Boundary Element Methods

Judit Gonzalez-Santana
Machel Higgins
Kirsten Stephens

EGEE 520 – Spring 2018 Final Project

Introduction

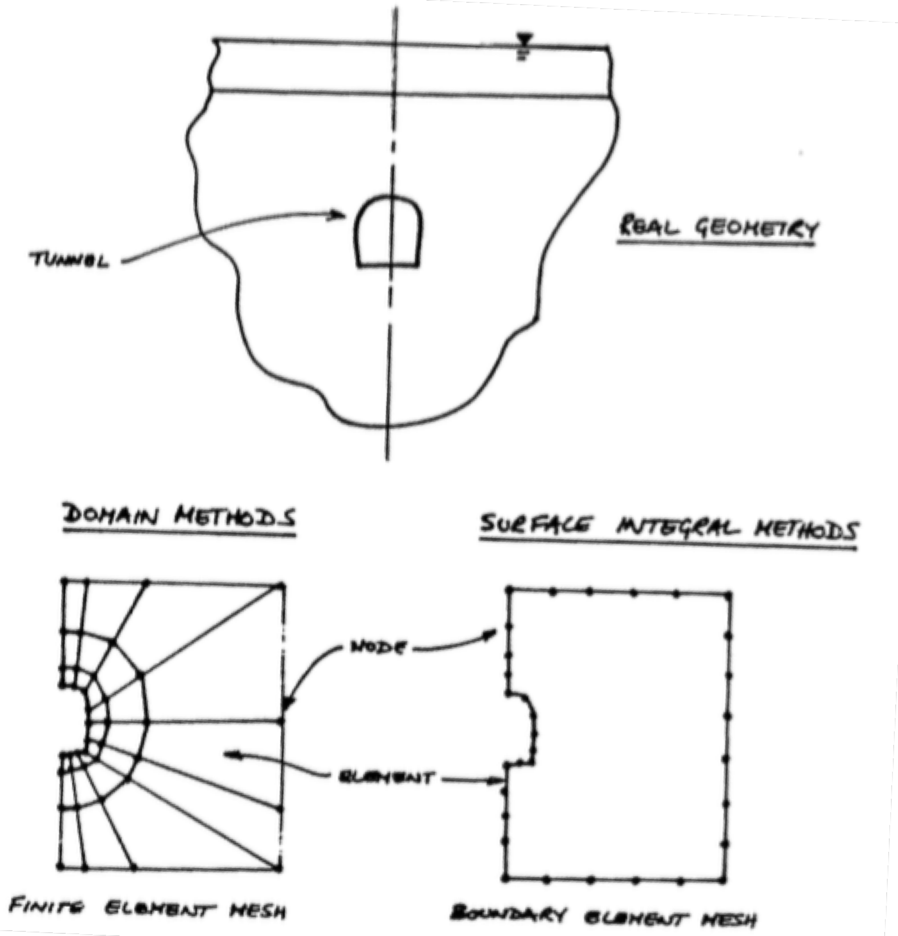
Introduction

- BEM is a numerical method that solves PDEs by transforming them into boundary integral equations (BIE)

Introduction

Boundary Element Method (BEM)	Finite Element Method (FEM)
Boundary solution method Infinite, semi-infinite	Domain method finite
Solves integral equations (BIEs)	Solves differential equations (PDEs)
Mesh boundaries only	Mesh entire domain
Small, filled-in matrix $Vh = Hv$	Large, sparse matrix $Ku = F$
Homogeneous, linear problems	Heterogeneous, nonlinear problems

LaForce (2006), Standford PE281 notes


TUNNEL
REAL GEOMETRY
DOMAIN METHODS
SURFACE INTEGRAL METHODS
NODE
ELEMENT
FINITE ELEMENT MESH
BOUNDARY ELEMENT MESH

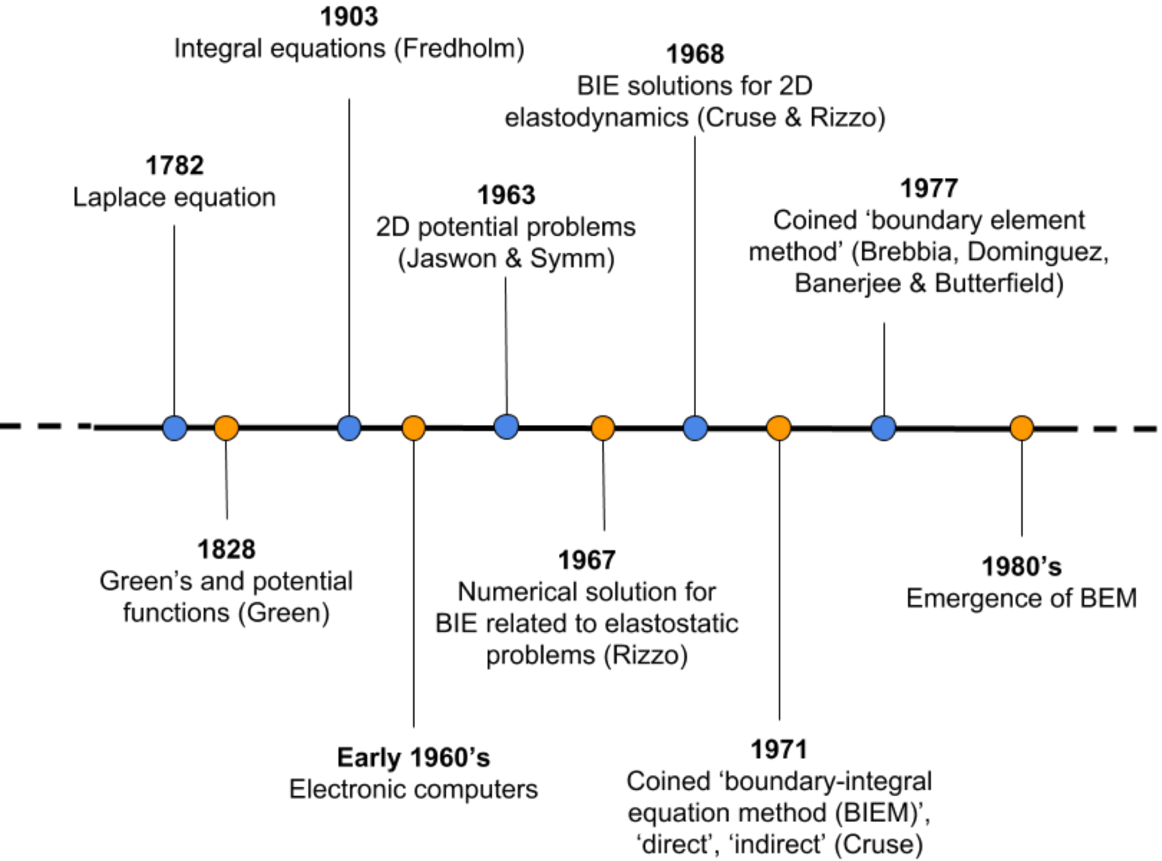
Introduction

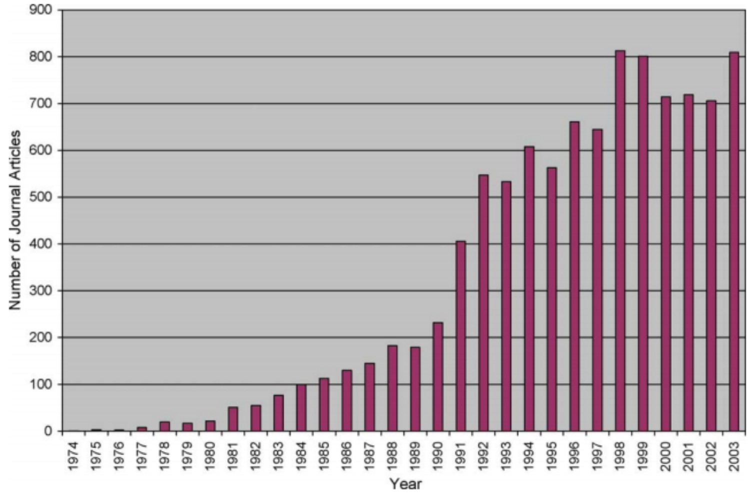
- Advantages
  - Only the boundaries have to be discretized
    - Reduces complexity and dimensions → reduces computational time
  - Good for solving linear problems, stress concentration problems involving incompressible materials, steady state

- Disadvantages
  - Halfspace has to be homogeneous
  - Not efficient for non-linear and/or transient problems
  - Requires explicit knowledge of a fundamental solution of the PDE

Historical Perspectives

Historical Perspectives





Cheng & Cheng (2005) EABE

General Principles & Equations

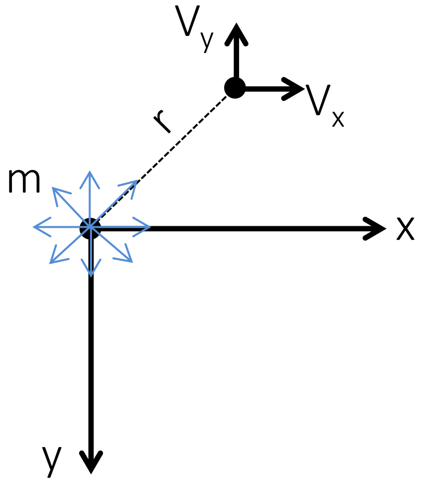
Derivation of potential from Laplace equation

- Laplace equation: $\nabla^2 u=0$
- In 2D:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

- Fundamental solution for $-\infty < x < \infty, -\infty < y < \infty$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\delta(\xi - x, \eta - y)=0$$



Solving for potential

- Laplace from Cartesian into radial co-ords:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

- Boundary conditions:

$$\delta \neq 0 \; @ \; r = 0$$
$$u \to \infty \; as \; r \to 0$$

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$$

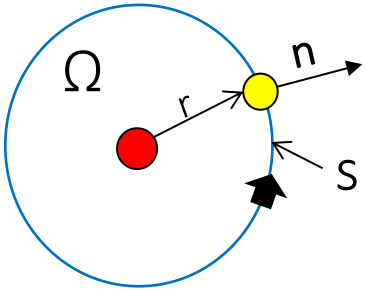
- Since in an infinite domain, $\frac{\partial^2 u}{\partial \theta^2} = 0$

- Applying those conditions, Laplace equation becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

- Solution for this equation: $u = A \ln(r) + B$

Solving for potential



- Integrating original Laplace equation (in Cartesian co-ords)

$$\int_{\Omega} \nabla^2 u \, d\Omega + \int_{\Omega} \delta(\xi - x, \eta - y) \, d\Omega = 0$$

- Applying Green-Gauss Theorem, and assuming circle of radius ε>0 @ r=0:

$$\int_{\Omega} \nabla^2 u \, d\Omega = \int_{d\Omega} \frac{\partial u}{\partial n} \, ds$$

- Equating *dn=dr* and since *u = Aln(r) + B*,

$$\int_0^{2\pi\varepsilon} \frac{A}{\varepsilon} \, ds => A = \frac{1}{2\pi}$$

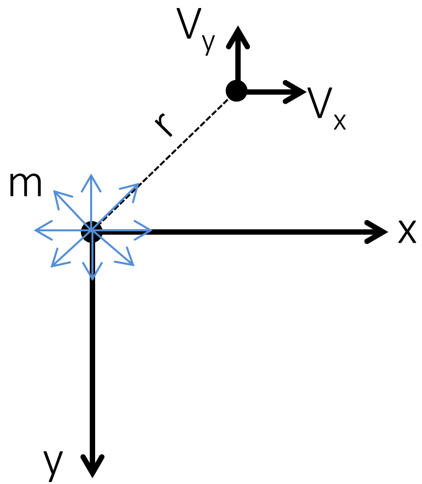
- Setting *B=0* for convenience, we get the equation for the potential:

$$u = \frac{1}{2\pi} ln(r)$$

Fluid flow problems

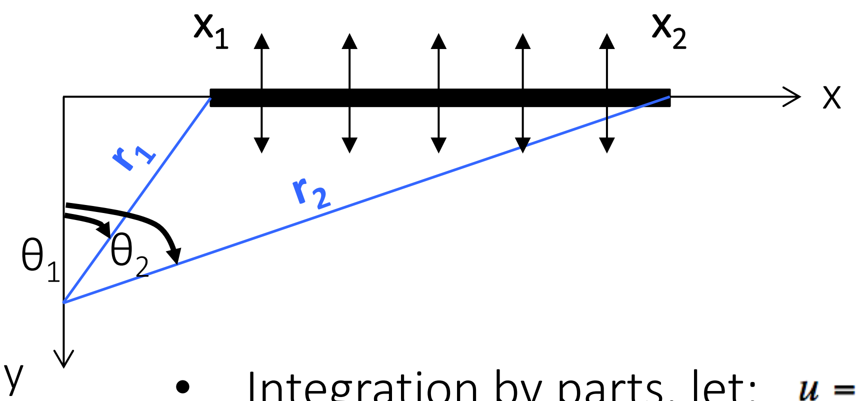
- For flow problems, *u=h*, and unit source strength is indicated as *m*

$$h = \frac{m}{2\pi} ln(r)$$



- Velocity in x-direction: $v_x = -K\frac{\partial h}{\partial x} = -K\frac{m}{2\pi}\left(\frac{x}{r^2}\right)$
- Velocity in y-direction: $v_y = -K\frac{\partial h}{\partial y} = -K\frac{m}{2\pi}\left(\frac{y}{r^2}\right)$

Evaluating head along line segment



- Integrate *h* with respect to *x* (polar to Cartesian):

$$\oint \frac{m}{2\pi} ln\left(\sqrt{x^2 + y^2}\right)dx \rightarrow \frac{m}{4\pi} \oint ln\left(x^2 + y^2\right)dx$$

- Integration by parts, let: $u = x^2 + y^2,\ \frac{dv}{dx} = 1 \rightarrow \frac{du}{dx} = \frac{2x}{x^2 + y^2},\ v = x$

$$\rightarrow \oint \frac{m}{2\pi} ln\left(\sqrt{x^2 + y^2}\right)dx = \frac{m}{4\pi}\left[xln(x^2 + y^2) - \int \frac{2x^2}{x^2 + y^2}dx\right]$$

$$\rightarrow \frac{m}{4\pi}\left(x\,ln(x^2 + y^2) - \left[-2y\,tan^{-1}\left(\frac{x}{y}\right) + 2x\right]\right)$$

$$h = \frac{m}{2\pi}\left(x\,ln(r) + y\theta - x\right)$$

Evaluating velocity along line segment

$$h = \frac{m}{2\pi}\left(x\,ln(r) + y\theta - x\right)$$

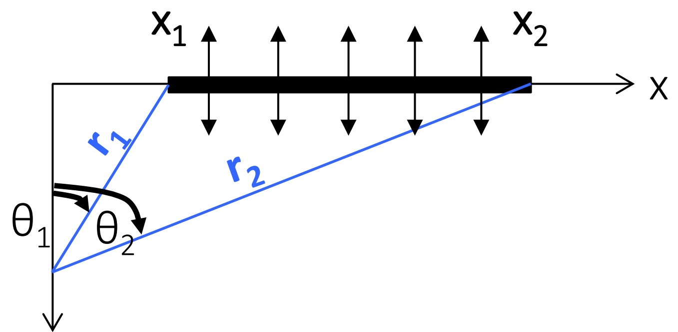
- Similarly to before, we can calculate the velocities in the x-direction and y-direction at ea

$$v_x = -K\frac{\partial h}{\partial x} = -K\frac{m}{2\pi}ln(r)$$

$$v_y = -K\frac{\partial h}{\partial y} = -K\frac{m}{2\pi}\theta$$

Hand-Calculation Example

Systems of equations:

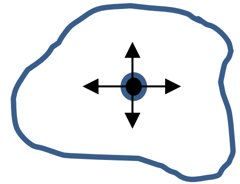


Solving

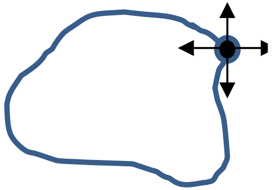
$$h = \frac{m}{2\pi}\left[x\ln(r) - x + y\theta\right]_1^2$$

$$v_x = -K\frac{\partial h}{\partial x} = -K\frac{m}{2\pi}\left[\ln(r)\right]_1^2$$

$$v_y = -K\frac{\partial h}{\partial y} = -K\frac{m}{2\pi}\left[\theta\right]_1^2$$



$$c_{ij}(p) = \delta_{ij} = 1$$



$$c_{ij}(p) = \frac{1}{2}\delta_{ij} = \frac{1}{2}$$

$$c_{ij}(p)h_j(p) + \oint V_{ij}(p,q)h_{ij}(q)\ dr = \oint H_{ij}(p,q)\vec{v}_j(q)\bullet\hat{n}\ dr$$

$$\mathbf{V\ h = H\ v}$$

- $\oint V_{ij}$ = integrated effect of a unit source at element i on the resulting normal flux at boundary element j.
- $\oint H_{ij}$ = integrated effect of a unit source at element i on the resulting head at boundary element j.
- c_{ij} = free term due to bringing the source to the boundary

Direct vs Indirect method:

Direct – solves the system of equations directly:

$$\mathbf{V}\ \mathbf{h} = \mathbf{H}\ \mathbf{v}$$

Indirect – has an intermediate step whereby the source strength (m) required at each point to reproduce the known BCs (b) is calculated. These source strengths are then applied to calculate the unknown BCs (d).

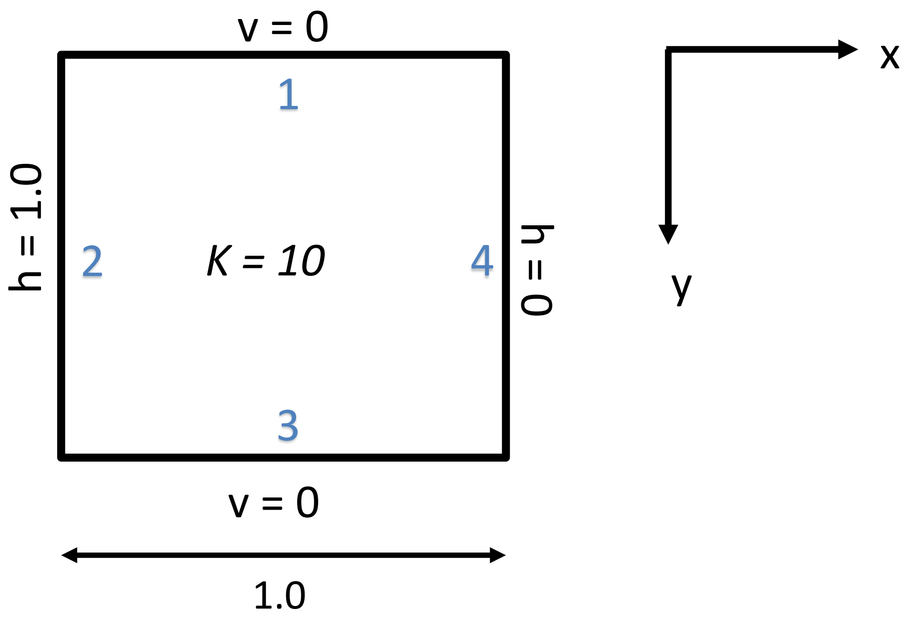
$$b = A_1 m$$

$$m = A_1^{-1} b$$

$$d = A_2 m$$

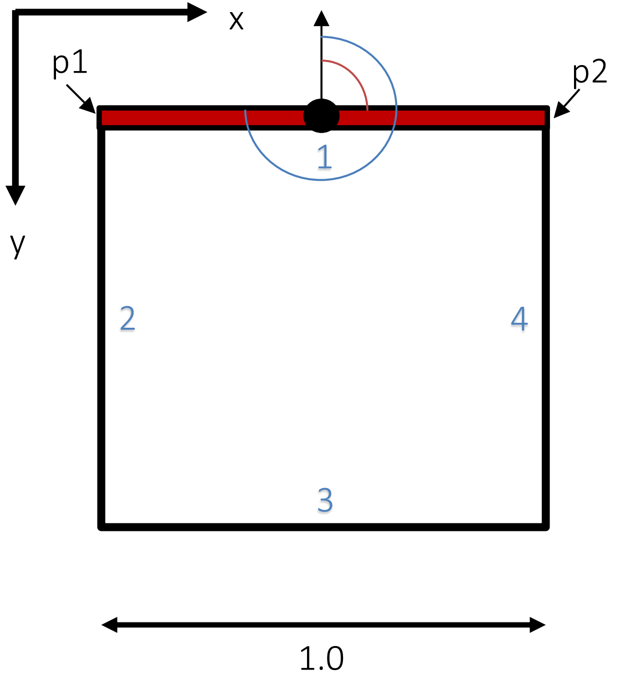
Where A_1 is a matrix of influence coefficients for the known BCs (b) and A_2 is that for the unknown BCs (d), ie. they are combinations of the terms in V and H.

Example:



Obtaining the H_{ij} values:

$$h = \frac{m}{2\pi}\left[x\ln(r) - x + y\theta\right]_1^2$$



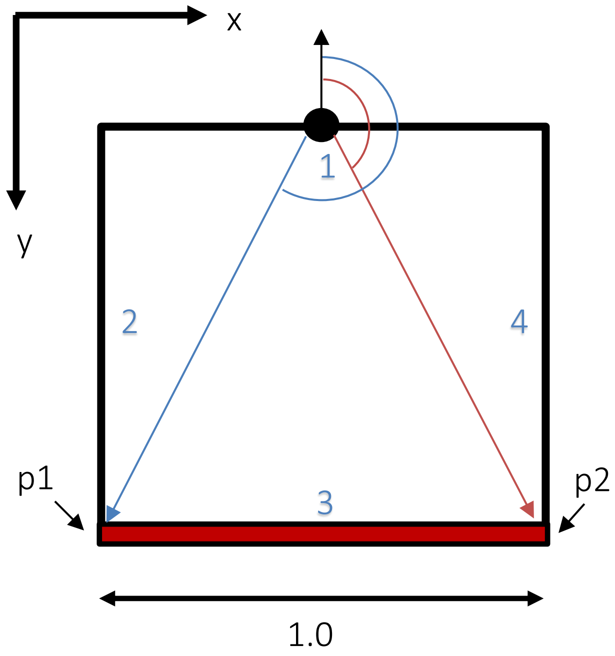
Eg. Effect of a unit source on node 1 on element 1:

- At p2:
  - x = 0.5
  - y = 0
  - r = 0.5
  - $\theta = \frac{\pi}{2}$
- At p1:
  - x = -0.5
  - y = 0
  - r = 0.5
  - $\theta = \frac{3\pi}{2}$

$$h_{11} = \frac{1}{2\pi}\left[(0.5\ln(0.5) - 0.5) - (-0.5\ln(0.5) + 0.5)\right] = -0.269$$

Obtaining the H_{ij} values:

$$h = \frac{m}{2\pi}\left[x\ln(r) - x + y\theta\right]_1^2$$



Eg. Effect of a unit source on node 1 on element 3:

- At p2:
  - x = 0.5
  - y = -1
  - $r = \frac{\sqrt{5}}{2}$
  - $\theta = 1.11715$
- At p1:
  - x = -0.5
  - y = -1
  - $r = \frac{\sqrt{5}}{2}$
  - $\theta = 2.03444$

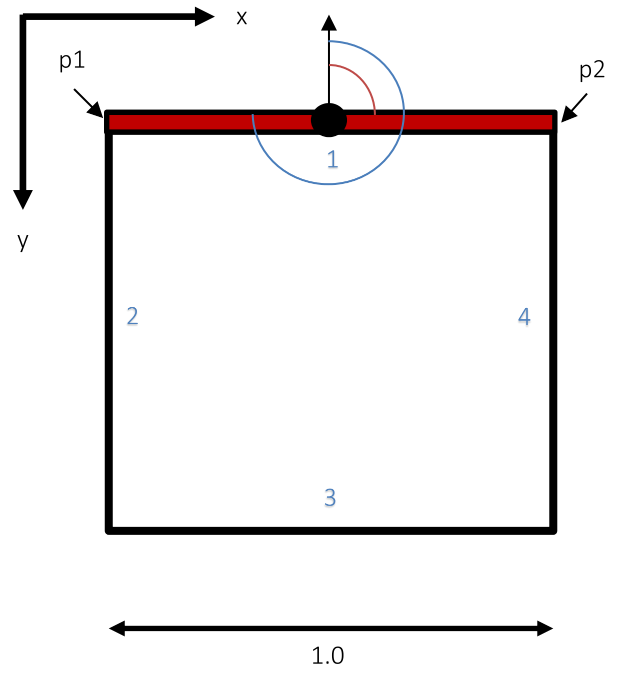
$$h_{31} = \frac{1}{2\pi}\left[\left(0.5\ln\left(\frac{\sqrt{5}}{2}\right) - 0.5 - 1.11715\right) - \left(-0.5\ln\left(\frac{\sqrt{5}}{2}\right) + 0.5 - 2.03444\right)\right]$$

h = 0.006185

H_{ij} matrix:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} = \begin{bmatrix} -0.270 & -0.053 & 0.006 & -0.053 \\ -0.053 & -0.270 & -0.053 & 0.006 \\ 0.006 & -0.053 & -0.270 & -0.053 \\ -0.053 & 0.006 & -0.053 & -0.270 \end{bmatrix}$$

Obtaining the V_{ij} values:



We are interested in the value of flow normal to the element:

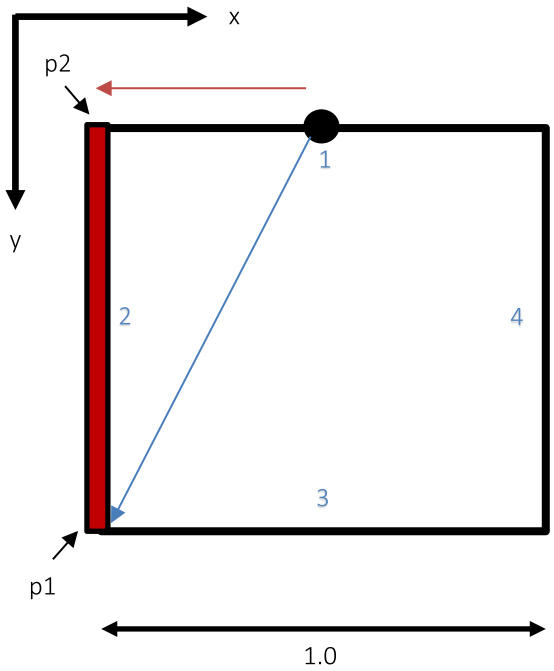
- V_x on elements 2 and 4
- V_y on elements 1 and 3

Eg. Unit source at 1 on element 1:

- At p2:
  - $\theta = \frac{\pi}{2}$
- At p1:
  - $\theta = \frac{3\pi}{2}$

$$v_{11} = -K\frac{m}{2\pi}\left[\theta\right]_1^2 = -\frac{10}{2\pi}\left[\frac{\pi}{2} - \frac{3\pi}{2}\right] = 5.000$$

Obtaining the V_{ij} values:



Eg. Unit source at 1 on element 2:

- At p2:
  - r = 0.5
- At p1:
  - $r = \frac{\sqrt{5}}{2}$

$$v_{12} = -K\frac{m}{2\pi}\left[ln(r)\right]_1^2 = -\frac{10}{2\pi}\left[ln(0.5) - ln\left(\frac{\sqrt{5}}{2}\right)\right] = -1.281$$

V_{ij} matrix:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} = \begin{bmatrix} 5.000 & -1.281 & -1.476 & -1.281 \\ -1.281 & 5.000 & -1.281 & -1.476 \\ -1.476 & -1.281 & 5.000 & -1.281 \\ -1.281 & -1.476 & -1.281 & 5.000 \end{bmatrix}$$

Direct method:

$$c_{ij}(p)h_j(p) + \oint V_{ij}(p,q)h_{ij}(q)\ dr = \oint H_{ij}(p,q)\vec{v_j}(q) \bullet \hat{n}\ dr$$

$$\mathbf{V}\ \mathbf{h} = \mathbf{H}\ \mathbf{v}$$

Where $\delta = c_{ij}(p)h_j(p) = 1/2$

$$\begin{bmatrix} v_{11}+\delta & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22}+\delta & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33}+\delta & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44}+\delta \end{bmatrix} \begin{bmatrix} h_1 \\ 1.000 \\ h_3 \\ 0.000 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} 0.000 \\ v_2 \\ 0.000 \\ v_4 \end{bmatrix}$$

Direct method:

$$\mathbf{V}\ \mathbf{h} = \mathbf{H}\ \mathbf{v}$$

Substituting in the pre-calculated values of v_{ij} and h_{ij}:

$$\begin{bmatrix} 5.500 & -1.281 & -1.476 & -1.281 \\ -1.281 & 5.500 & -1.281 & -1.476 \\ -1.476 & -1.281 & 5.500 & -1.281 \\ -1.281 & -1.476 & -1.281 & 5.500 \end{bmatrix} \begin{bmatrix} h_1 \\ 1.000 \\ h_3 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -0.270 & -0.053 & 0.006 & -0.053 \\ -0.053 & -0.270 & -0.053 & 0.006 \\ 0.006 & -0.053 & -0.270 & -0.053 \\ -0.053 & 0.006 & -0.053 & -0.270 \end{bmatrix} \begin{bmatrix} 0.000 \\ v_2 \\ 0.000 \\ v_4 \end{bmatrix}$$

Rearranging for known values on the right, unknowns on the left:

$$\begin{bmatrix} h_1 \\ v_2 \\ h_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 5.500 & 0.050 & -1.480 & 0.050 \\ -1.280 & 0.270 & -1.280 & -0.006 \\ -1.480 & 0.050 & 5.500 & 0.050 \\ -1.280 & -0.006 & -1.280 & 0.270 \end{bmatrix}^{-1} \begin{bmatrix} -0.270 & 1.280 & 0.006 & 1.280 \\ -0.050 & -5.500 & -0.050 & 1.480 \\ 0.006 & 1.280 & -0.270 & 1.280 \\ -0.050 & 1.480 & -0.050 & -5.500 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.409 \\ -16.284 \\ 0.409 \\ 8.987 \end{bmatrix}$$

Indirect method:

$$v_i = \sum_{j=1}^{M} V_{ij}.m_j$$

$$h_i = \sum_{j=1}^{M} H_{ij}.m_j$$

$$\longrightarrow \begin{bmatrix} v_1 \\ h_2 \\ v_3 \\ h_4 \end{bmatrix} = A_1 \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} \longrightarrow \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = A_1^{-1} \begin{bmatrix} v_1 \\ h_2 \\ v_3 \\ h_4 \end{bmatrix} \longrightarrow \begin{bmatrix} h_1 \\ v_2 \\ h_3 \\ v_4 \end{bmatrix} = A_2 \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix} \qquad A_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix}$$

Indirect method:

$$b = A_1 m$$

$$\begin{bmatrix} v_1 \\ h_2 \\ v_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -1.28 & -1.48 & -1.28 \\ -0.05 & -0.27 & -0.05 & 0.0062 \\ -1.48 & -1.28 & 5 & -1.28 \\ -0.05 & 0.0062 & -0.05 & -0.27 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

$$m = A_1^{-1} b$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 5 & -1.28 & -1.48 & -1.28 \\ -0.05 & -0.27 & -0.05 & 0.0062 \\ -1.48 & -1.28 & 5 & -1.28 \\ -0.05 & 0.0062 & -0.05 & -0.27 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.06 \\ -3.27 \\ -1.06 \\ 0.35 \end{bmatrix}$$

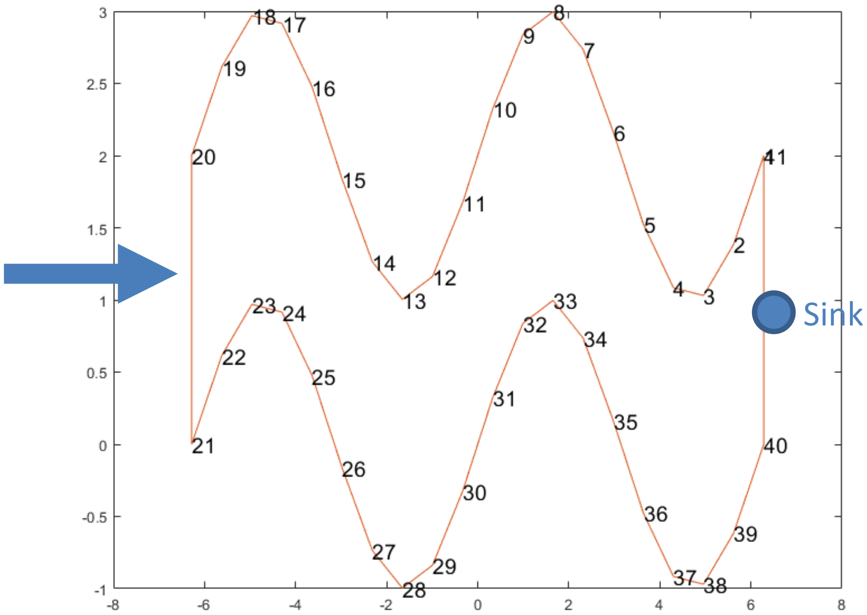
$$d = A_2 m$$

$$\begin{bmatrix} h_1 \\ v_2 \\ h_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -0.27 & -0.05 & 0.0062 & -0.05 \\ -1.28 & 5 & -1.28 & -1.48 \\ 0.0062 & -0.05 & -0.27 & -0.05 \\ -1.28 & -1.48 & -1.28 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -1.06 \\ -3.27 \\ -1.06 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.437 \\ -14.17 \\ 0.437 \\ 9.296 \end{bmatrix}$$

Numerical Example

BEM Numerical Solutions

Snaking Pipe Suspected to Have a Hole



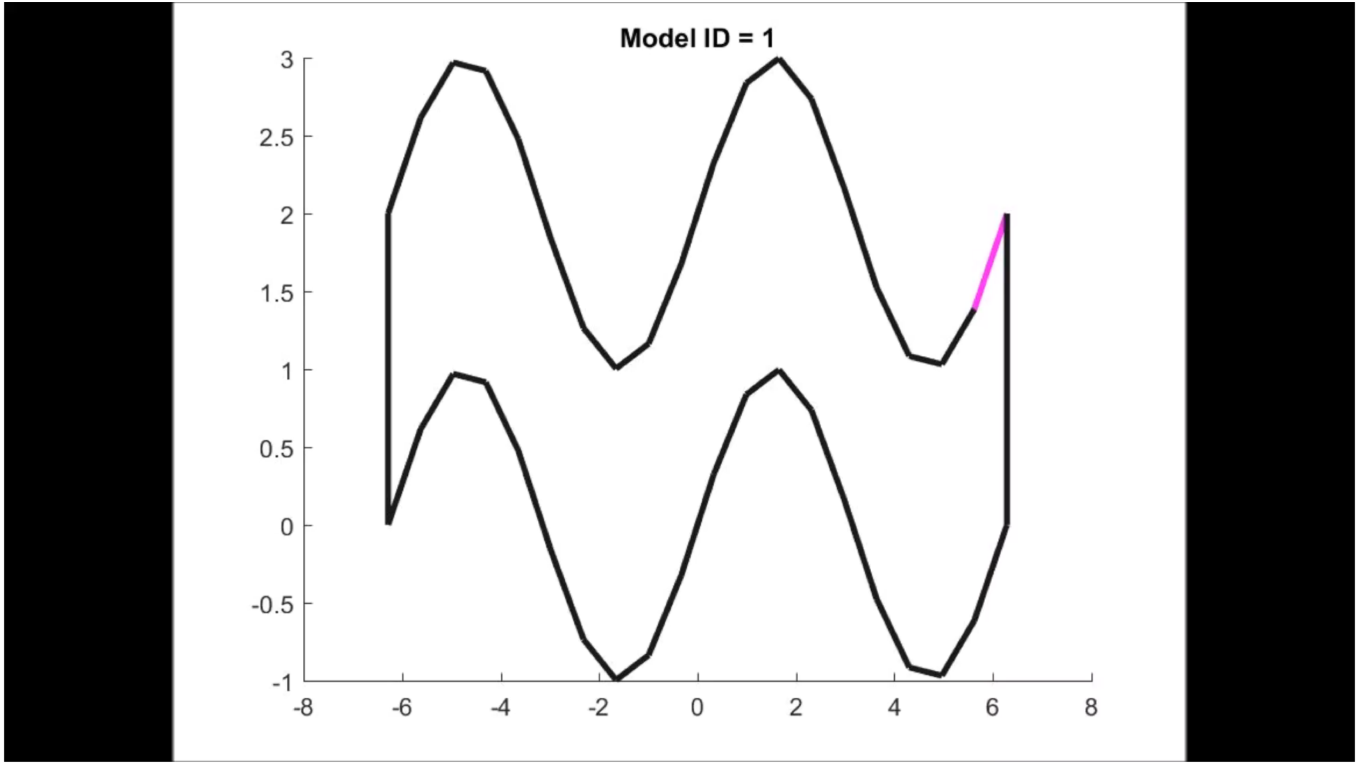
Problem: Given that one should know the velocity out the right end of the pipe when the pipe is intact, if the velocity were to change could we numerically figure out where a hole in the pipe could be?

Method: Use Prof Elsworth's General Direct Boundary Element Routine for Potential Flow code

- 40 Nodes/ 20 Elements
- Boundary Conditions:
  - Head at right-hand-side = 1 (source)
  - Head at left-hand-side = 0 (sink)
  - Sides of 2D pipe should be impermeable, i.e. have velocity = 0 except at hole.
  - One element other than rightmost or leftmost element will be another sink (i.e. h = 0)

BEM Numerical Solutions

Model Setup



BEM Numerical Solutions

Model Setup

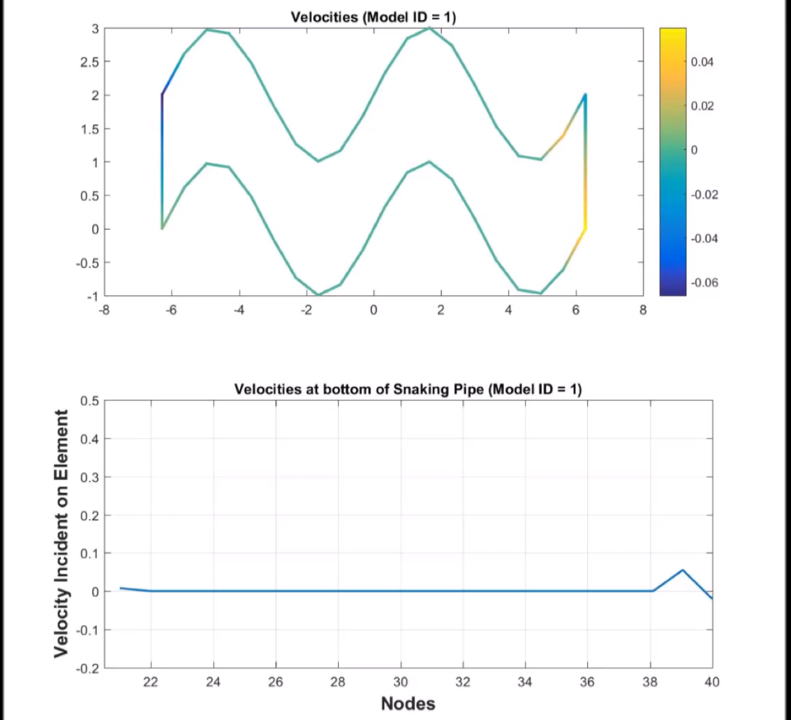
```
C.......................................................................
C
C        GENERAL DIRECT BOUNDARY ELEMENT ROUTINE FOR POTENTIAL FLOW.
C
C
C        D. ELSWORTH, UNIVERSITY OF TORONTO. MAY 1985 VERSION.
C
C        REVISIONS AS OF MARCH 1990
C
C.......................................................................
C
C        INPUT FILES MODIFIED FOR PC SEE PROGRAM BEM
C
C.......................................................................
C
C        INPUT DATA:
C
C        CARD  1 (20A4)  HEADING
C
C        CARD  2 (3I5,E15.8)      CONTROL DATA
C                NUMNP - NUMBER OF NODAL POINTS
C                NUMEL - NUMBER OF ELEMENTS
C                MDX   - MODE OF EXECUTION
C                                1 = GENERAL BEM
C                                2 = SUPERELEMENT BEM
C                HYDC  - ISOTROPIC HYDRAULIC CONDUCTIVITY
C
C        CARD  3 TO (NUMNP+2)    NODAL INPUT DATA  (I5,2F10.0,I5,F10.0,I5)
C                NODE  - GLOBAL NODE NUMBER
C                X-X   - X COORDINATE FO NODE
C                Y-Y   - Y COORDINATE OF NODE
C                IBC   - INTEGER BOUNDARY CONDITION
C                                1 = NODAL POTENTIAL SPECIFIED
C                                2 = NODAL VELOCITY SPECIFIED
C                RBC   - REAL BOUNDARY CONDITION (MAGNITUDE)
C                IGLOB - GLOBAL NUMBER OF NODE FOR SUPERELEMENT IF REQUIRED
C
C        CARD (NUMNP+3) TO (NUMNP+2+NUMEL)       ELEMENT INPUT DATA (5I5)
C                NEL   - ELEMENT NUMBER
C                II    - GLOBAL NODE NO. OF LOCAL NODE II
C                JJ    - GLOBAL NODE NO. OF LOCAL NODE JJ
C                KK    - GLOBAL NODE NO. OF LOCAL NODE KK
C                ELTYP - ELEMENT TYPE CODE
C                                0 = BOUNDARY ELEMENT
```

```
Snaking Pipe w Holes
   40   20    1  1.0
    1     6.2832     2.0000     1   0.0
    2     5.6218     1.3858     1   0.0
    3     4.9604     1.0306     2   0.0
    4     4.2990     1.0842     2   0.0
    5     3.6376     1.5241     2   0.0
    6     2.9762     2.1646     2   0.0
    7     2.3149     2.7357     2   0.0
    8     1.6535     2.9966     2   0.0
    9     0.9921     2.8372     2   0.0
   10     0.3307     2.3247     2   0.0
   11    -0.3307     1.6753     2   0.0
   12    -0.9921     1.1628     2   0.0
   13    -1.6535     1.0034     2   0.0
   14    -2.3149     1.2643     2   0.0
   15    -2.9762     1.8354     2   0.0
   16    -3.6376     2.4759     2   0.0
   17    -4.2990     2.9158     2   0.0
   18    -4.9604     2.9694     2   0.0
   19    -5.6218     2.6142     2   0.0
   20    -6.2832     2.0000     1   1.0
   21    -6.2832     0.0000     1   1.0
   22    -5.6218     0.6142     2   0.0
   23    -4.9604     0.9694     2   0.0
   24    -4.2990     0.9158     2   0.0
   25    -3.6376     0.4759     2   0.0
   26    -2.9762    -0.1646     2   0.0
   27    -2.3149    -0.7357     2   0.0
   28    -1.6535    -0.9966     2   0.0
   29    -0.9921    -0.8372     2   0.0
   30    -0.3307    -0.3247     2   0.0
   31     0.3307     0.3247     2   0.0
   32     0.9921     0.8372     2   0.0
   33     1.6535     0.9966     2   0.0
   34     2.3149     0.7357     2   0.0
   35     2.9762     0.1646     2   0.0
   36     3.6376    -0.4759     2   0.0
   37     4.2990    -0.9158     2   0.0
   38     4.9604    -0.9694     2   0.0
   39     5.6218    -0.6142     2   0.0
   40     6.2832    -0.0000     1   0.0
    1    3    1    2    0
    2    5    3    4    0
```

Matlab formats input and plots output.

BEM Numerical Solutions

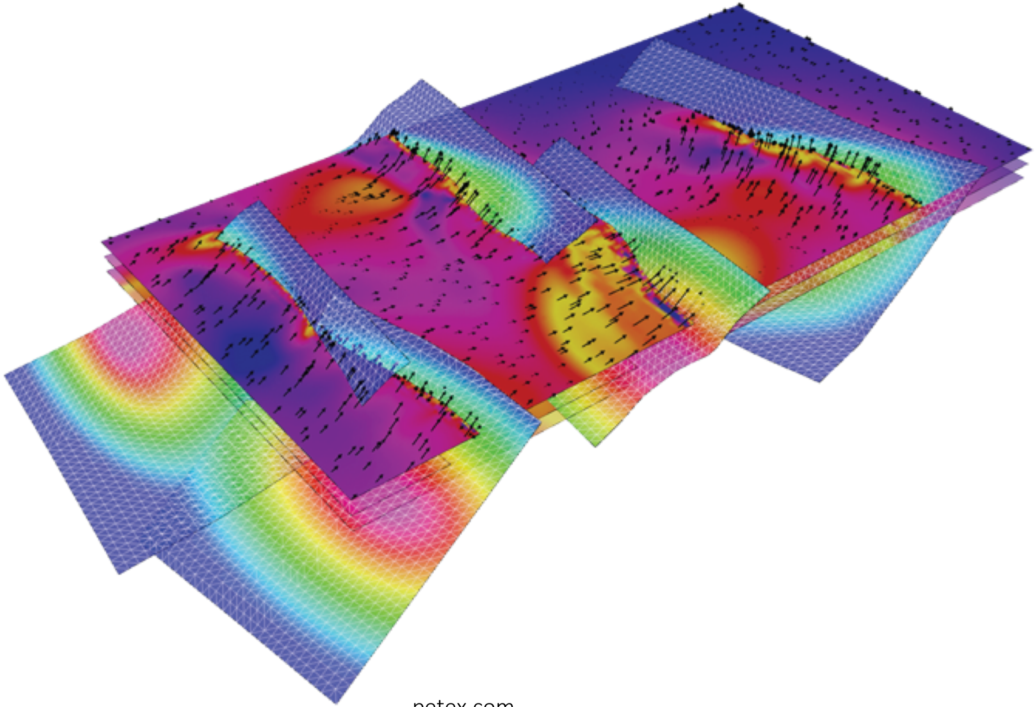
Results



Example Applications

BEM Examples

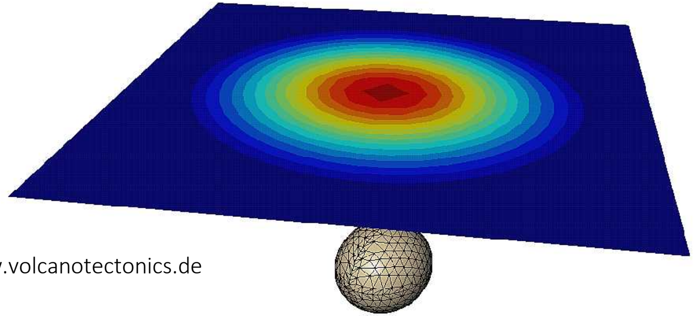
Displacement strain and stress on a Fault System



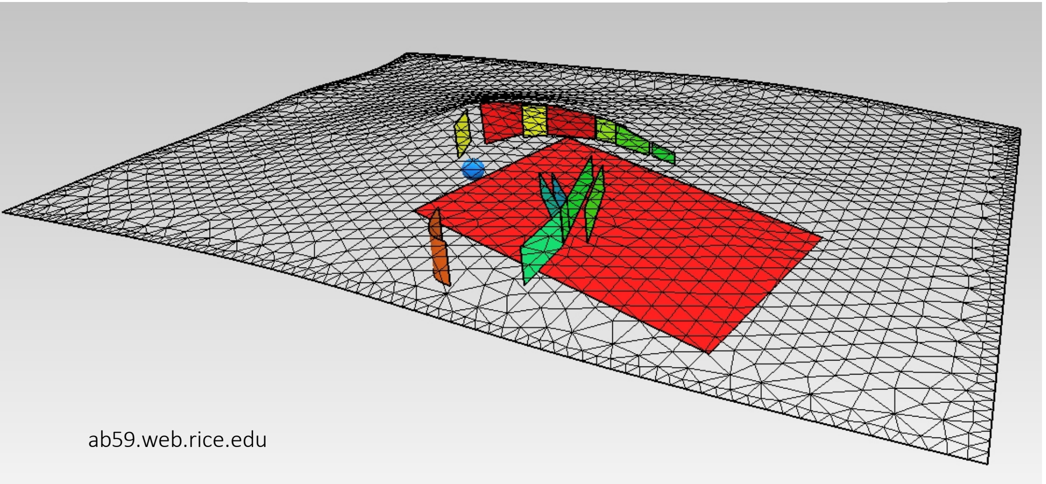
petex.com

BEM Examples

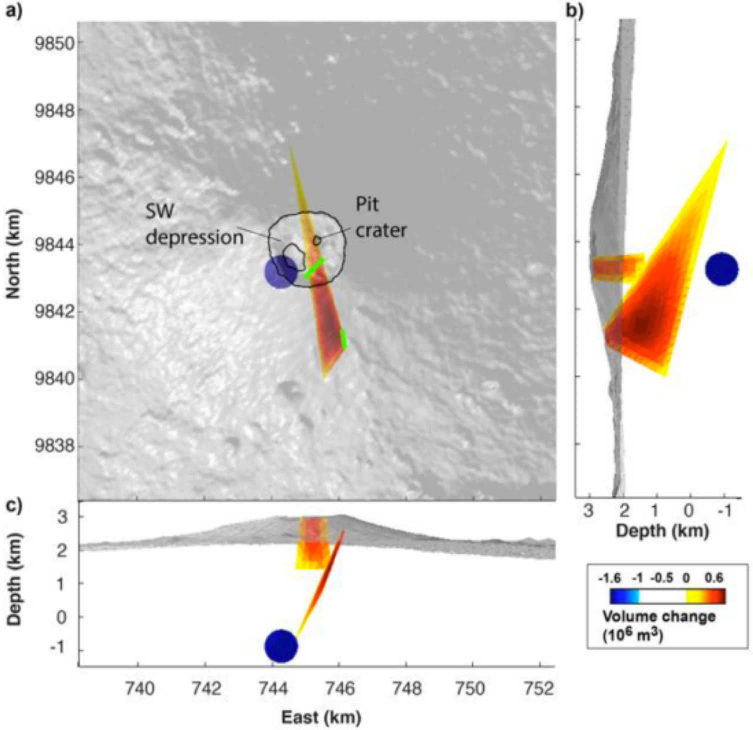
Volcano Deformation



www.volcanotectonics.de



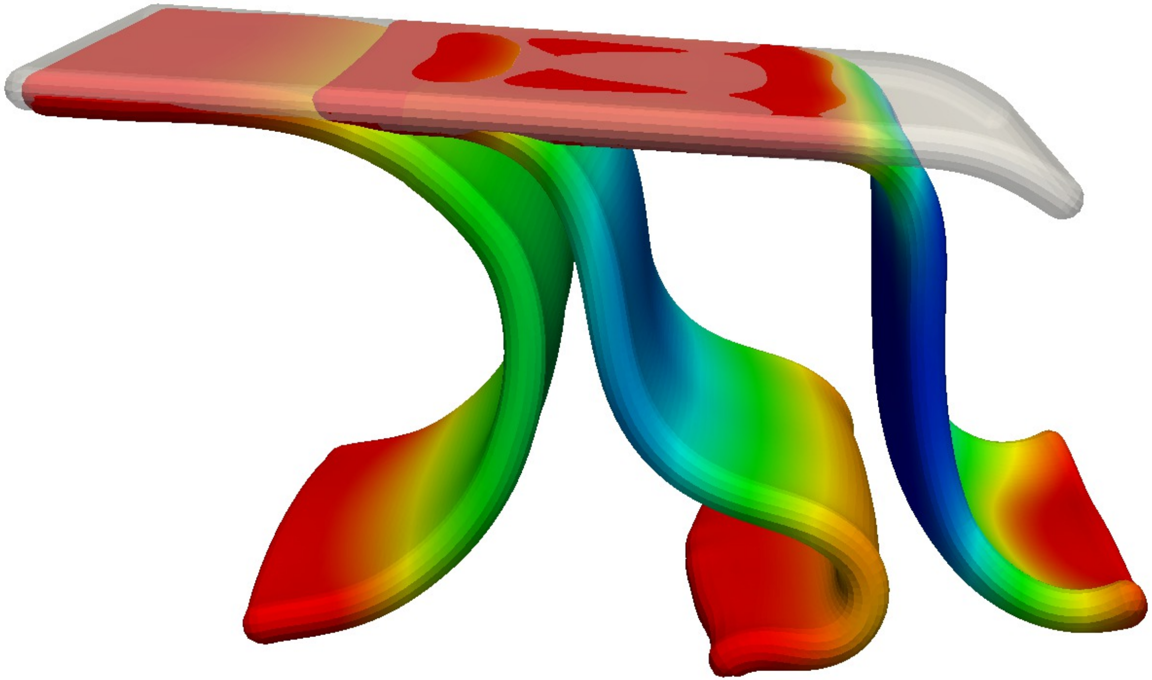
ab59.web.rice.edu



Wauthier et al. (2015) RS

BEM Examples

Geodynamics: 3D Flow BEM

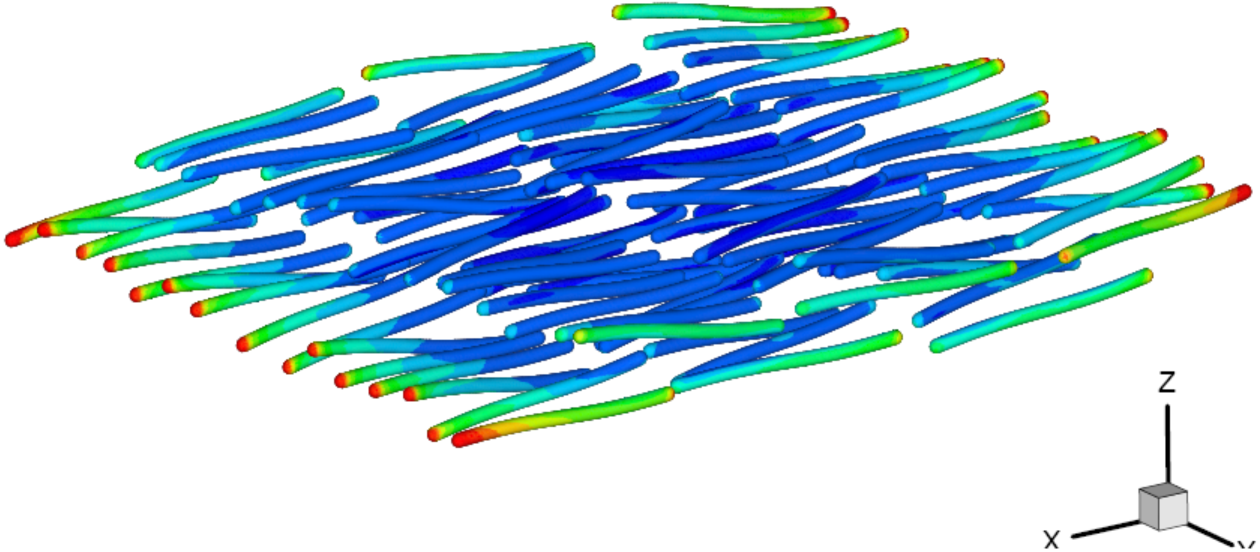


Subducting slabs with three different ratios of viscosity of the slab-mantle.

fast.u-psud.fr/~ribe

BEM Examples

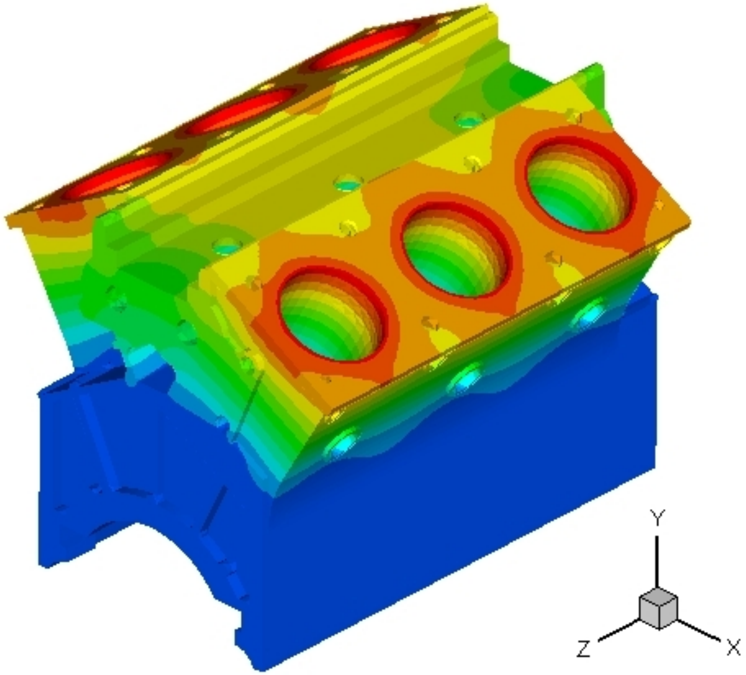
Stokes Flow



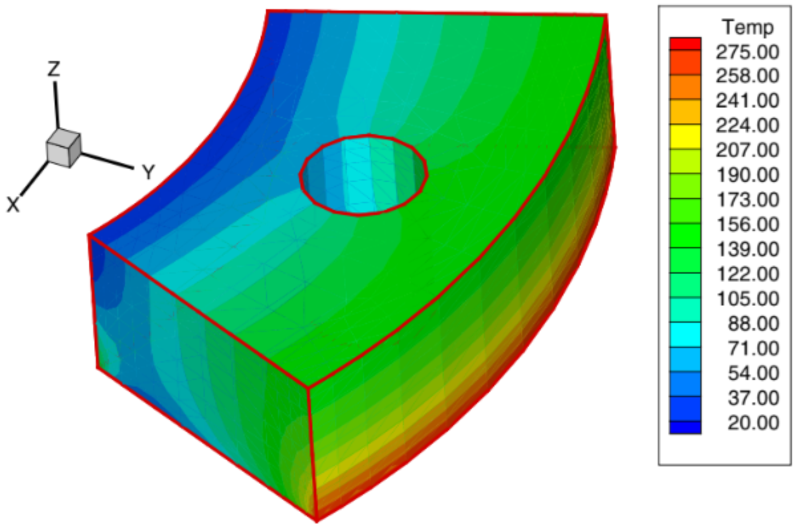
Stokes flow interacting with fibers

BEM Examples

Heat Transfer



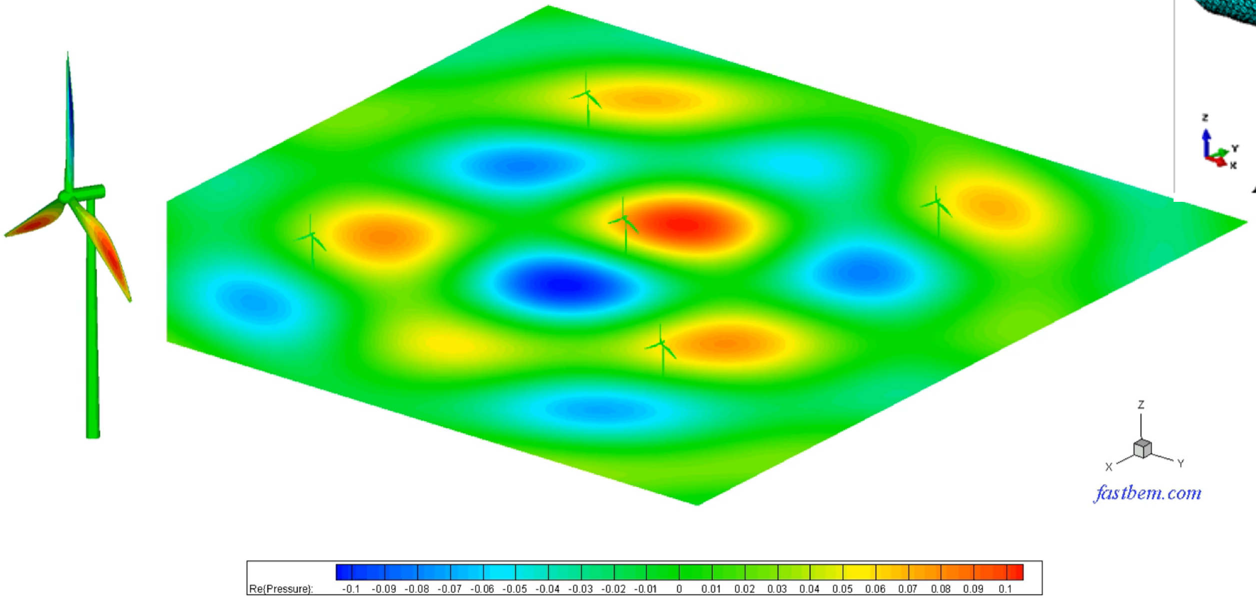
Fastbem.com

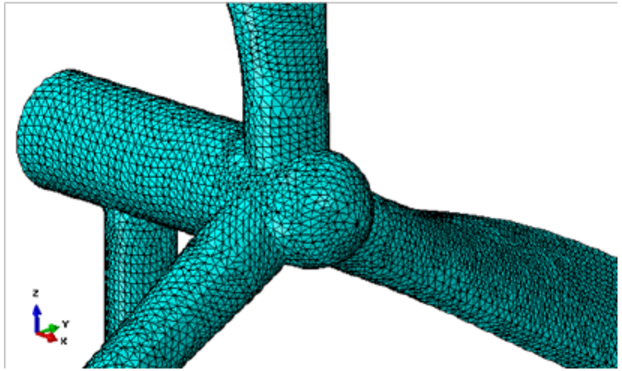


Sutradhar et al. (2004)

BEM Examples

Acoustics: Wind Turbine Noise





BEM Examples

Acoustics: Noise From a Landing Commercial Jet