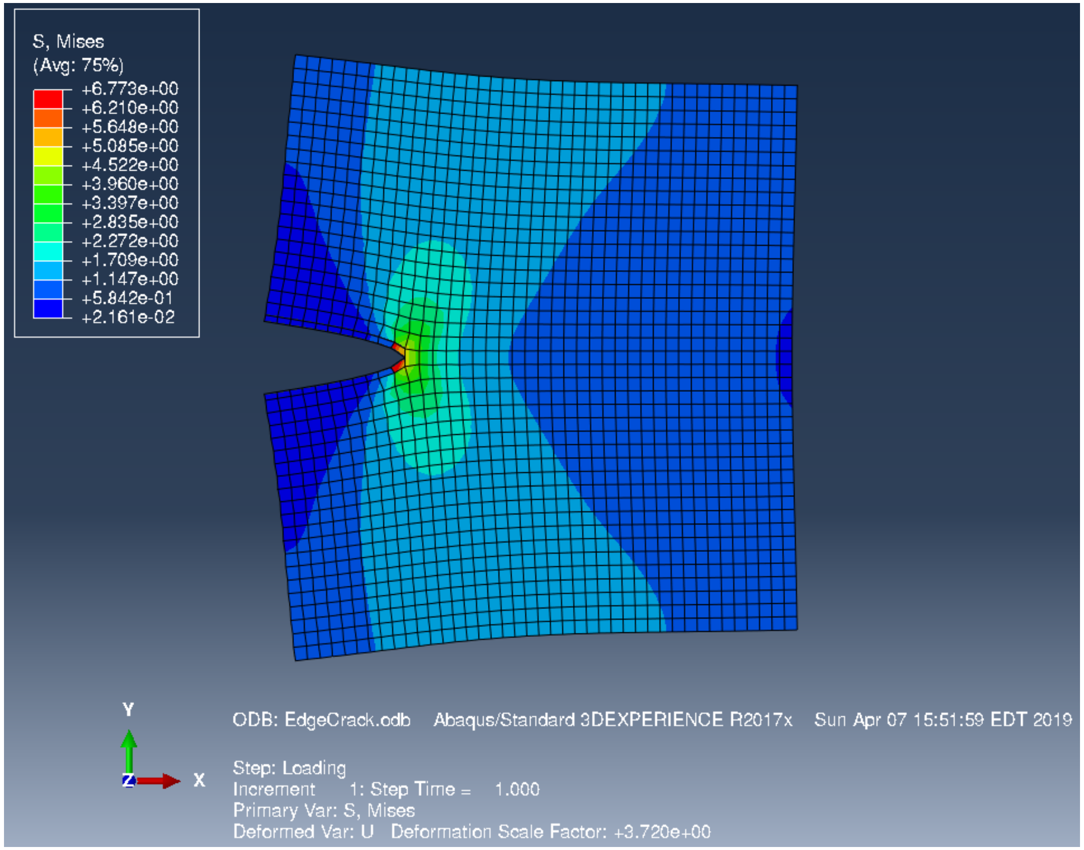
Numerical Example



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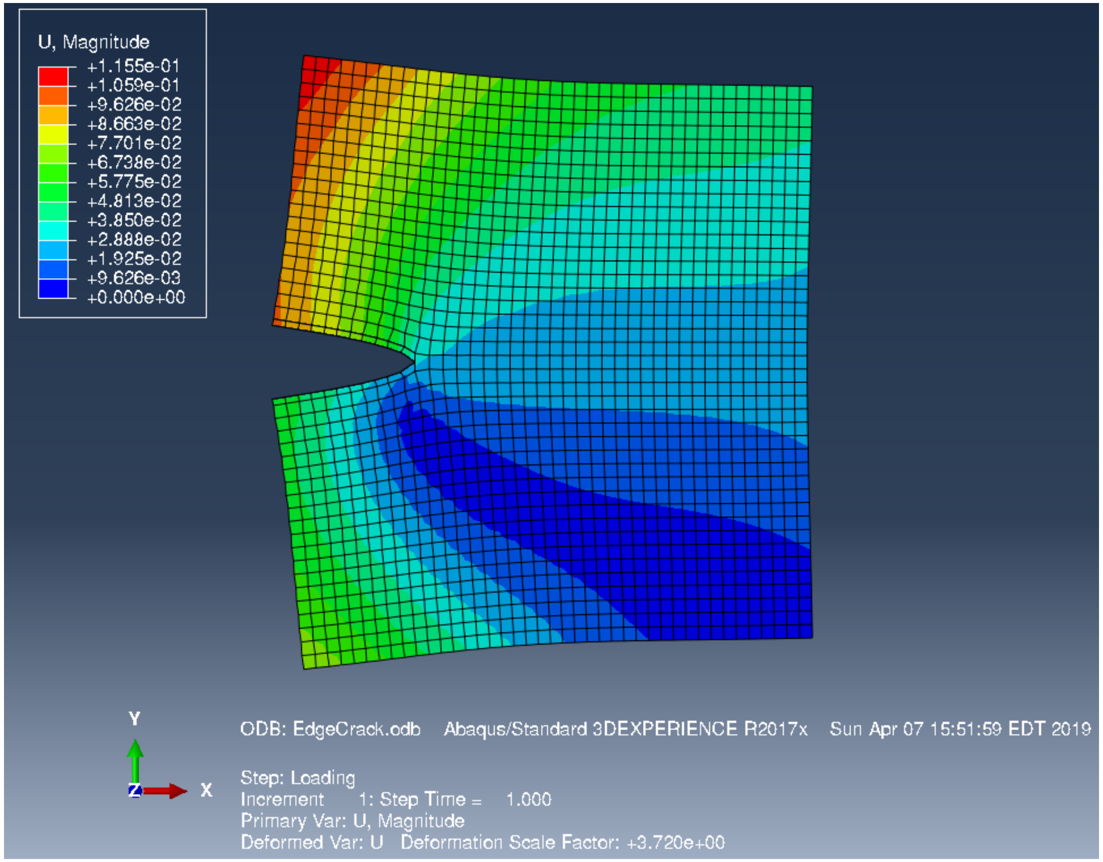
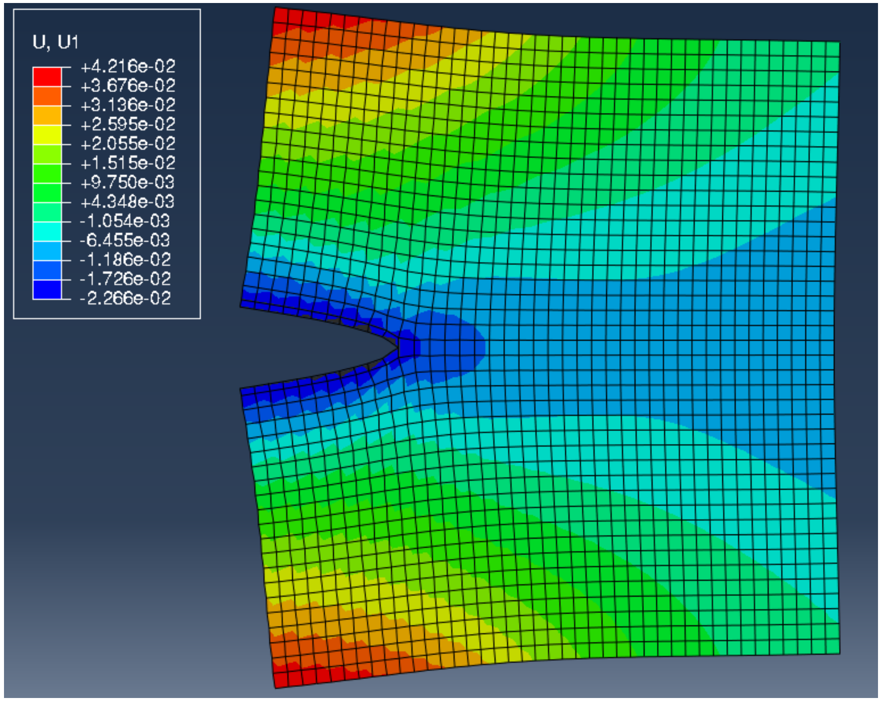
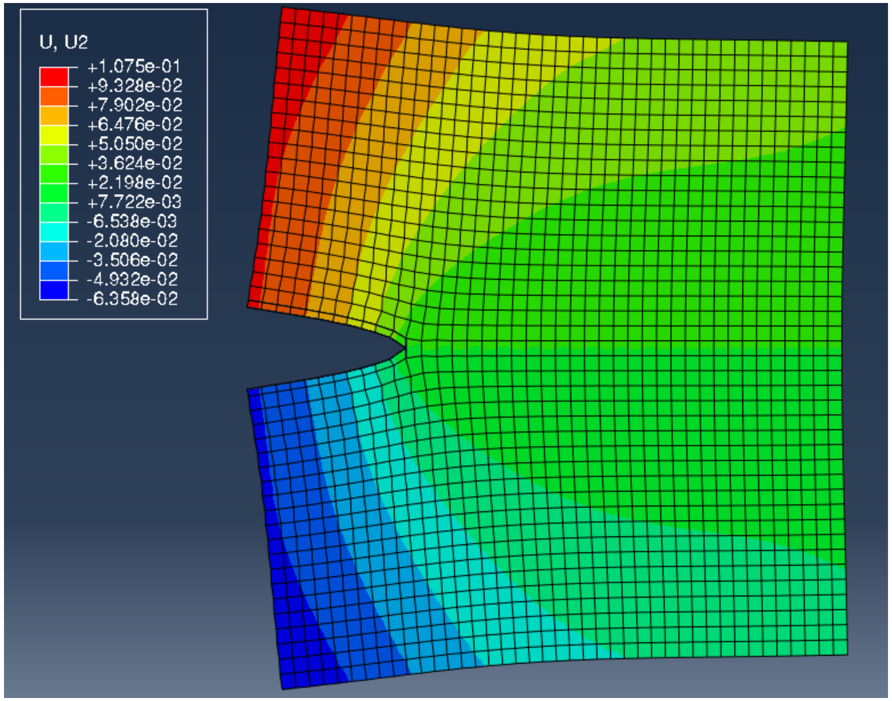
Numerical Example

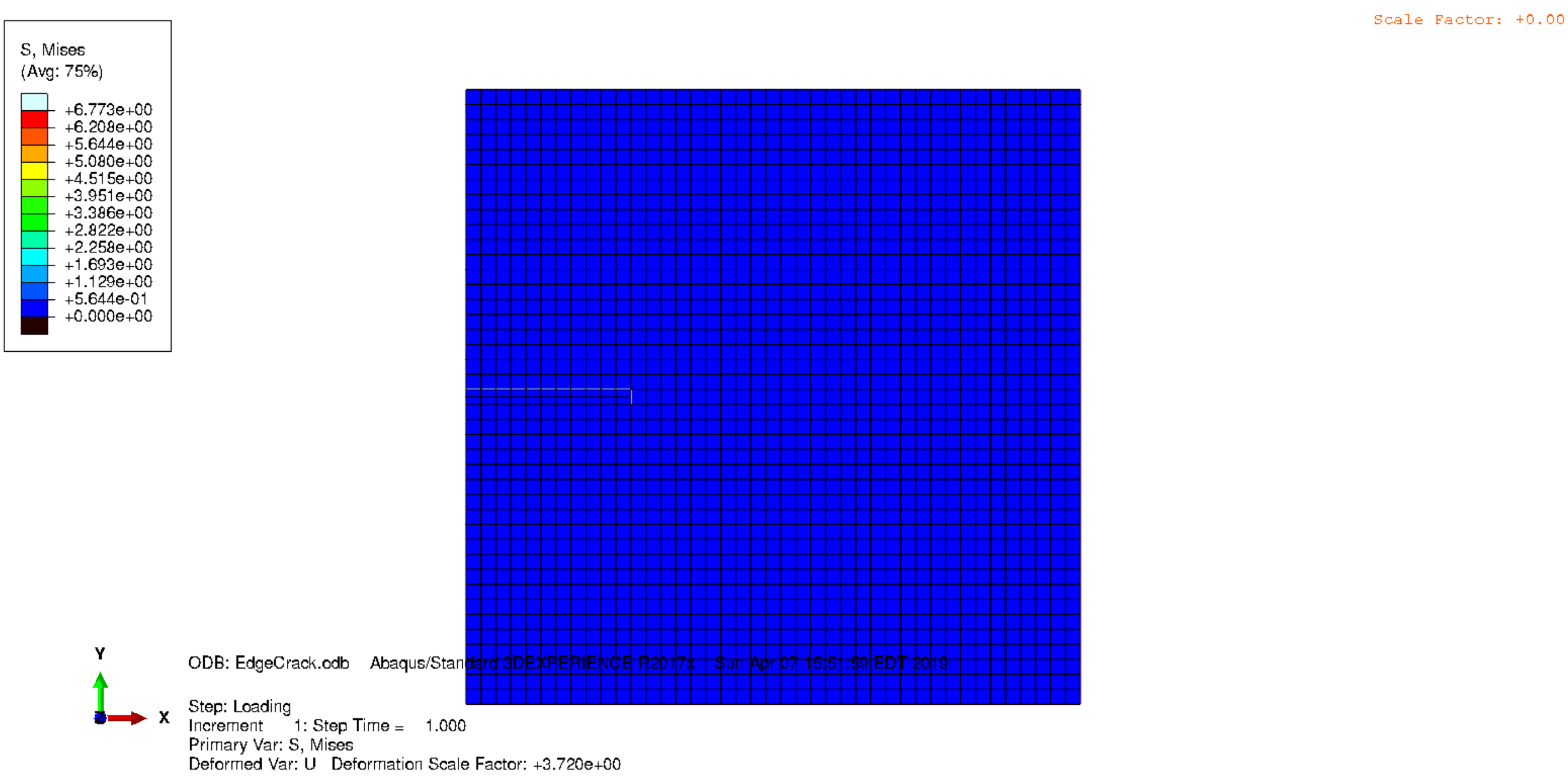


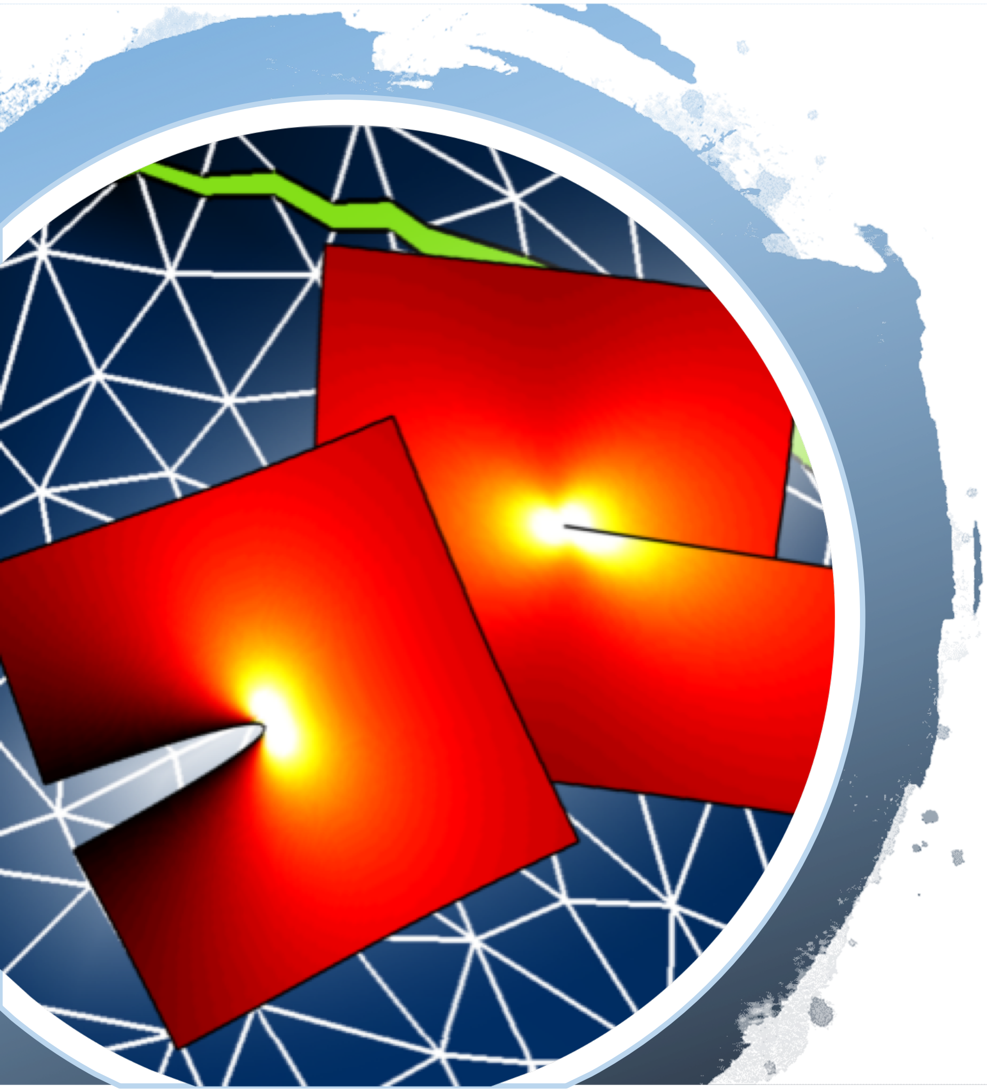
Numerical Example





Numerical Example

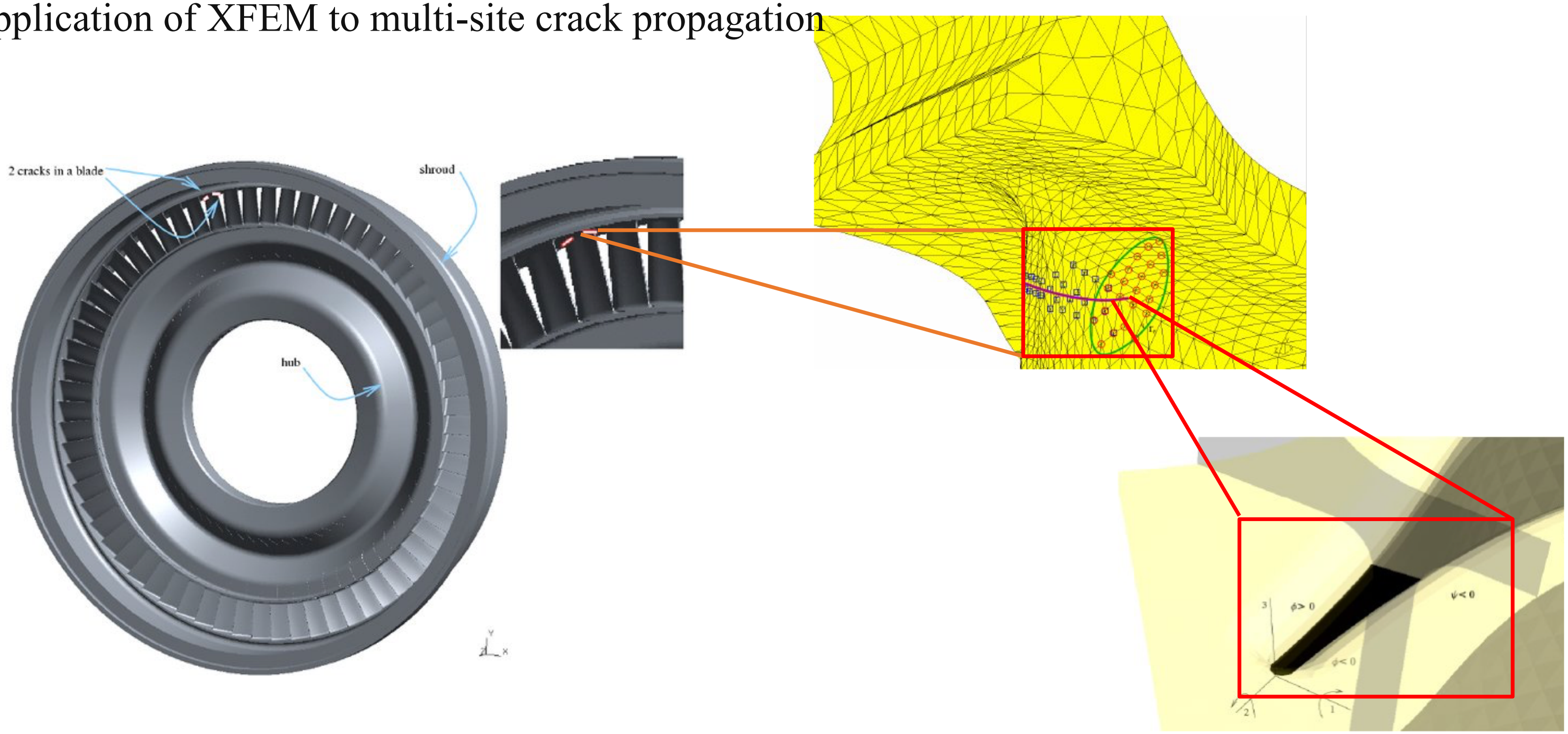




Example Applications

Example Applications

Application of XFEM to multi-site crack propagation

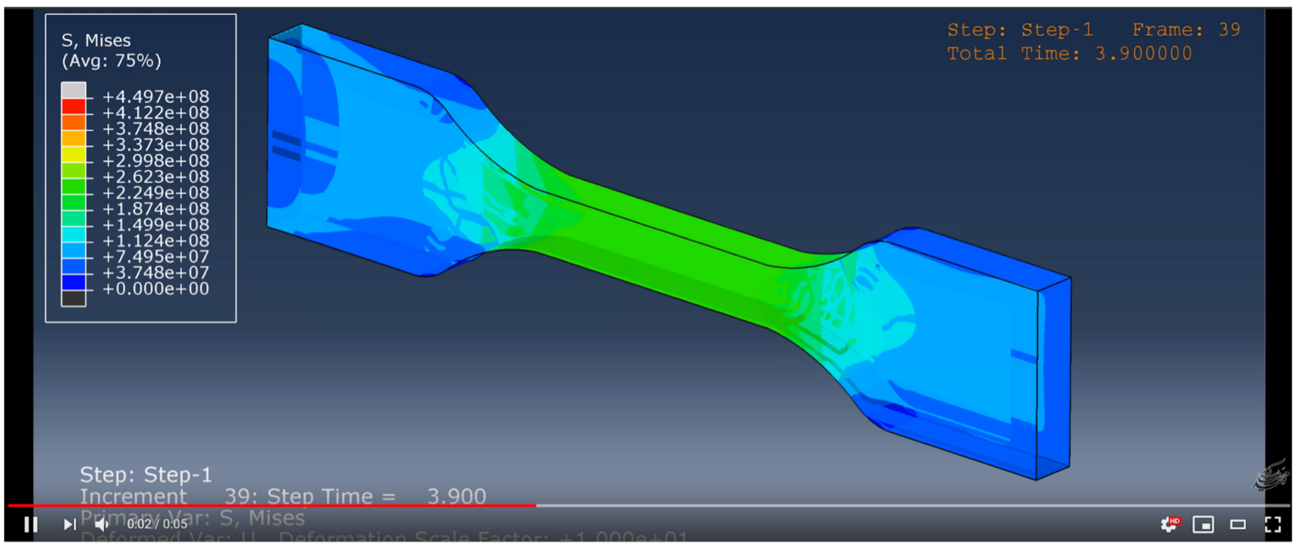


Duflot, Marc, et al. "Application of XFEM to multi-site crack propagation." Engng Fract Mech, submitted for publication(2008).

Example Applications

Abaqus XFEM simulation for tensile test

https://www.youtube.com/watch?v=QJws0SaGdII



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Thank you!