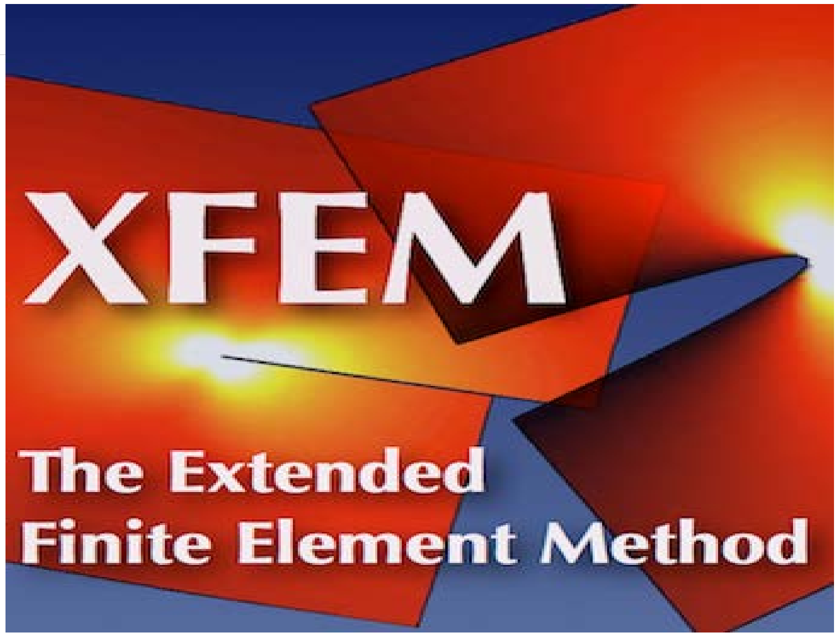
EXTENDED FINITE ELEMENT METHOD



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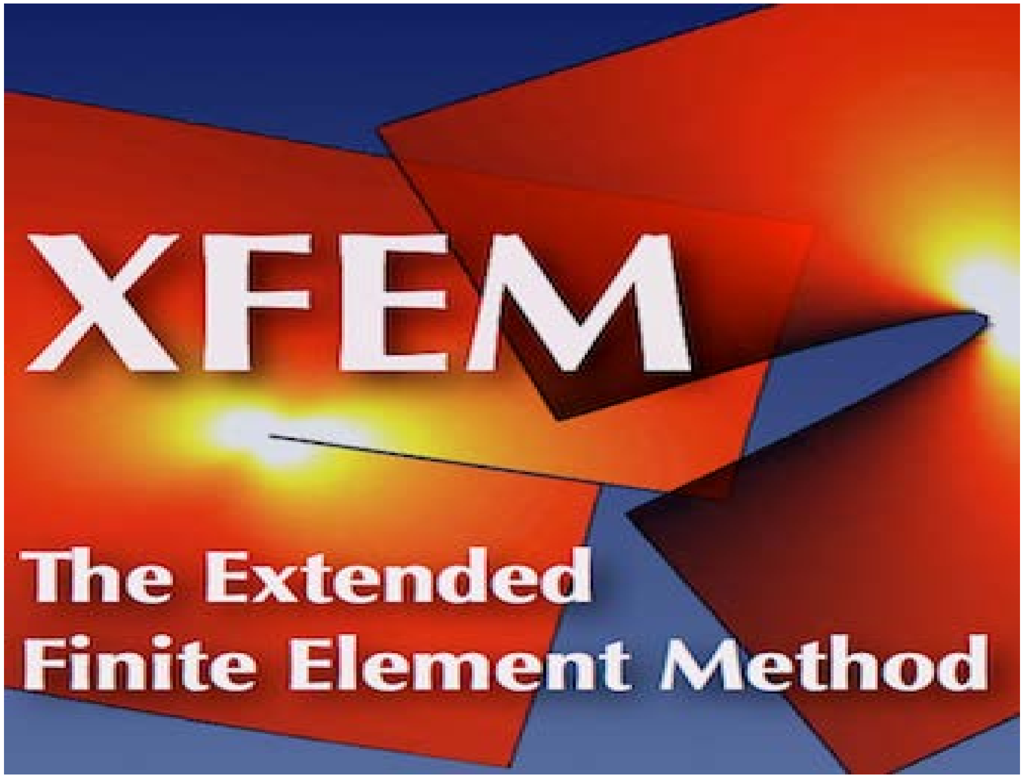
INTRODUCTION



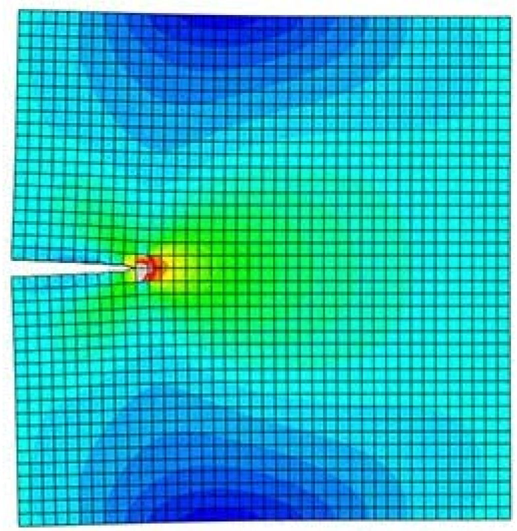
XFEM - Context

- The difficulty of 3D Crack propagation
- Conventional fracture mechanics is not valid for crack propagation with complex geometry
- Experiments are mostly empirical and phenomenological and mainly focus on planar cracks.
- Rapid development of numerical simulations along with the development of computer technology
- Many computational fracture mechanics methods appeared
- X-FEM proposed in the late 1990s and became very powerful for complicated fracture problems.

Introduction

- Conventional FEM shortcomings:
  - Element boundary must coincide with geometrics of the defects
  - Mesh size dependent on the geometric size of the small defects then decides critical stable time increment
  - Cracks can only propagate along the element not natural path

- XFEM: a numerical technique based on generalized finite element method(GFEM) and partition of unity method(PUM)



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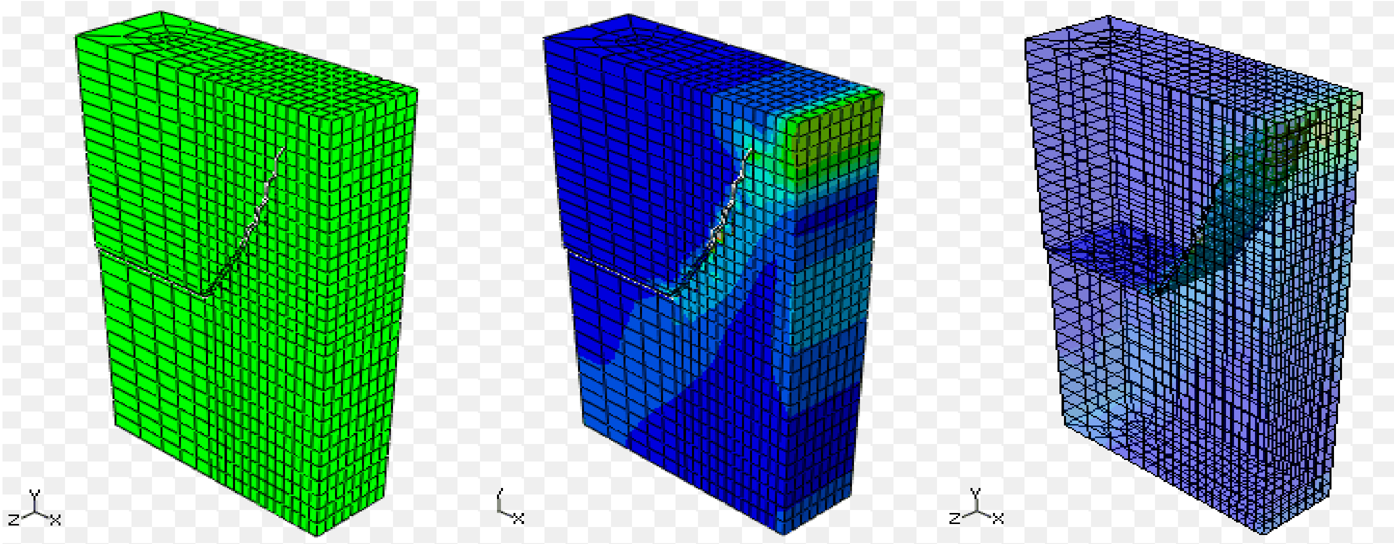
Introduction

- Performs best for pronounced non-smooth characteristics in small parts of the computational domain, e.g. near discontinuities and singularities, it has optimal convergence rates for these applications



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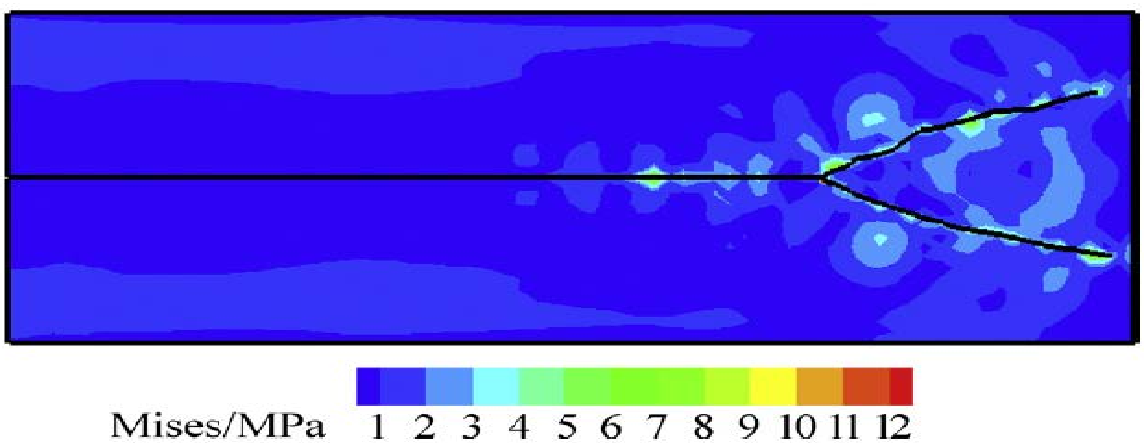
Characterizations of XFEM

- Crack location and crack propagation inside the elements:
  - Cracks with complicated geometries can be modeled by structured meshes
  - Cracks can propagate element by element without remeshing (save computational cost significantly)

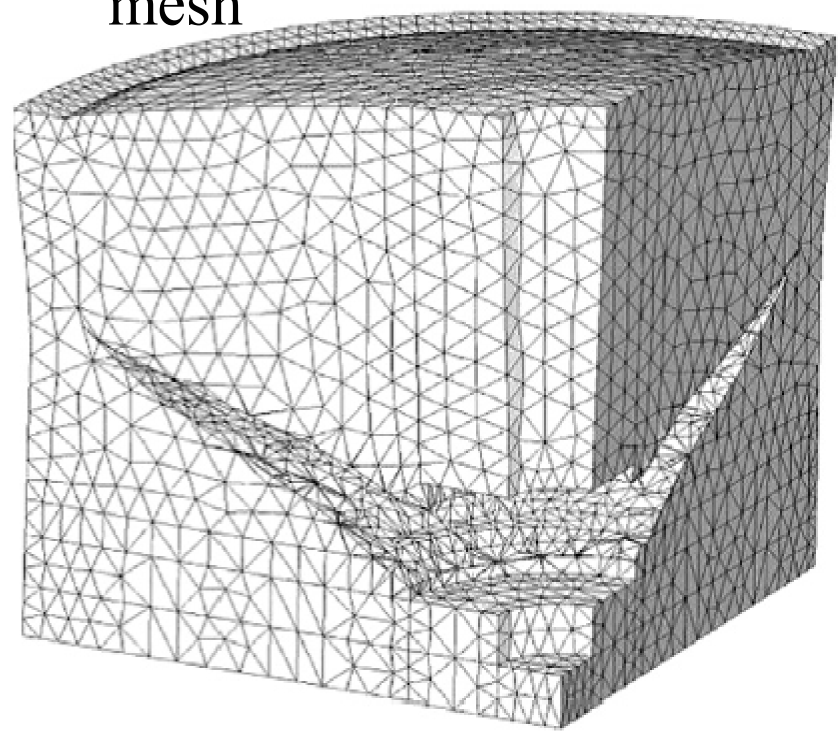


One example of crack branching simulation (Xu et al., 2013)

Characterizations of XFEM

- Enriched elements with additional degrees of freedom at crack surface and crack tips
- The discontinuous shape function is used to capture the singularity of the stress field near the crack tip
- Convenient to implement in commercial software and with parallel computing.

- The description for the discontinuous field is entirely independent of mesh



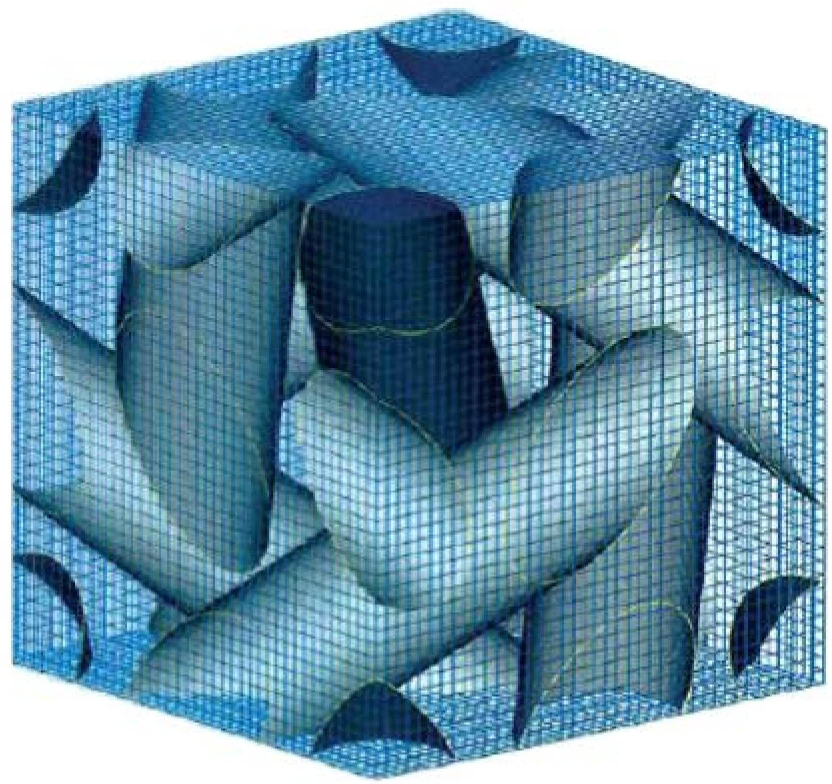
3D fracture X-FEM modeling
(Areias and Belytschko, 2005b)

Characterizations of XFEM

- Not only simulate cracks, but also heterogeneous materials with voids and inclusions

- Displacement: strong discontinuity
- Strain: weak discontinuity

Mesh boundary does not have to coincide with material interfaces, representative volume element(RVE) can be meshed by brick elements



X-FEM modeling of nanotube composites (Belytschko et al., 2003b)

Comparing with Other Methods

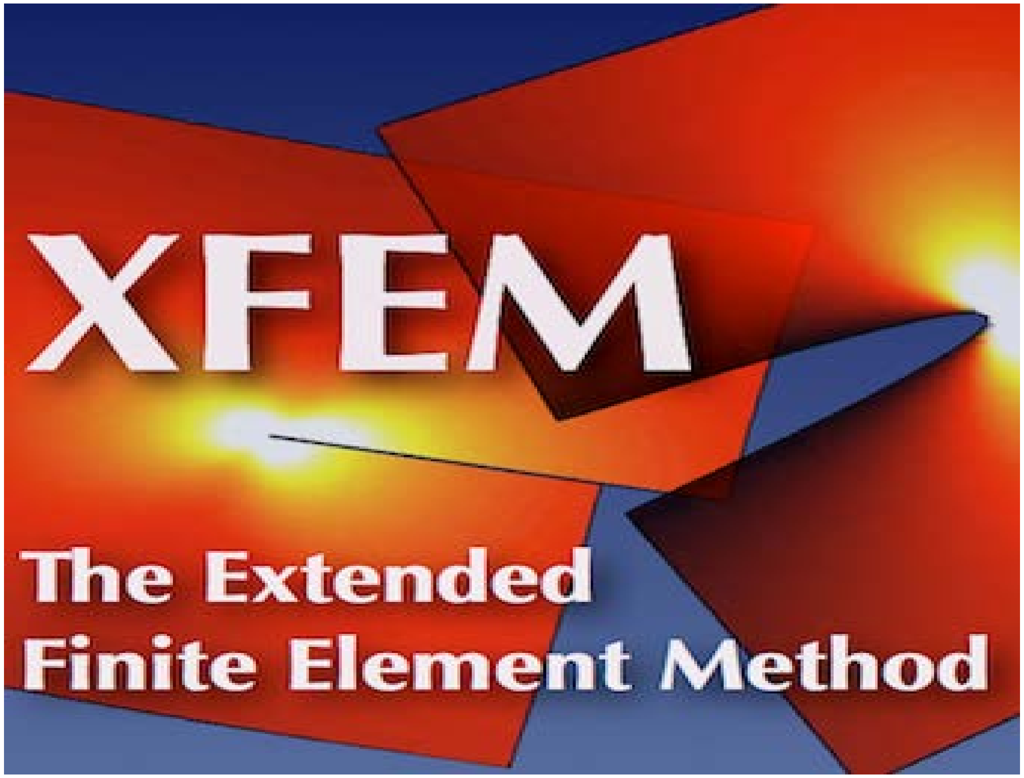
- VS. remeshing technique in FEM
  - Unnecessary in X-FEM for mapping of field variables after crack propagation

- VS. boundary element method,
  - X-FEM can solve multi-material or multi-phase problems

Shortcomings and Assumptions

- Only for linear elastic materials
- multiple initial cracks can be defined
- XFEM analysis is only assumed to be quasi-static
- Time stepping needs to be small enough to capture crack propagation
- Difficulties in localization of fracture initiation

HISTORICAL PERSPECTIVE



X – FEM Research Status

- The improvement of related theories:
  - blending elements
  - Subdomain integration of a discontinuous field
  - Stability of explicit time integration
  - Extent of the basis of enrichment shape function

- The development of element types
  - 2D plane element
  - 3D solid element
  - Shell element

Development of XFEM Theory

- Melenk and Babuska (1996) and Duarte and Oden (1996)
  - the mathematical foundation of the partition of unity finite element method(PUFEM) were discussed
- Belytschko and Black (1999)
  - minimal remeshing finite element method by adding discontinuous enrichment functions to the finite element approximation to account for the presence of a crack.
- Moës *et al*. (1999) and Dolbow (1999) and eXtended Finite Element Method (XFEM) was named
  - Heaviside function and crack tip function as the enrichment shape function of elements including the crack surface and tip respectively

Development of XFEM Theory

- Daux *et al.* (2000)
  - Crack branching by more than one enrichment shape function in crack tip elements
- Belytschko et al. (2003a)
  - New crack nucleation criterion, the loss of hyperbolas
- Chessa et al. (2003)
  - Improved blending element performance using assumed strain method

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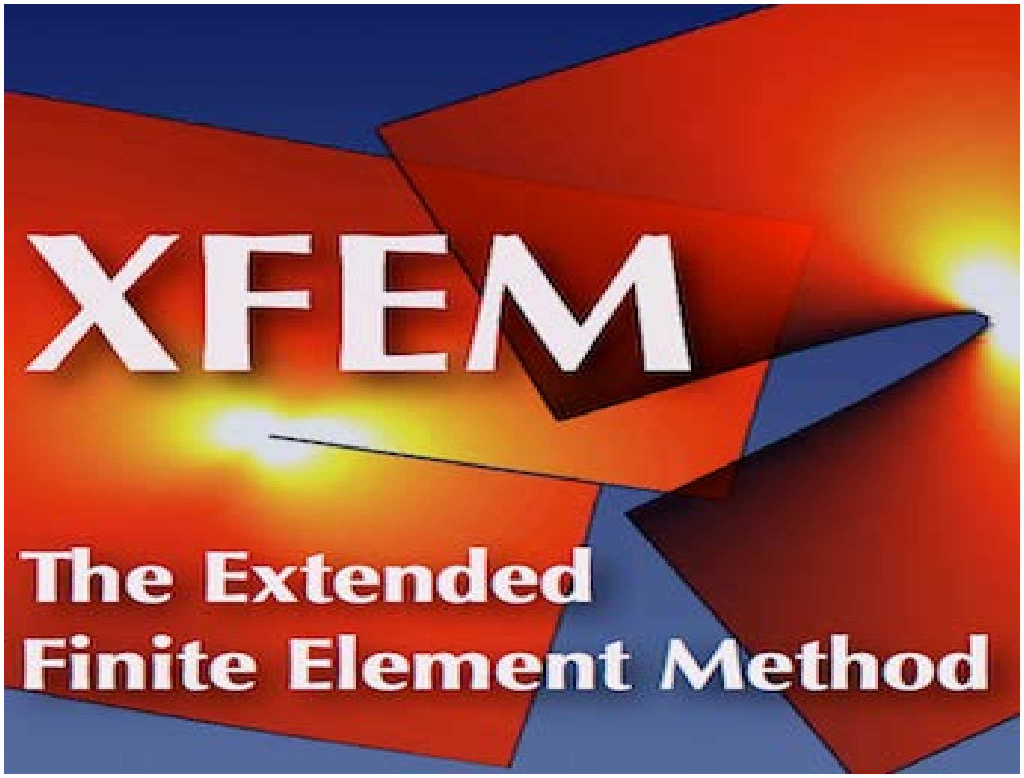
Alternative Techniques

- Element-free Galerkin method (EFG) (Belytschko *et al.* 1994)
- Meshless local Petrov–Galerkin(MLPG) (Atluri and Shen 2002)
- Smoothed particle hydrodynamics (SPH) (Belytschko *etal.* 1996)
- Finite point method (FPM) (Onate *et al.* 1995)
- Reproducing kernel particle method (RKPM) (Liu *et al.* 1996)
- Equilibrium online method (ELM) (Sadeghirad and Mohammadi 2007)

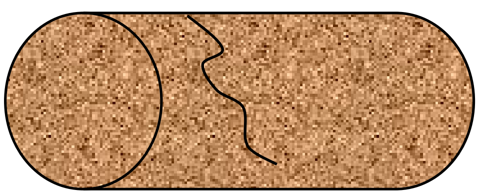
GENERAL PRINCIPLES



What is X-FEM?

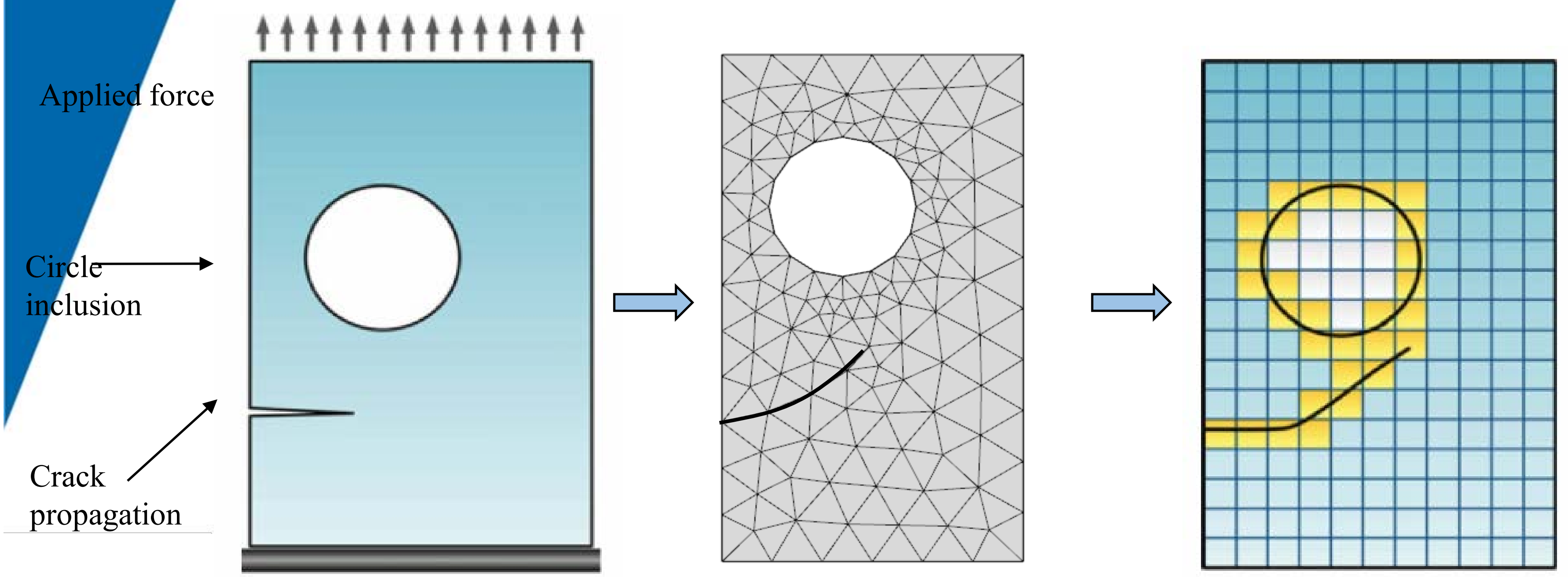
EXtended Finite Element Method

Extended? → Discontinuity



The original displacement functions fails to model the crack

How does ‘Extended’ works?



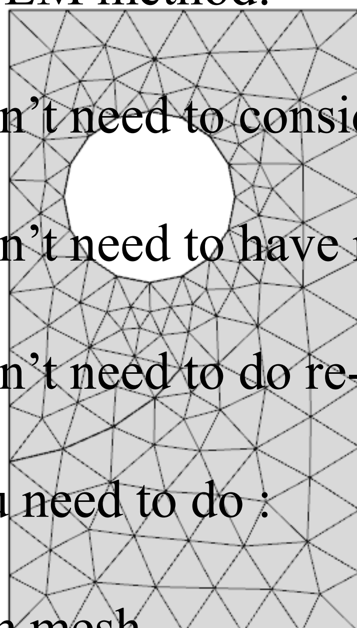
FEM

X-FEM

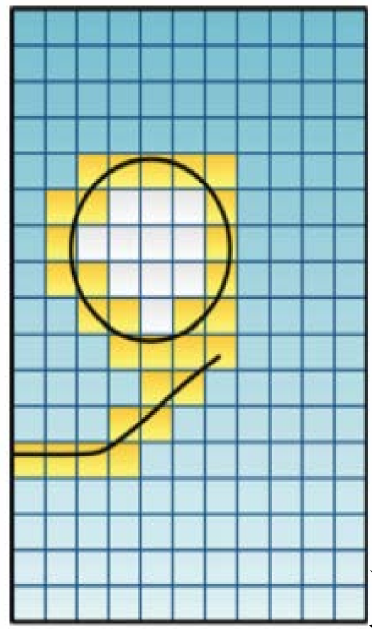
How does 'Extended' works?

With X-FEM method:

- You don't need to consider the ... ntinuity anymore!
- You don't need to have nodes ... the crack/interface !
- You don't need to do re-mesh!



extended



What you need to do :

- uniform mesh
- enrich the element where discontinuity exist.

How ?

How does X-FEM enriched the approximation?

$$U(x) = U^{FE} \quad \boxed{+ \; U^{enrichment}}$$

$$\boxed{\sum_{I} N_I(x) * U_I} \qquad \boxed{\sum_{j} \psi_j(x) * u_j}$$

enriched shape function   newly added degrees of freedom

Strong discontinuity   weak discontinuity

Discontinuity

Weak discontinuity

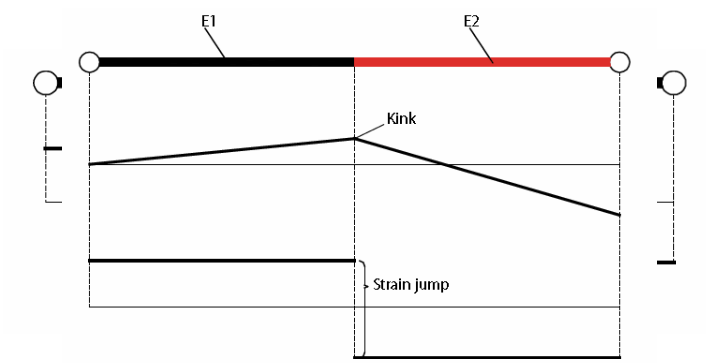
discontinuity in the derivative of problem

displacement is weak which makes a kink

Strong discontinuity

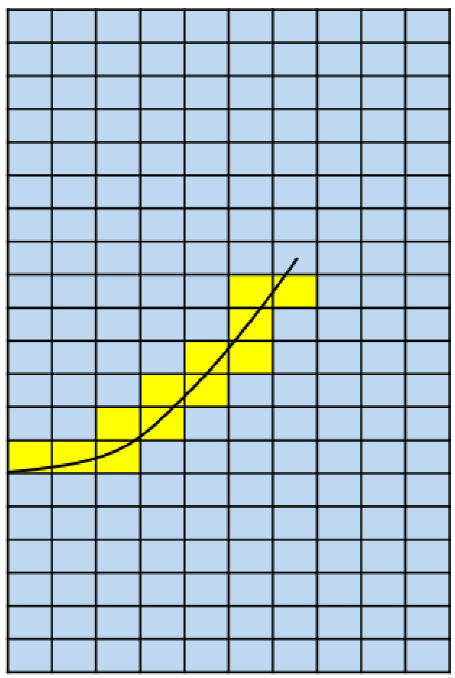
discontinuity in the solution variable

displacement is so strong that makes a jump

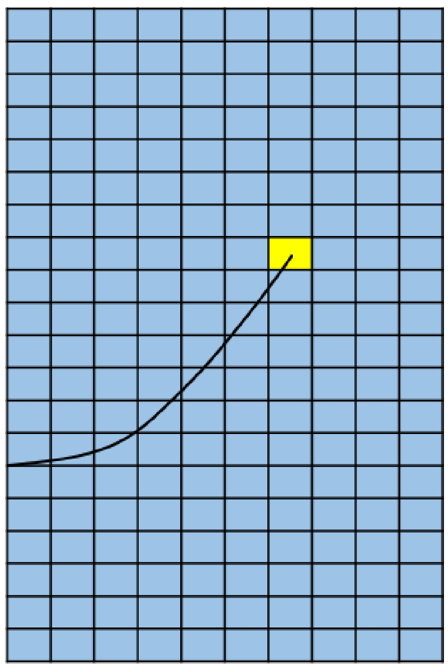


Bimaterial Bar

Enriched shape function construction-strong discontinuity



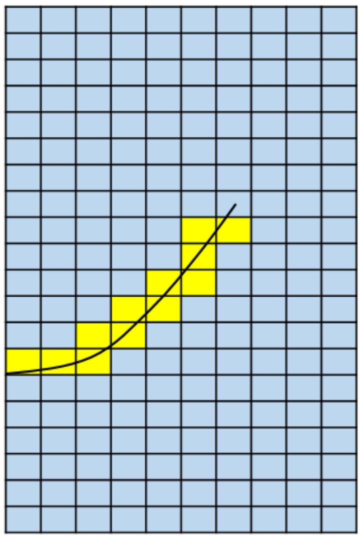
Crack-cross element



Crack –embedded element

Enriched shape function constriction-strong discontinuity crack-cross element

the displacement field on both sides of crack are discontinuity



Crack-cross element

$$\psi_j(x) = N_j(x) * H(f(x))$$

H(x)is the Heaviside step function

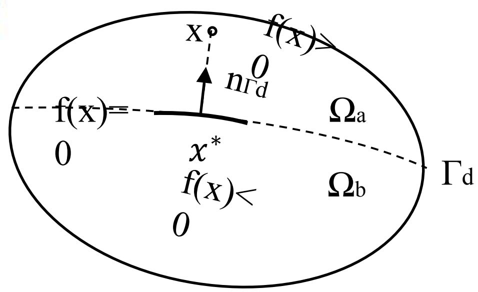
$$H(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

f(x)is the signed distance function

Enriched shape function constriction-strong discontinuity signed distance function

f(x) is the signed distance function

$$f(x)=\|\boldsymbol{x}-\boldsymbol{x}^*\|\,\mathrm{sign}(\mathbf{n}_{\Gamma_d}\cdot(\boldsymbol{x}-\boldsymbol{x}^*))$$



$$f(x)\begin{cases} >0 & if \quad x\in\Omega_a \\ =0 & if \quad x\in\Gamma_d \\ <0 & if \quad x\in\Omega_b \end{cases}$$

When x is lies the same direction as n, f(x)>0

When x is lies on the interface, f(x)=0

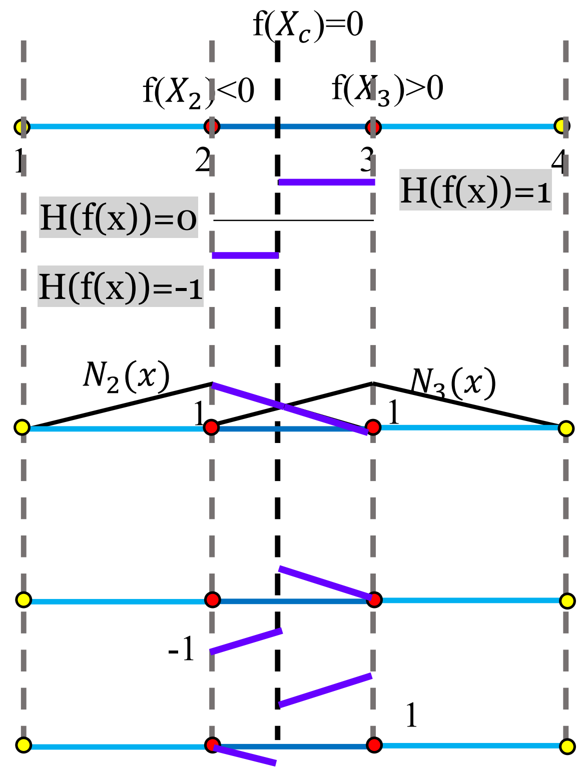
When x is lies the opposite direction as n, f(x)<0

Enriched shape function constriction-strong discontinuity crack-cross element

$$\psi_3(x) = N_3(x) * H(f(x))$$

$$H(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

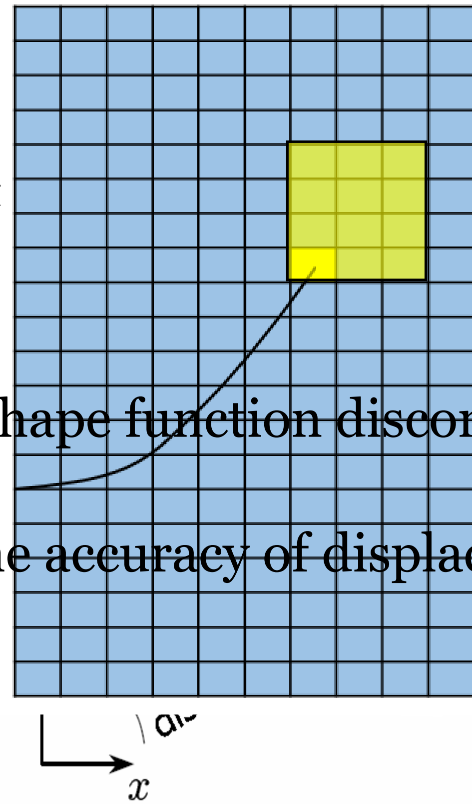
Γ_d



Enriched shape function constriction-strong discontinuity crack-embedded element(crack tip)

(x)

$$\phi(x) = \Big[\sqrt{r}\,\mathrm{si} \quad \ldots \quad \sqrt{r}\cos\frac{\theta}{2}, \quad \sqrt{r}\cos\frac{\theta}{2}\sin\theta \Big]$$



build the shape function discontinuity

improve the accuracy of displacement field

tip)

r and Θ are position parameters in polar coordinates.

2D Displacement function for the strong discontinuity

$$U\ (x) = U^{FE} + U^{enrichment}$$

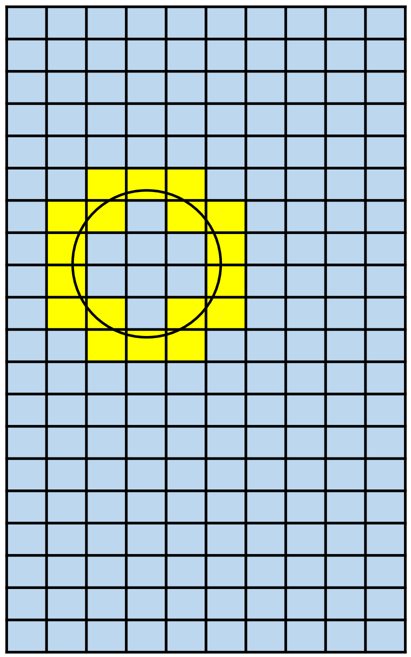
$$= U^{FE} + U^{enrichment(crack-cross)} + U^{enrichment(crack-embedded)}$$

$$= \sum_{\mathrm{I}} N_I(x) * U_I + \sum_j \psi_j(x) * \mathrm{u}'_j + \sum_j \phi_k(x) * \mathrm{u}''_k$$

$$= \sum_{\mathrm{I}} N_I(x) * U_I + \sum_j N_j(x) * H(f(x)) * \mathrm{u}'_j + \sum_{\mathrm{k}} N_k(x) * \phi_k(x) * \mathrm{u}''_k$$

$$\boxed{U(x) = \sum_{\mathrm{I}} N_I(x) * U_I + \sum_j N_j(x) * H(f(x)) * \mathrm{u}'_j + \sum_{\mathrm{k}} N_k(x) * \phi_k(x) * \mathrm{u}''_k}$$

Enriched shape function constriction-weak discontinuity



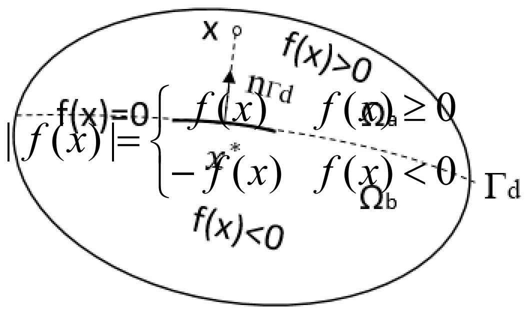
Ramp function: $|f(x)|$

$$\psi_k(x) = N_k(x) * |f(x)|$$

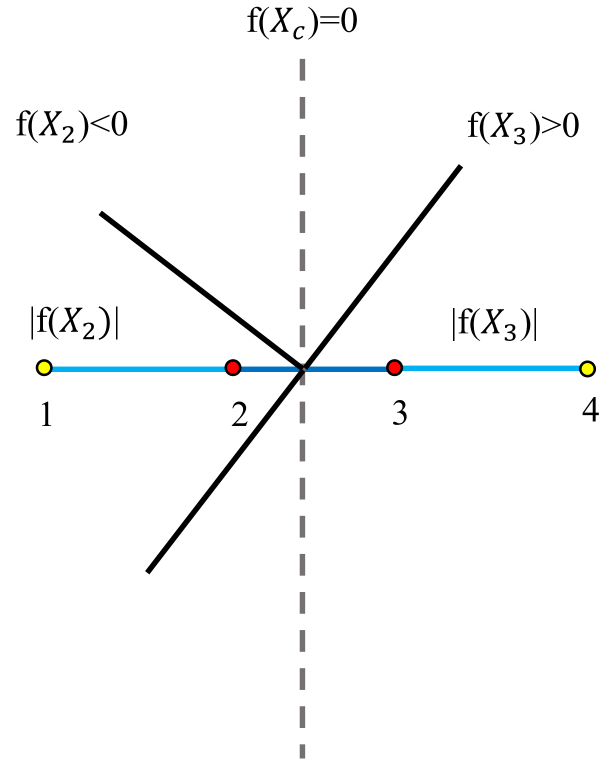
Crack-cross element

$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

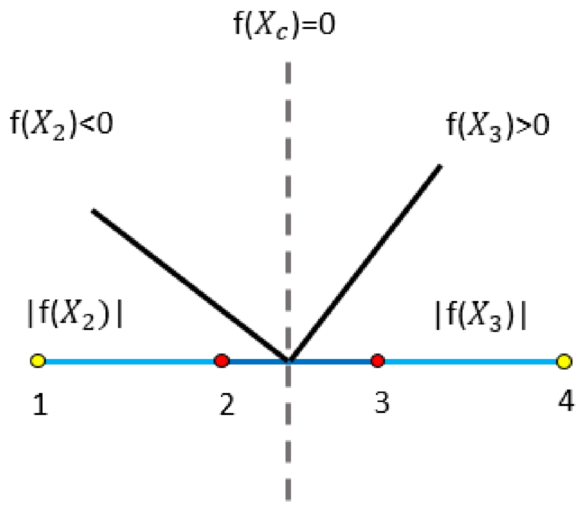
Enriched shape function constriction-rump function

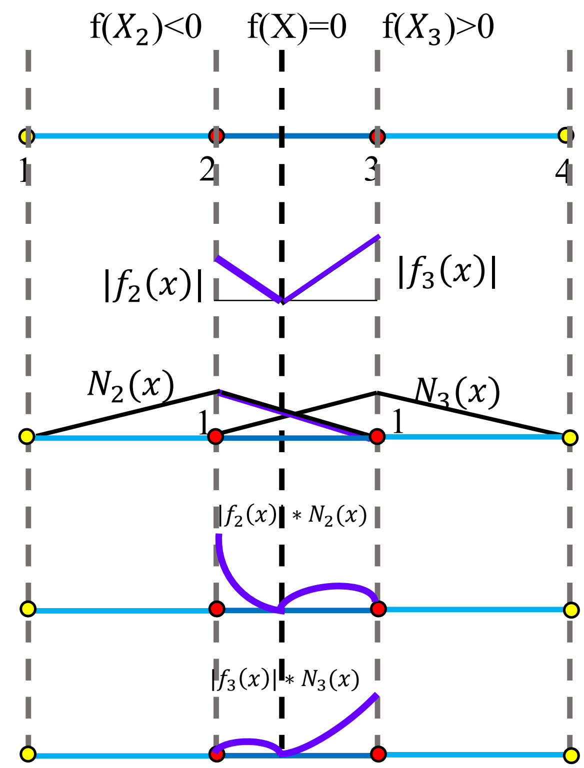


$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

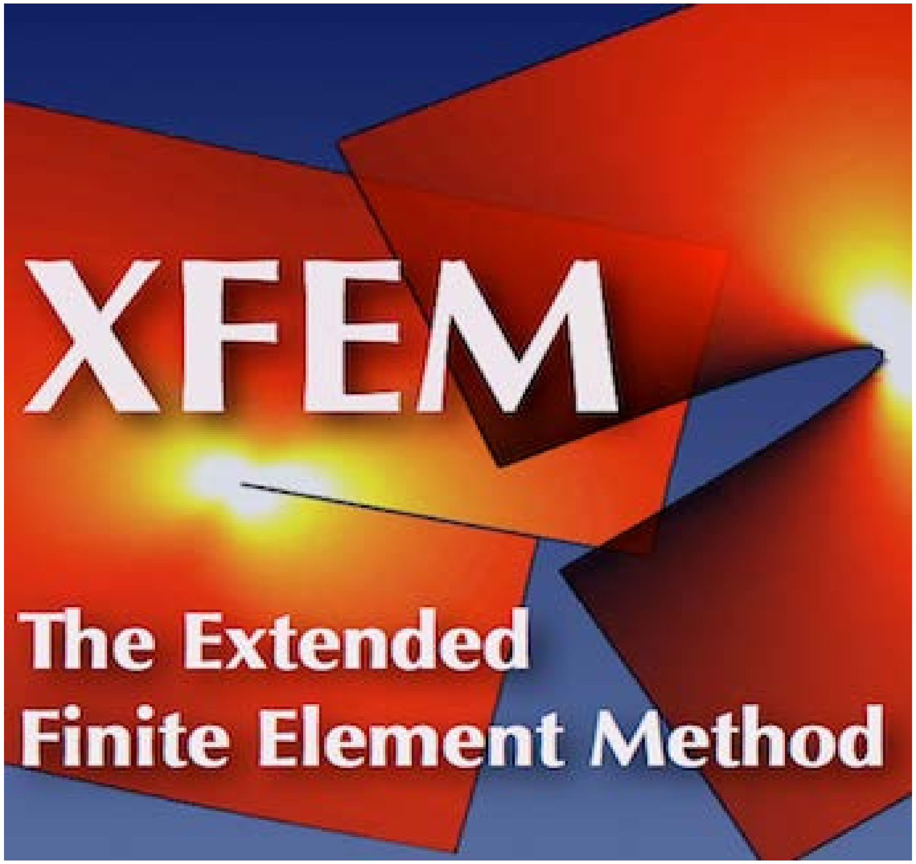


Enriched shape function constriction-Weak discontinuity





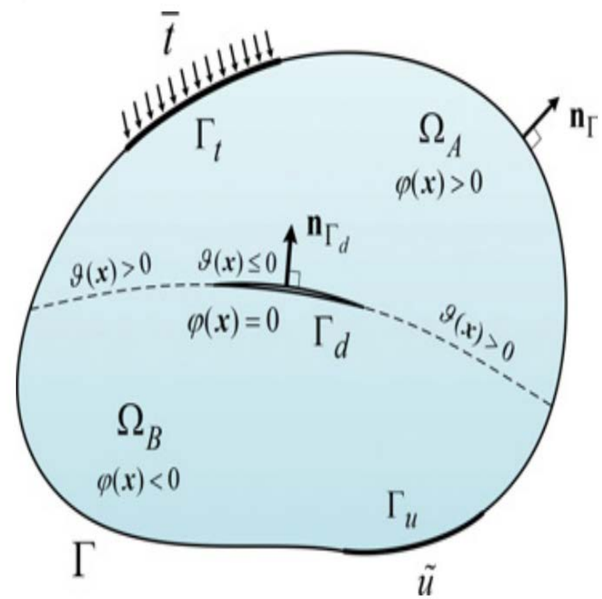
GOVERNING EQUATION



Governing Equation

➤ **The strong form of the equilibrium equation:**

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$$



Boundary conditions

- Displacement (Dirichlet) boundary condition: $\mathbf{u} = \tilde{\mathbf{u}}$ on Γ_u
- Traction (Neumann) boundary condition: $\boldsymbol{\sigma} \cdot \mathbf{n}_\Gamma = \bar{\mathbf{t}}$ on Γ_t
- Internal boundary condition: $\boldsymbol{\sigma} \cdot \mathbf{n}_{\Gamma_d} = \bar{\mathbf{t}}_d$ on the discontinuity Γ_d

Governing Equation

- **The weak form of equilibrium equation:**

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$$

Multiply by the test functions δu(x, t) and integrate over the domain Ω

$$\int_{\Omega} \delta\mathbf{u}(\boldsymbol{x}, t)\,(\nabla \cdot \boldsymbol{\sigma} + \mathbf{b})\,d\Omega = 0$$

Integration by parts
& Divergence theorem for discontinuous problems

$$\int_{\Omega} \nabla\delta\mathbf{u} : \boldsymbol{\sigma}\,d\Omega + \boxed{\int_{\Gamma_d} [\![\delta\mathbf{u} \cdot \boldsymbol{\sigma}]\!]\mathbf{n}_{\Gamma_d}\,d\Gamma} - \int_{\Gamma_t} \delta\mathbf{u} \cdot \bar{\mathbf{t}}\,d\Gamma - \int_{\Omega} \delta\mathbf{u} \cdot \mathbf{b}\,d\Omega = 0$$

can be eliminated if a weak discontinuity or traction free crack is used

$$\int_{\Omega} \nabla\delta\mathbf{u} : \boldsymbol{\sigma}\,d\Omega - \int_{\Gamma_t} \delta\mathbf{u} \cdot \bar{\mathbf{t}}\,d\Gamma - \int_{\Omega} \delta\mathbf{u} \cdot \mathbf{b}\,d\Omega = 0$$

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Governing Equation

- **Discretization of governing equation**

$$\int_{\Omega} \nabla \delta \mathbf{u} : \boldsymbol{\sigma} \, d\Omega - \int_{\Gamma_t} \delta \mathbf{u} \cdot \bar{\mathbf{t}} \, d\Gamma - \int_{\Omega} \delta \mathbf{u} \cdot \mathbf{b} \, d\Omega = 0$$

$$\mathbf{K}\bar{\mathbf{U}} - \mathbf{F} = 0$$

Displacement approximation

$$\mathbf{u}(\boldsymbol{x}, t) = \sum_{i=1}^{\mathcal{N}} N_i(\boldsymbol{x})\, \bar{\mathbf{u}}_i + \sum_{j=1}^{\mathcal{M}} N_j(\boldsymbol{x}) \left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right) \bar{\mathbf{a}}_j$$

$$\equiv \mathbf{N}^{std}(\boldsymbol{x})\,\bar{\mathbf{u}} + \mathbf{N}^{enr}(\boldsymbol{x})\,\bar{\mathbf{a}}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{x}, t) = \sum_{i=1}^{\mathcal{N}} \frac{\partial N_i}{\partial \boldsymbol{x}} \bar{\mathbf{u}}_i + \sum_{j=1}^{\mathcal{M}} \left[\frac{\partial N_j}{\partial \boldsymbol{x}} \left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right) + N_j(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{x}} \left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right) \right] \bar{\mathbf{a}}_j$$

$$\equiv \mathbf{B}^{std}(\boldsymbol{x})\,\bar{\mathbf{u}} + \mathbf{B}^{enr}(\boldsymbol{x})\,\bar{\mathbf{a}}$$

Constitutive relation $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

K: global stiffness matrix
U: vector of degrees of nodal freedom (both standard and enriched)
f: vector of external force

Governing Equation

➢ **Discretization of governing equation**

$$\mathbf{K}\bar{\mathbf{U}} - \mathbf{F} = 0$$

Standard degrees of nodal freedom

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{ua} \\ \mathbf{K}_{au} & \mathbf{K}_{aa} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{a}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u \\ \mathbf{F}_a \end{Bmatrix}$$

Enriched degrees of nodal freedom

$$\mathbf{F} = \begin{Bmatrix} \int_{\Gamma_t} \left(\mathbf{N}^{std}\right)^T \bar{\mathbf{t}}\, d\Gamma + \int_{\Omega} \left(\mathbf{N}^{std}\right)^T \mathbf{b}\, d\Omega \\ \int_{\Gamma_t} \left(\mathbf{N}^{enr}\right)^T \bar{\mathbf{t}}\, d\Gamma + \int_{\Omega} \left(\mathbf{N}^{enr}\right)^T \mathbf{b}\, d\Omega \end{Bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \int_{\Omega} \left(\mathbf{B}^{std}\right)^T \mathbf{D}\mathbf{B}^{std} d\Omega & \int_{\Omega} \left(\mathbf{B}^{std}\right)^T \mathbf{D}\mathbf{B}^{enr} d\Omega \\ \int_{\Omega} \left(\mathbf{B}^{enr}\right)^T \mathbf{D}\mathbf{B}^{std} d\Omega & \int_{\Omega} \left(\mathbf{B}^{enr}\right)^T \mathbf{D}\mathbf{B}^{enr} d\Omega \end{bmatrix}$$

For 2D problems:

$$\mathbf{B}_i^{std} = \begin{bmatrix} \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y \\ \partial N_i/\partial y & \partial N_i/\partial x \end{bmatrix}$$

$$\mathbf{B}_j^{enr} = \begin{bmatrix} \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial x & 0 \\ 0 & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial y \\ \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial y & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial x \end{bmatrix}$$

Governing Equation

- The Heaviside fu $\psi(x) = H(\xi)$

$$H(\xi) = \text{sign}(\xi) = \begin{cases} 1 & \forall \xi > 0 \\ -1 & \forall \xi < 0 \end{cases}$$

$$\mathbf{B}_j^{enr} = \begin{bmatrix} \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial x & 0 \\ 0 & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial y \\ \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial y & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_j)\right)\right]/\partial x \end{bmatrix}$$

$$\Downarrow$$

$$H_{,i}(\xi) = \begin{cases} 1 & \text{at crack tip} \\ 0 & \text{otherwise} \end{cases}$$

$$\Downarrow$$

$$\mathbf{B}_i^{\mathrm{a}} = \begin{bmatrix} N_{i,x}\left[H(\xi) - H(\xi_i)\right] & 0 \\ 0 & N_{i,y}\left[H(\xi) - H(\xi_i)\right] \\ N_{i,y}\left[H(\xi) - H(\xi_i)\right] & N_{i,x}\left[H(\xi) - H(\xi_i)\right] \end{bmatrix}$$

- The weak discontinuity $\psi(\mathbf{x}) = \chi(\mathbf{x}) = |\xi(\mathbf{x})|$

$$\psi_{,i}(\mathbf{x}) = sign(\xi)\ \xi_{,i}(\mathbf{x})$$

$$\xi_{,i}(\mathbf{x}) = \sum_{j=1}^{4} N_{j,i}(\mathbf{x})\ \xi_j$$

Governing Equation

- The near tip enrichment f $\psi = F_\alpha(r,\theta)$

①　②　③　④

$$F_\alpha(r,\theta) = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\theta\sin\frac{\theta}{2}, \sqrt{r}\sin\theta\cos\frac{\theta}{2}\right\}$$

(in terms of the local crack tip coordinate system)

$$\mathbf{B}_j^{enr} = \begin{bmatrix} \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial x & 0 \\ 0 & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial y \\ \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial y & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial x \end{bmatrix}$$

Asymptotic near tip displacement field

$$u(x,y) = \frac{K_{\mathrm{I}}}{2\mu}\sqrt{r/2\mu}\cos(\theta/2)[\kappa - 1 + 2\sin^2(\theta/2)] + \frac{K_{\mathrm{II}}}{2\mu}\sqrt{r/2\mu}\sin(\theta/2)[\kappa + 1 + 2\cos^2(\theta/2)]$$

$$v(x,y) = \frac{K_{\mathrm{I}}}{2\mu}\sqrt{r/2\mu}\sin(\theta/2)[\kappa + 1 - 2\cos^2(\theta/2)] - \frac{K_{\mathrm{II}}}{2\mu}\sqrt{r/2\mu}\cos(\theta/2)[\kappa - 1 - 2\sin^2(\theta/2)]$$

Derivatives of $F_\alpha(r,\theta)$ with respect to the crack tip polar coordinates become

$$F_{1,r} = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}, \qquad F_{1,\theta} = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2}$$

$$F_{2,r} = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}, \qquad F_{2,\theta} = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2}$$

$$F_{3,r} = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\sin\theta, \quad F_{3,\theta} = \sqrt{r}\left(\frac{1}{2}\cos\frac{\theta}{2}\sin\theta + \sin\frac{\theta}{2}\cos\theta\right)$$

$$F_{4,r} = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}\sin\theta, \quad F_{4,\theta} = \sqrt{r}\left(-\frac{1}{2}\sin\frac{\theta}{2}\sin\theta + \cos\frac{\theta}{2}\cos\theta\right)$$

Governing Equation

$$F_{1,r}=\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2},\qquad F_{1,\theta}=\frac{\sqrt{r}}{2}\cos\frac{\theta}{2}$$

$$F_{2,r}=\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2},\qquad F_{2,\theta}=-\frac{\sqrt{r}}{2}\sin\frac{\theta}{2}$$

$$F_{3,r}=\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\sin\theta,\quad F_{3,\theta}=\sqrt{r}\left(\frac{1}{2}\cos\frac{\theta}{2}\sin\theta+\sin\frac{\theta}{2}\cos\theta\right)$$

$$F_{4,r}=\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}\sin\theta,\quad F_{4,\theta}=\sqrt{r}\left(-\frac{1}{2}\sin\frac{\theta}{2}\sin\theta+\cos\frac{\theta}{2}\cos\theta\right)$$

$$\mathbf{B}_j^{enr}=\begin{bmatrix}\partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial x & 0\\ 0 & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial y\\ \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial y & \partial\left[N_j(\boldsymbol{x})\left(\psi(\boldsymbol{x})-\psi(\boldsymbol{x}_j)\right)\right]/\partial x\end{bmatrix}$$

Derivatives of $F_\alpha(r,\theta)$ with respect to the local crack coordinate system (x', y')

$$F_{1,x'}=-\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2},\qquad F_{1,y'}=\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}$$

$$F_{2,x'}=\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2},\qquad F_{2,y'}=\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}$$

$$F_{3,x'}=-\frac{1}{2\sqrt{r}}\sin\frac{3\theta}{2}\sin\theta,\quad F_{3,y'}=\frac{1}{2\sqrt{r}}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\cos\theta\right)$$

$$F_{4,x'}=-\frac{1}{2\sqrt{r}}\cos\frac{3\theta}{2}\sin\theta,\quad F_{4,y'}=\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\cos\theta\right)$$

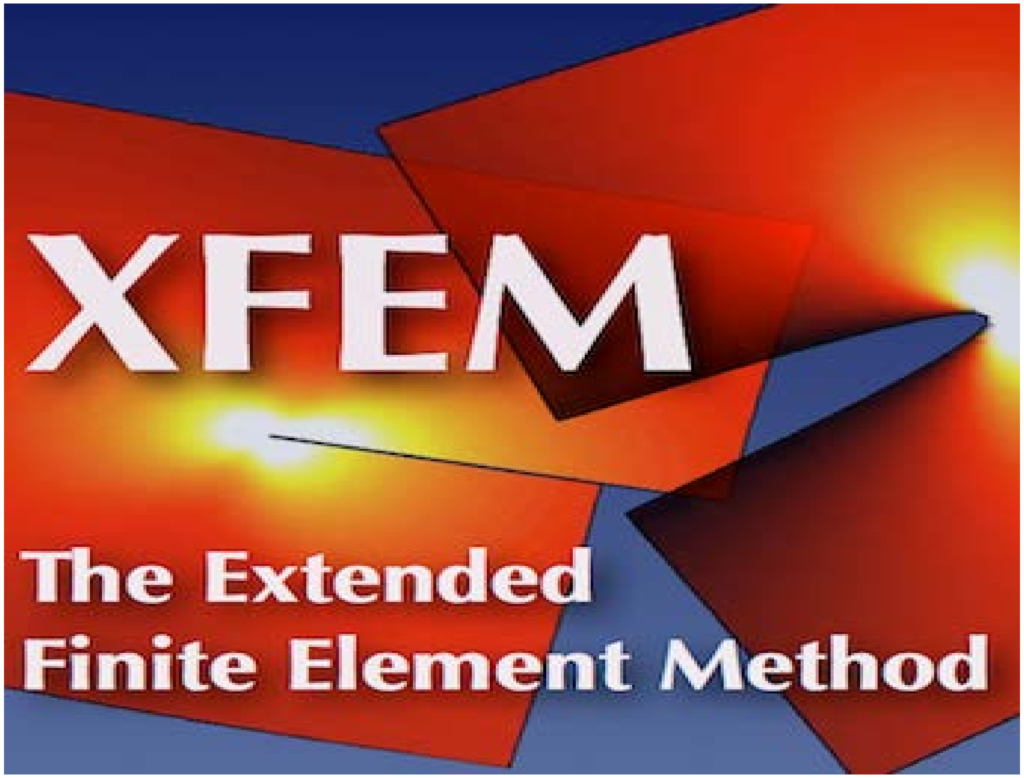
The derivatives in the global coordinate system

$$F_{\alpha,x}=F_{\alpha,x'}\cos(\alpha)-F_{\alpha,y'}\sin(\alpha)$$

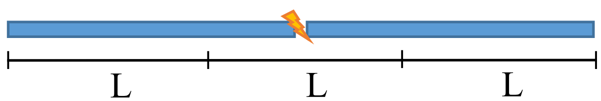
$$F_{\alpha,y}=F_{\alpha,x'}\sin(\alpha)+F_{\alpha,y'}\cos(\alpha)$$

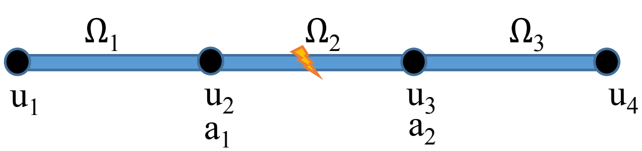
α is the angle of crack path with respect to the x axis.

HAND-CALCULATION EXAMPLE



1D Calculation





Example: 1D Crack Propagation

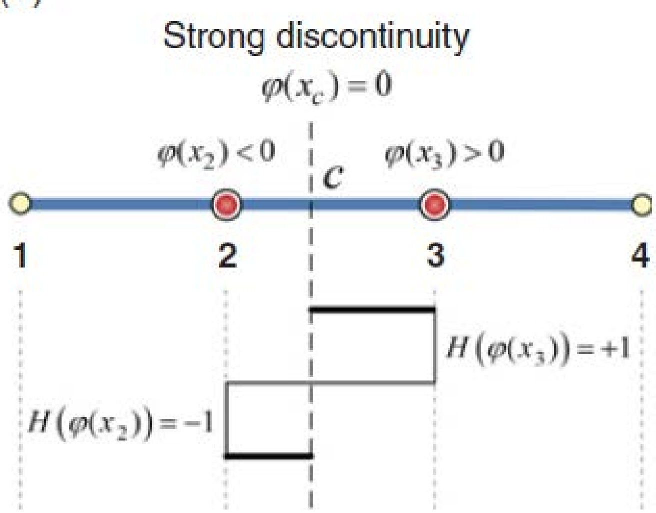
- Bar example
- Three elements
- Four nodes, four standard degrees of freedom
- Two enriched degree of freedoms
- One crack – strong discontinuity

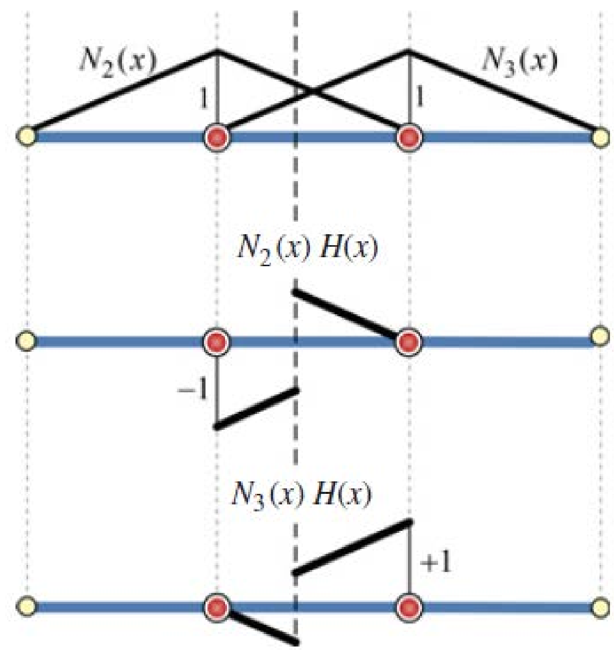
XFEM Discretization

Signed Distance Function

$$|\varphi(\boldsymbol{x})| = \begin{cases} -\varphi(\boldsymbol{x}) & \text{if } \varphi(\boldsymbol{x}) < 0 \\ +\varphi(\boldsymbol{x}) & \text{if } \varphi(\boldsymbol{x}) \geq 0 \end{cases}$$

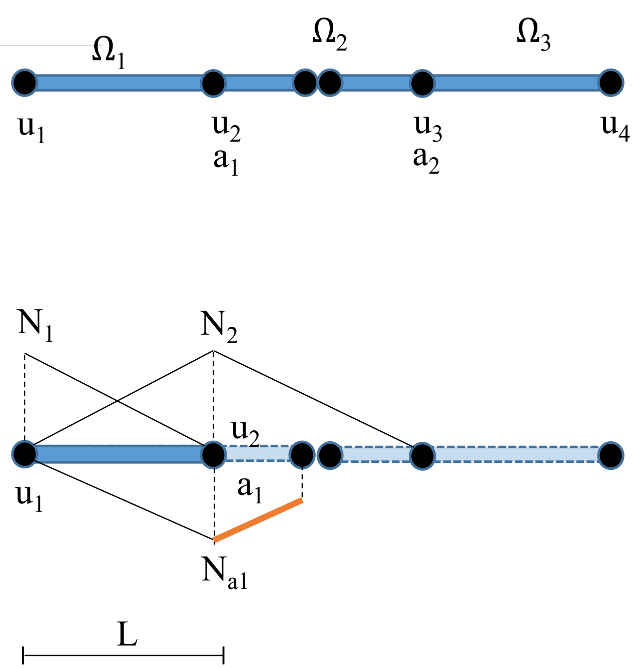
(a)





Source: Khoei, 2015

Nodes and Elements



H= -1

$$N_u^{std} = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

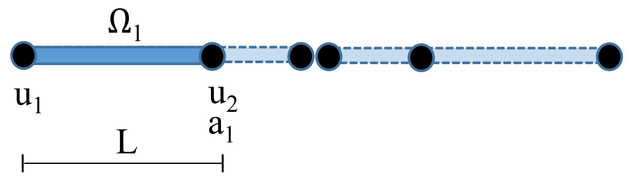
$$B_u^{std} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

$$N_a^{enr} = H\left[\frac{x}{L} - 1\right] = \left[1 - \frac{x}{L}\right]$$

$$B_a^{enr} = H\left[-\frac{1}{L}\right] = \left[\frac{1}{L}\right]$$

$$u(x) = N_i u_i + H N_i a_j$$

Element 1



H = -1

$$K = \int \underline{B}^T \, \underline{D} \, \underline{B} \, dx$$

$$K_{uu} = EA \int_0^L (B_u^{std})^T B_u^{std} dx$$
$$= \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1; \quad 1]$$

$$K_{ua} = EA \int_0^L (B_u^{std})^T B_a^{enr} dx$$
$$= \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [1]$$

$$K_{au} = EA \int_0^L (B_a^{enr})^T B_u^{std} dx$$
$$= \frac{EA}{L} [1][-1; \quad 1]$$

$$K_{au} = EA \int_0^L (B_a^{enr})^T B_a^{enr} dx$$
$$= \frac{EA}{L} [-1][-1]$$

$$\boxed{\begin{aligned} B_u^{std} &= \frac{1}{L}[-1 \quad 1] \\ B_a^{enr} &= \frac{1}{L}[1] \end{aligned}}$$

$$K_{uu} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{ua} = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{au}^T = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{aa} = \frac{EA}{L}$$

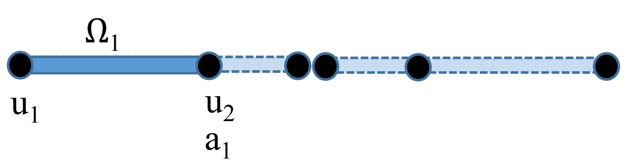
$$K_{uu} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathrm{K_{ua}} = \frac{EA}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{au}^{T} = \frac{EA}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{aa} = \frac{EA}{L}$$

Element 1

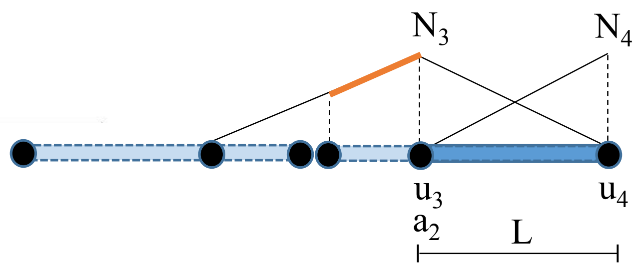


H = -1

$$K = \begin{bmatrix} K_{uu} & K_{ua} \\ K_{au} & K_{aa} \end{bmatrix}$$

$$K_{element\ 1} = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & a_1 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \end{array}$$

Element 3



H= +1

$$N_u^{std} = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$B_u^{std} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

$$N_a^{enr} = H\left[\frac{x}{L} - 1\right] = \left[\frac{x}{L}\right]$$

$$B_a^{enr} = H\left[\frac{1}{L}\right] = \left[\frac{1}{L}\right]$$

$$K_{uu} = EA\int_0^L \left(B_u^{std}\right)^T B_u^{std}\,dx$$

$$= \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[-1; \quad 1]$$

$$K_{ua} = EA\int_0^L \left(B_u^{std}\right)^T B_a^{enr}\,dx$$

$$= \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[1]$$

$$K_{au} = EA\int_0^L (B_a^{enr})^T B_u^{std}\,dx$$

$$= \frac{EA}{L}\,[1][-1; \quad 1]$$

$$K_{au} = EA\int_0^L (B_a^{enr})^T B_a^{enr}\,dx$$

$$= \frac{EA}{L}\,[-1][-1]$$

$$\boxed{\begin{gathered} B_u^{std} = \frac{1}{L}[-1 \quad 1] \\ B_a^{enr} = \frac{1}{L}[1] \end{gathered}}$$

$$K = \int \underline{B}^T\,\underline{D}\,\underline{B}\,dx$$

$$K_{uu} = \frac{EA}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

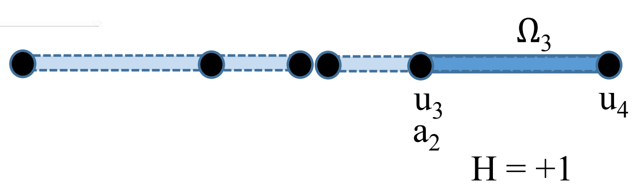
$$K_{ua} = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$K_{au}^T = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$K_{aa} = \frac{EA}{L}$$

Local Level

Element 3



$$\begin{array}{cccc} & u_3 & u_4 & a_2 \\ K_{element\ 3} = & \left[\begin{matrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{matrix}\right. & & \left.\right] \end{array}$$

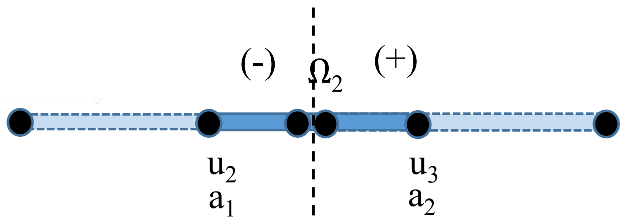
$$K_{uu} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathrm{K_{ua}} = \frac{EA}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{au}^{T} = \frac{EA}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$K_{aa} = \frac{EA}{L}$$

Element 2 (enriched nodes)



$$\mathbf{N}^{enh} = [\mathbf{N}^{std}, \mathbf{N}^{enr}]$$

$$N_u^{std} = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$B_u^{std} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

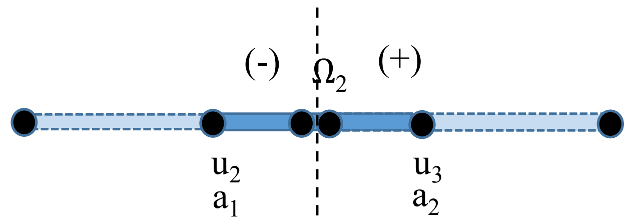
$$N_a^{enr} = H\left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$B_a^{enr} = H\left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

Element 2 (enriched nodes)

Ω^-

Ω^+



$$N_u^{std} = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$B_u^{std} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

$$N_a^{enr}(-) =$$

$$\left[\frac{x}{L} - 1 \quad -\frac{x}{L}\right]$$

$$B_a^{enr}(-) = \left[\frac{1}{L} \quad -\frac{1}{L}\right]$$

$$B_u^{std} = \frac{1}{L}[-1 \quad 1]$$
$$B_a^{enr}(+) = \frac{1}{L}[1 \quad -1]$$
$$B_a^{enr}(-) = \frac{1}{L}[-1 \quad 1]$$

$$N_u^{std} = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$B_u^{std} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

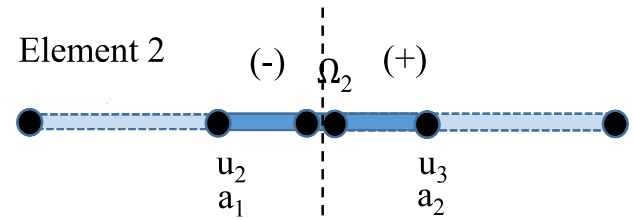
$$N_a^{enr}(+) =$$

$$H\left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$B_a^{enr}(+) = H\left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

Local Level

$$K_{uu} = EA\int_0^L \left(B_u^{std}\right)^T B_u^{std}\,dx = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[-1;\quad 1]$$



$$K_{ua}^{-} = EA\int_0^{\frac{L}{2}} \left(B_u^{std}\right)^T B_a^{enr^-}\,dx = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[-1;\quad 1]$$

$$K_{au}^{-} = EA\int_0^{\frac{L}{2}} (B_a^{enr-})^T B_u^{std}\,dx = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[-1;\quad 1]$$

$$K_{aa}^{-} = EA\int_0^{\frac{L}{2}} (B_a^{enr-})^T B_a^{enr^-}\,dx = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[-1;\quad 1]$$

$$K_{ua} = EA\int_{\frac{L}{2}}^{L} \left(B_u^{std}\right)^T B_a^{enr}\,dx = \frac{EA}{L}\begin{bmatrix}-1\\1\end{bmatrix}[1;\quad -1]$$

$$K_{au} = EA\int_{\frac{L}{2}}^{L} (B_a^{enr})^T B_u^{std}\,dx = \frac{EA}{L}\begin{bmatrix}1\\-1\end{bmatrix}[-1;\quad 1]$$

$$K_{au} = EA\int_{\frac{L}{2}}^{L} (B_a^{enr})^T B_a^{enr}\,dx = \frac{EA}{L}\begin{bmatrix}1\\-1\end{bmatrix}[1;\quad -1]$$

$$K_{uu} = \frac{EA}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\boxed{\begin{aligned} B_u^{std} &= \tfrac{1}{L}[-1\quad 1]\\ B_a^{enr}(+) &= \tfrac{1}{L}[1\quad -1]\\ B_a^{enr}(-) &= \tfrac{1}{L}[-1\quad 1]\end{aligned}}$$

$$K_{ua}^{-} = \frac{EA}{2L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$K_{au}^{T-} = \frac{EA}{2L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$K_{aa}^{-} = \frac{EA}{2L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$K_{ua}^{+} = \frac{EA}{2L}\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$$

$${K_{au}^{+}}^{T} = \frac{EA}{2L}\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$$

$$K_{aa}^{+} = \frac{EA}{2L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Local Level

$$K_{ua} = K_{ua}^{-} + K_{ua}^{+}$$
$$K_{au} = K_{au}^{-} + K_{au}^{+}$$
$$K_{aa} = K_{aa}^{-} + K_{aa}^{+}$$

$$K_{ua} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{EA}{2L}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$K_{au} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{EA}{2L}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$K_{aa} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{element\ 2} = \frac{EA}{L}\begin{array}{c} \begin{array}{cccc} u_2 & u_3 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\boxed{K_{uu} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad K_{aa} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

All Element

$$\begin{bmatrix} K_{uu} & K_{ua} \\ K_{au} & K_{aa} \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{a} \end{Bmatrix} = \begin{bmatrix} F_u \\ F_a \end{bmatrix}$$

Integration

$$K = \int \underline{B}^T \, \underline{D} \, \underline{B} \, dx$$

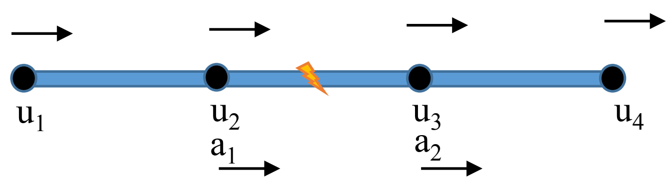
Fill in stiffness matrix

$$\begin{bmatrix} K_{uu} & K_{ua} \\ K_{au} & K_{aa} \end{bmatrix}, \text{6x6}$$

Moving to Global Level

$$K = \frac{EA}{L} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ & 2 & -1 & 0 & 1 & 0 \\ & & 2 & -1 & 0 & -1 \\ & \text{Symmetric} & & 1 & 0 & 1 \\ & & & & 2 & -1 \\ & & & & & 2 \end{bmatrix} \end{array}$$

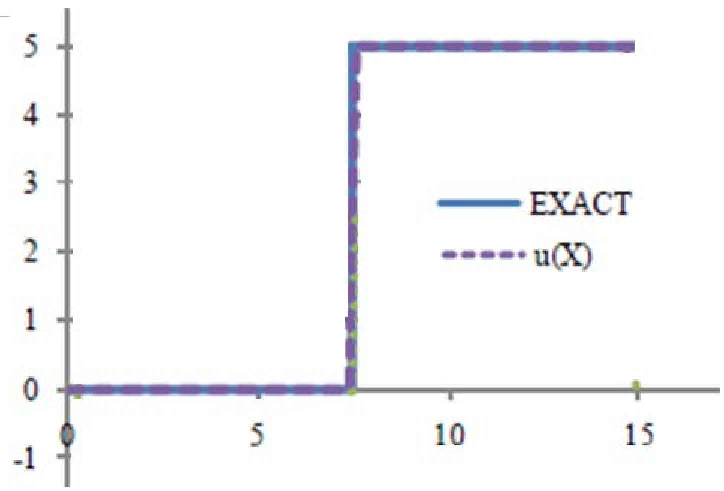
Stiffness Matrix

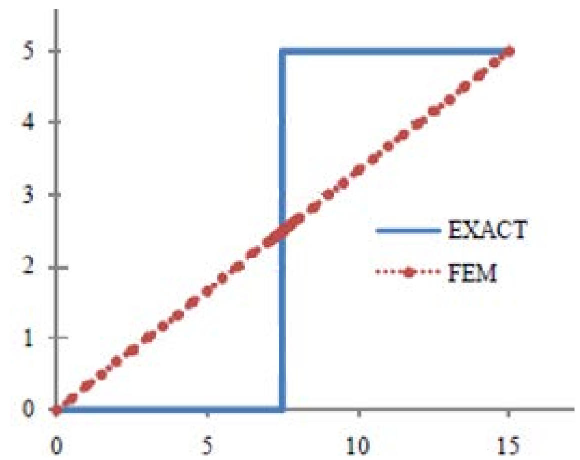


$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{u} \\ 0 \\ -\bar{u} \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\bar{u}}{2} \\ \frac{\bar{u}}{2} \\ \bar{u} \\ -\frac{\bar{u}}{2} \\ -\frac{\bar{u}}{2} \end{bmatrix}$$

XFEM Result

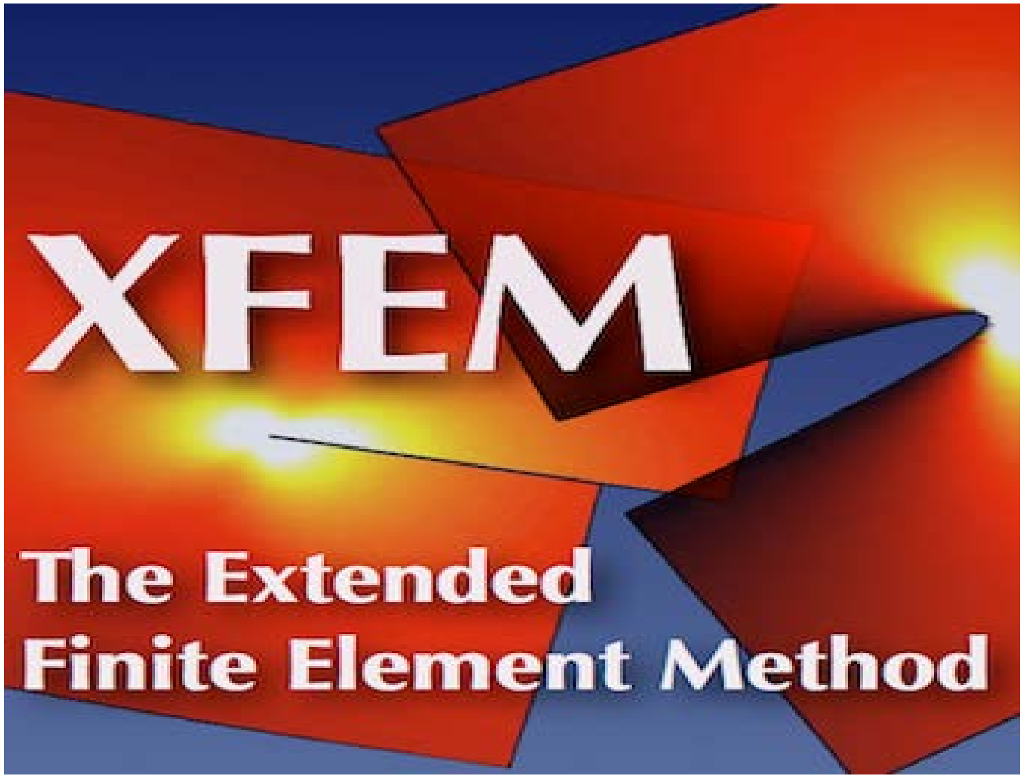
$$\mathbf{K}\bar{\mathbf{U}} - \mathbf{F} = 0$$





Source: Kermani et.al presentation, 2014

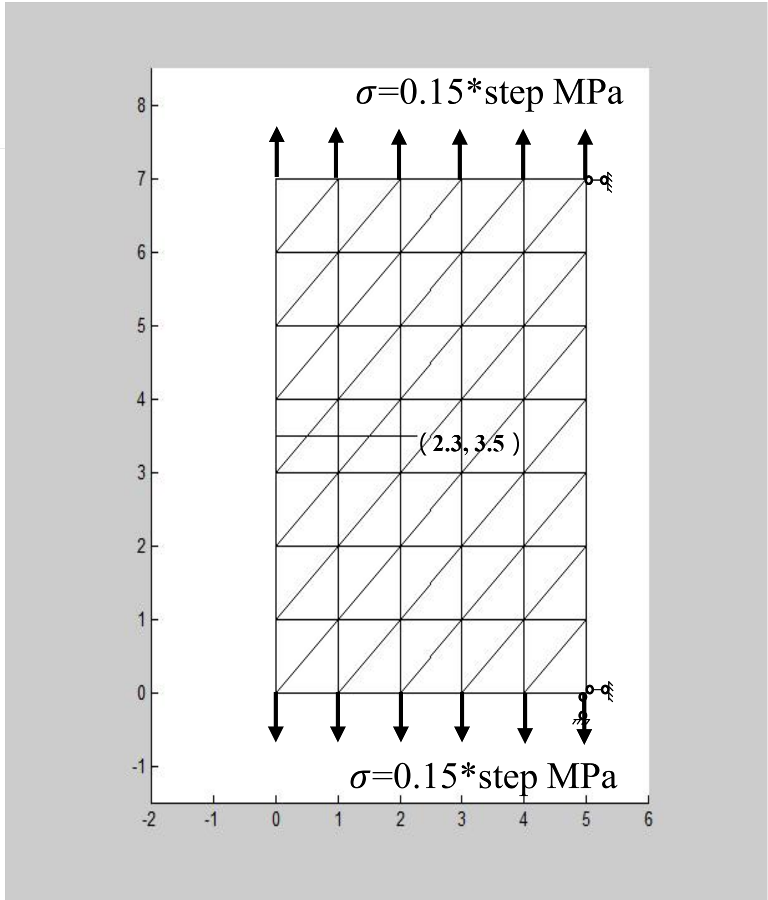
NUMERICAL EXAMPLE

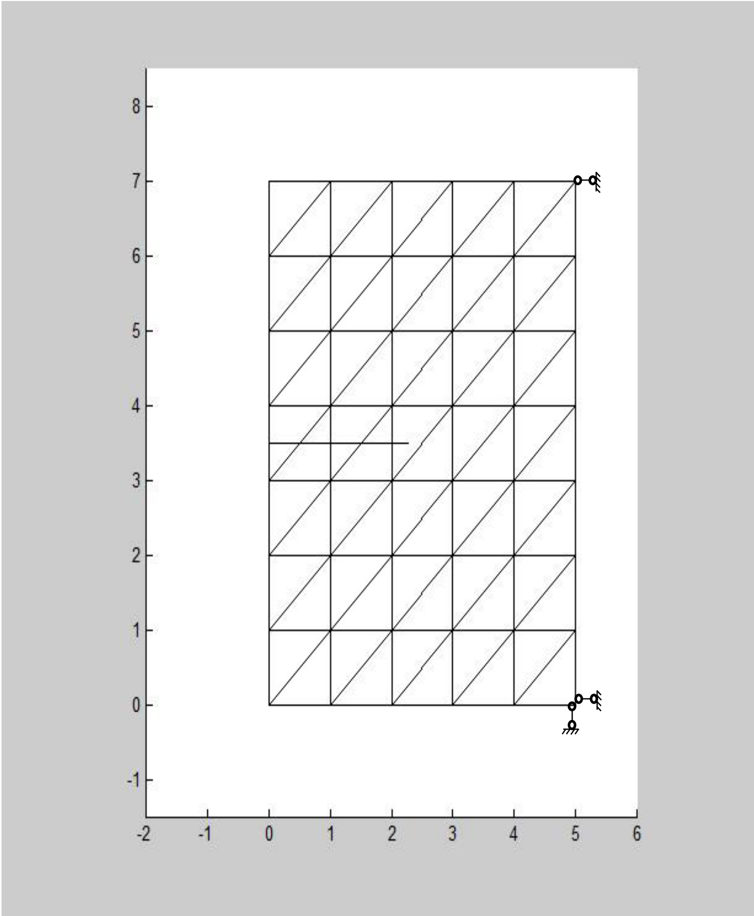


Numerical Example

- **2D Static Edge Crack**

E=10 MPa
v=0.3



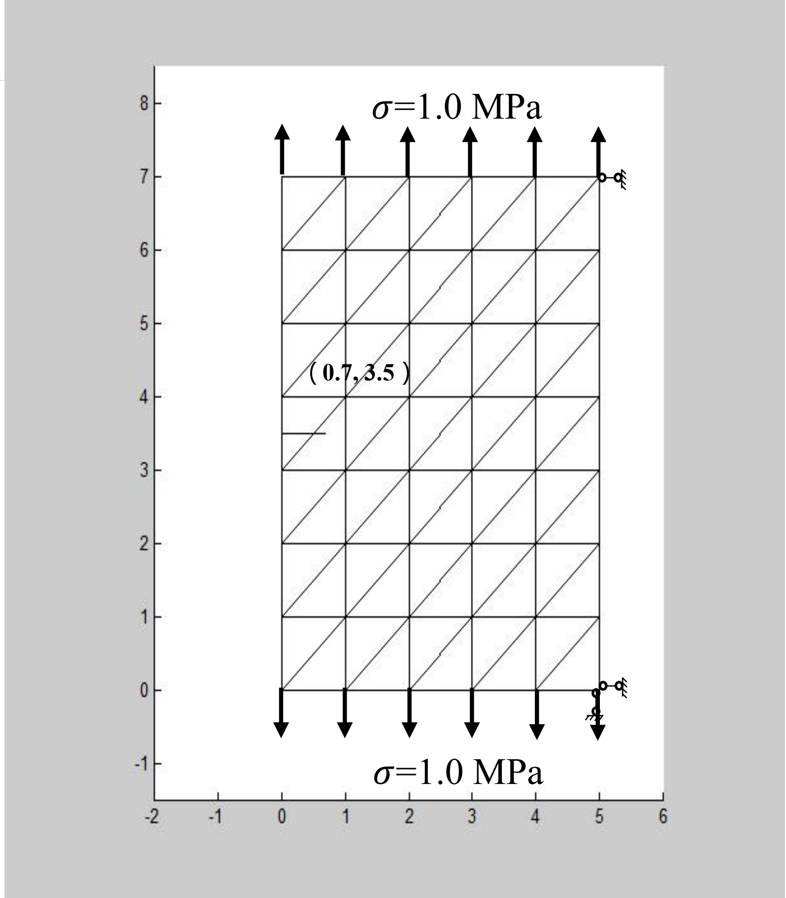


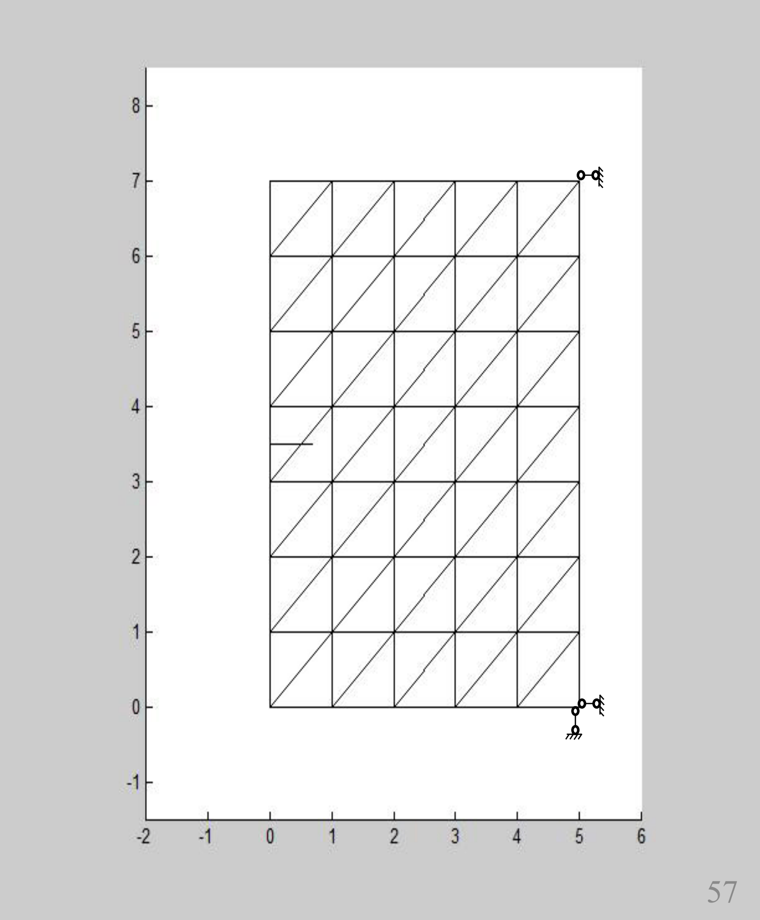
Numerical Example

➢ **2D Edge Crack Growth**

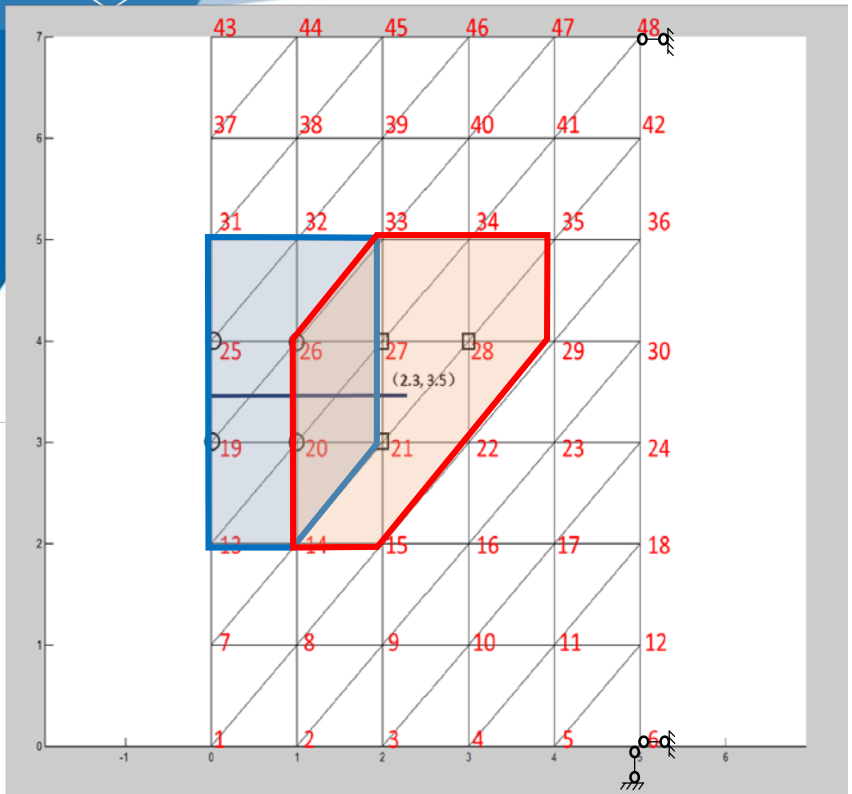
E=10 MPa
v=0.3
Tensile strength=0.1 MPa

Crack propagation criterion: Maximum stress>Tensile strength
Crack propagation direction: Vertical to the direction of maximum stress
Magnitude of the incremental crack growth: 1/10 of the initial crack length





Numerical Example



$$\mathbf{K}\mathbf{u}^h = \mathbf{f}$$

Size of Global Stiffness Matrix

$$= 41 \times 2 + 4 \times 4 + 3 \times 10 = 128$$

$$\begin{bmatrix} K_{(1,1)} & \cdots & K_{(1,128)} \\ \vdots & \ddots & \vdots \\ K_{(128,1)} & \cdots & K_{(128,128)} \end{bmatrix}_{128\times128} \begin{bmatrix} u^h_{(1,1)} \\ \vdots \\ u^h_{(128,1)} \end{bmatrix}_{128\times1} = \begin{bmatrix} f_{(1,1)} \\ \vdots \\ f_{(128,1)} \end{bmatrix}_{128\times1}$$

	Number of nodes	Number of unknowns per node
Standard nodes	41	2
Nodes with “H” enrichment	4	4
Nodes with “F” enrichment	3	10

Displacement approximation

$$\boldsymbol{u} = \sum_{i\in I} N_i(\boldsymbol{x})\boldsymbol{u_i} + \sum_{j\in J} N_j(\boldsymbol{x})H(\boldsymbol{x})\boldsymbol{a_i} + \sum_{k\in K}\sum_{\alpha=1}^{4} N_k(\boldsymbol{x})F_\alpha(\boldsymbol{x})\boldsymbol{b_k^\alpha}$$

Numerical Example

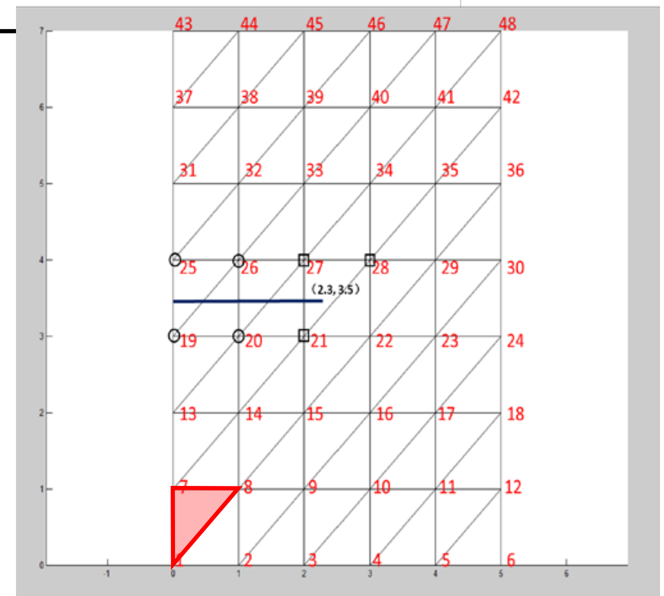
- Local Stiffness Matrix for Standard Elements

	Number of unknowns per node
Node 1	2
Node 8	2
Node 7	2

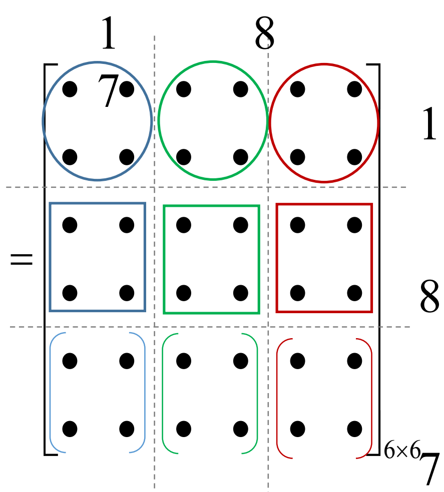
$$B = \begin{bmatrix} N_{1,x} & 0 & N_{8,x} & 0 & N_{7,x} & 0 \\ 0 & N_{1,y} & 0 & N_{8,y} & 0 & N_{7,y} \\ N_{1,y} & N_{1,x} & N_{8,y} & N_{8,x} & N_{7,y} & N_{7,x} \end{bmatrix}$$

$$K_{local} = \int [B]^T [D][B] dV$$

$$6\text{x}6 \;=\; 6\text{x}3 \;3\text{x}3 \;3\text{x}6$$



$K_{local} =$

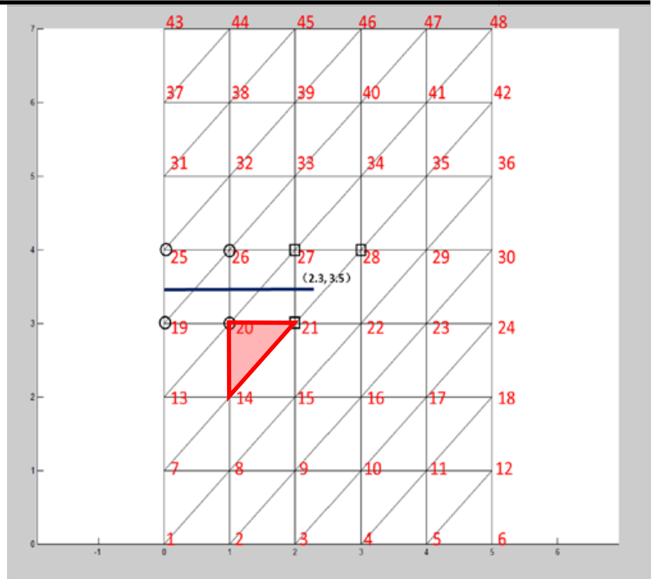


$$K_{global} = \begin{bmatrix} K_{(1,1)} & K_{(1,2)} & \cdots & K_{(1,13)} & K_{(1,14)} & K_{(1,15)} & K_{(1,16)} & \cdots & K_{(1,128)} \\ K_{(2,1)} & K_{(2,2)} & \cdots & K_{(2,13)} & K_{(2,14)} & K_{(2,15)} & K_{(2,16)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ K_{(13,1)} & K_{(13,2)} & \cdots & K_{(13,13)} & K_{(13,14)} & K_{(13,15)} & K_{(13,16)} & \cdots & \vdots \\ K_{(14,1)} & K_{(14,2)} & \cdots & K_{(14,13)} & K_{(14,14)} & K_{(14,15)} & K_{(14,16)} & \cdots & \vdots \\ K_{(15,1)} & K_{(15,2)} & \cdots & K_{(15,13)} & K_{(15,14)} & K_{(15,15)} & K_{(15,16)} & \cdots & \vdots \\ K_{(16,1)} & K_{(16,2)} & \cdots & K_{(16,13)} & K_{(16,14)} & K_{(16,15)} & K_{(16,16)} & \cdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{(128,1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & K_{(128,128)} \end{bmatrix}_{128\times128}$$

Numerical Example

➤ Local Stiffness Matrix for Enriched Elements

	Enrichment type	Number of unknowns per node
Node 14	unenriched	2
Node 21	“F”	10
Node 20	“H”	4



Node 14 (unenriched) Node 21 (enriched by “F”)

$$
B = \left[\begin{array}{cc|cccccccccc}
N_{14,x} & 0 & N_{21,x} & 0 & (N_{21}F^{1}{}_{21})_{,x} & 0 & (N_{21}F^{2}{}_{21})_{,x} & 0 & (N_{21}F^{3}{}_{21})_{,x} & 0 & (N_{21}F^{4}{}_{21})_{,x} & 0 \\
0 & N_{14,y} & 0 & N_{21,y} & 0 & (N_{21}F^{1}{}_{21})_{,y} & 0 & (N_{21}F^{2}{}_{21})_{,y} & 0 & (N_{21}F^{3}{}_{21})_{,y} & 0 & (N_{21}F^{4}{}_{21})_{,y} \\
N_{14,y} & N_{14,x} & N_{21,y} & N_{21,x} & (N_{21}F^{1}{}_{21})_{,y} & (N_{21}F^{1}{}_{21})_{,x} & (N_{21}F^{2}{}_{21})_{,y} & (N_{21}F^{2}{}_{21})_{,x} & (N_{21}F^{3}{}_{21})_{,y} & (N_{21}F^{3}{}_{21})_{,x} & (N_{21}F^{4}{}_{21})_{,y} & (N_{21}F^{4}{}_{21})_{,x}
\end{array}\right. \cdots
$$

$$
\cdots \left.\begin{array}{cccc}
N_{20,x} & & (N_{20}H_{20})_{,x} & \\
 & N_{20,y} & & (N_{20}H_{20})_{,y} \\
N_{20,y} & N_{20,x} & (N_{20}H_{20})_{,y} & (N_{20}H_{20})_{,x}
\end{array}\right]_{3\times 16}
$$

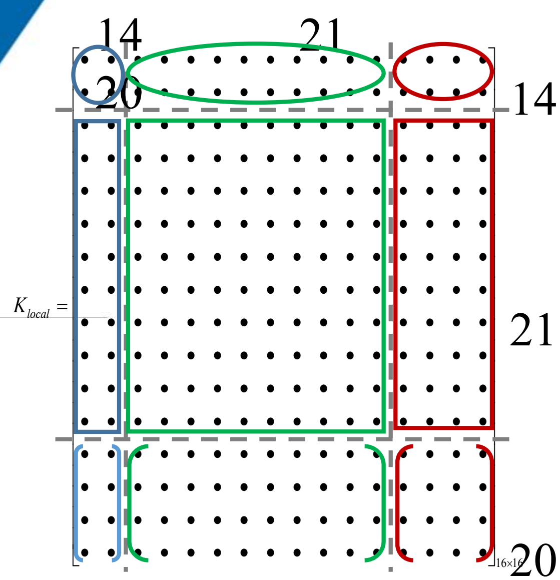
Node 20 (enriched by “H”)

$$K_{local} = \int [B]^{T} [D][B] dV$$

16x16 = 16x3 3x3 3x16

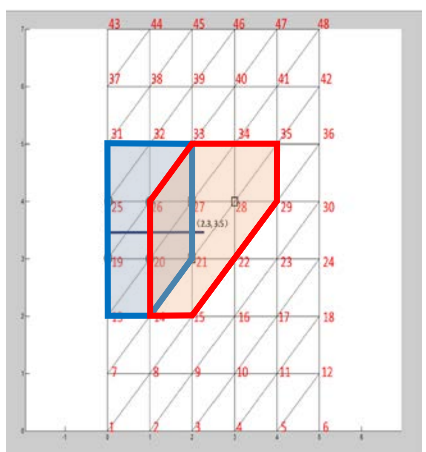
Numerical Example

Local Stiffness Matrix (16x16)



Global Stiffness Matrix (128x128)

$$K_{global} = \begin{bmatrix}
K_{(1,1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & K_{(1,128)} \\
\vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & K_{(27,27)} & K_{(27,28)} & \cdots & K_{(27,41)} & \cdots & K_{(27,44)} & K_{(27,45)} & \cdots & K_{(27,54)} & \cdots & \vdots \\
\vdots & \cdots & K_{(28,27)} & K_{(28,28)} & \cdots & K_{(28,41)} & \cdots & K_{(28,44)} & K_{(28,45)} & \cdots & K_{(28,54)} & \cdots & \vdots \\
\vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \cdots & K_{(41,27)} & K_{(41,28)} & \cdots & K_{(41,41)} & \cdots & K_{(41,44)} & K_{(41,45)} & \cdots & K_{(41,54)} & \cdots & \vdots \\
\vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\vdots & \cdots & K_{(44,27)} & K_{(44,28)} & \cdots & K_{(44,41)} & \cdots & K_{(44,44)} & K_{(44,45)} & \cdots & K_{(44,54)} & \cdots & \vdots \\
\vdots & \cdots & K_{(45,27)} & K_{(45,28)} & \cdots & K_{(45,41)} & \cdots & K_{(45,44)} & K_{(45,45)} & \cdots & K_{(45,54)} & \cdots & \vdots \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\vdots & \cdots & K_{(54,27)} & K_{(54,28)} & \cdots & K_{(54,41)} & \cdots & K_{(54,44)} & K_{(54,45)} & \cdots & K_{(54,54)} & \cdots & \vdots \\
\vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
K_{(128,1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & K_{(128,128)}
\end{bmatrix}_{128\times128}$$



Calculate K_{local} for every element

↓

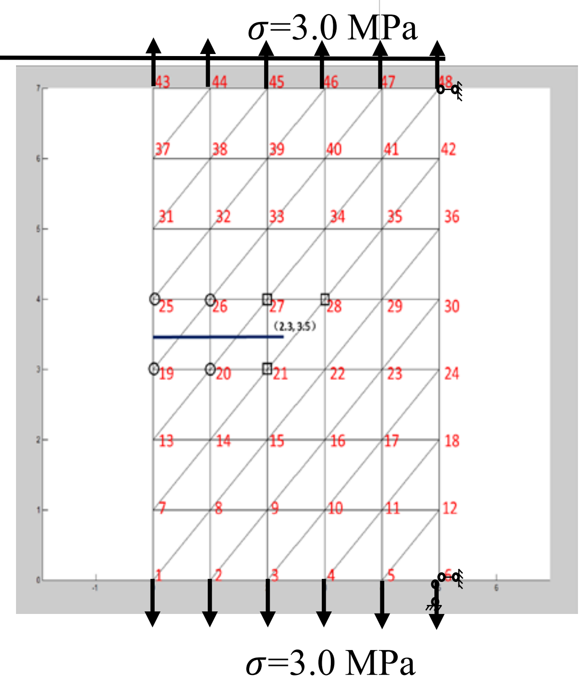
K_{global}

Numerical Example

- Apply boundary conditions

Prescribed displacement (equal to 0)

$$
\begin{bmatrix}
K_{(1,1)} & \cdots & K_{(1,11)} & K_{(1,12)} & \cdots & K_{(1,127)} & K_{(1,128)} \\
\vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\
K_{(11,1)} & \cdots & \ddots & \cdots & \cdots & \cdots & K_{(11,128)} \\
K_{(12,1)} & \cdots & \cdots & \ddots & \cdots & \cdots & K_{(12,128)} \\
\vdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\
K_{(127,1)} & \cdots & \cdots & \cdots & \cdots & \ddots & K_{(127,128)} \\
K_{(128,1)} & \cdots & K_{(128,11)} & K_{(128,12)} & \cdots & K_{(128,127)} & K_{(128,128)}
\end{bmatrix}_{128\times128}
\begin{bmatrix}
u^h_{(1,1)} \\ \vdots \\ u^h_{(11,1)} \\ u^h_{(12,1)} \\ \vdots \\ u^h_{(127,1)} \\ u^h_{(128,1)}
\end{bmatrix}_{128\times1}
=
\begin{bmatrix}
f_{(1,1)} \\ \vdots \\ f_{(11,1)} \\ f_{(12,1)} \\ \vdots \\ f_{(127,1)} \\ f_{(128,1)}
\end{bmatrix}_{128\times1}
$$

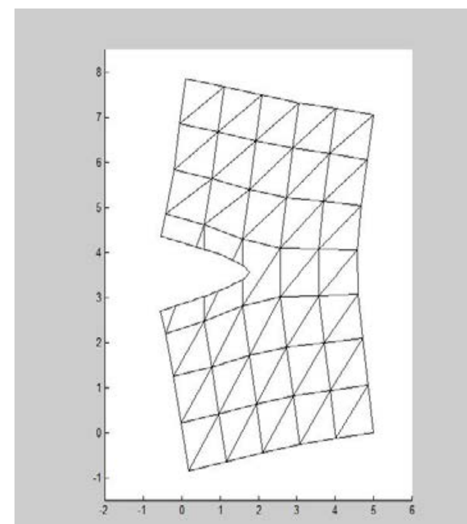


$$
\text{with}\quad
\begin{cases}
f_{(2,1)} = f_{(12,1)} = -1.5\times10^{6}\ \text{N} \\
f_{(4,1)} = f_{(6,1)} = f_{(8,1)} = f_{(10,1)} = -3.0\times10^{6}\ \text{N} \\
f_{(118,1)} = f_{(128,1)} = 1.5\times10^{6}\ \text{N} \\
f_{(120,1)} = f_{(122,1)} = f_{(124,1)} = f_{(126,1)} = 3.0\times10^{6}\ \text{N}
\end{cases}
$$

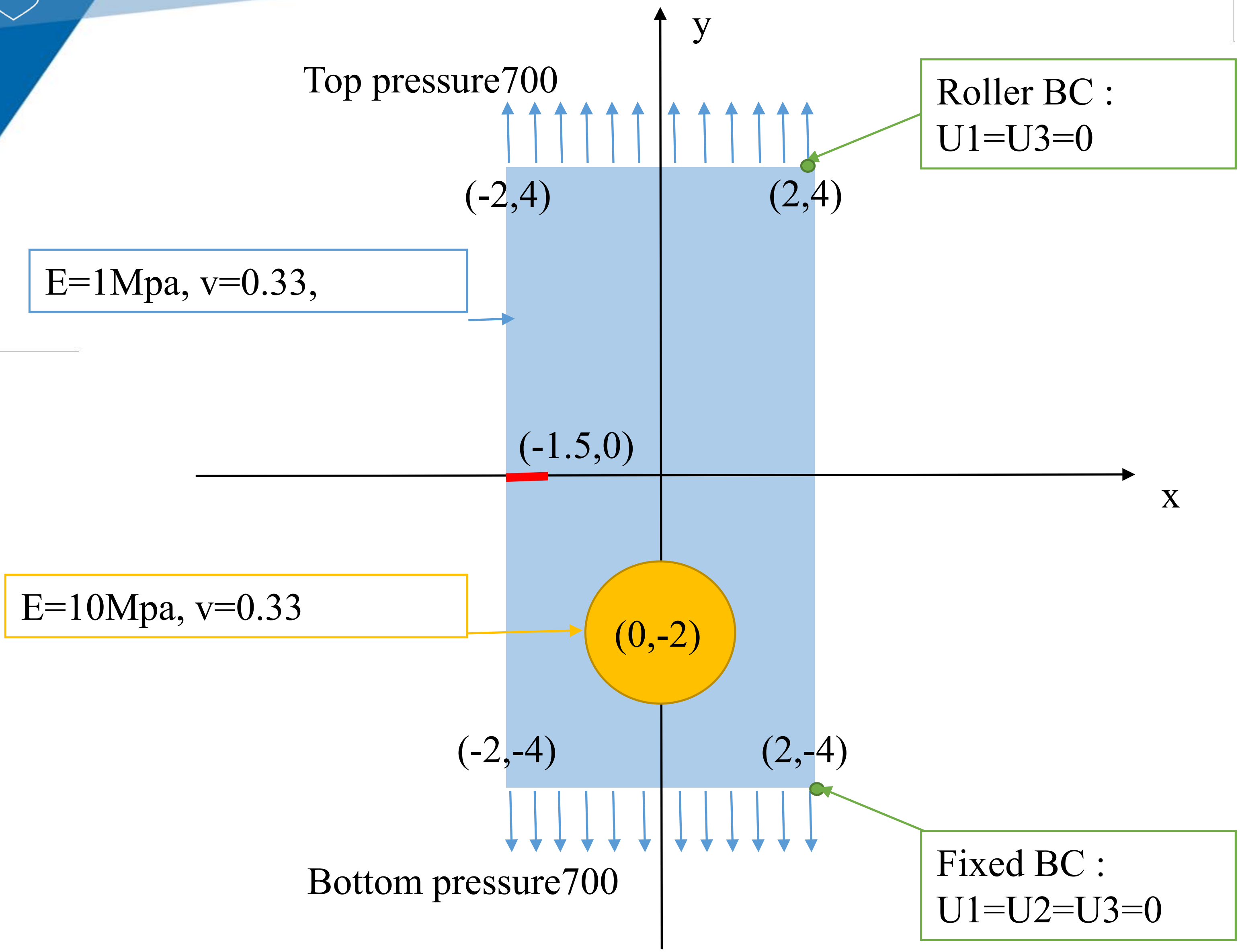
We can solve for all the unknowns!

Calculate displacement

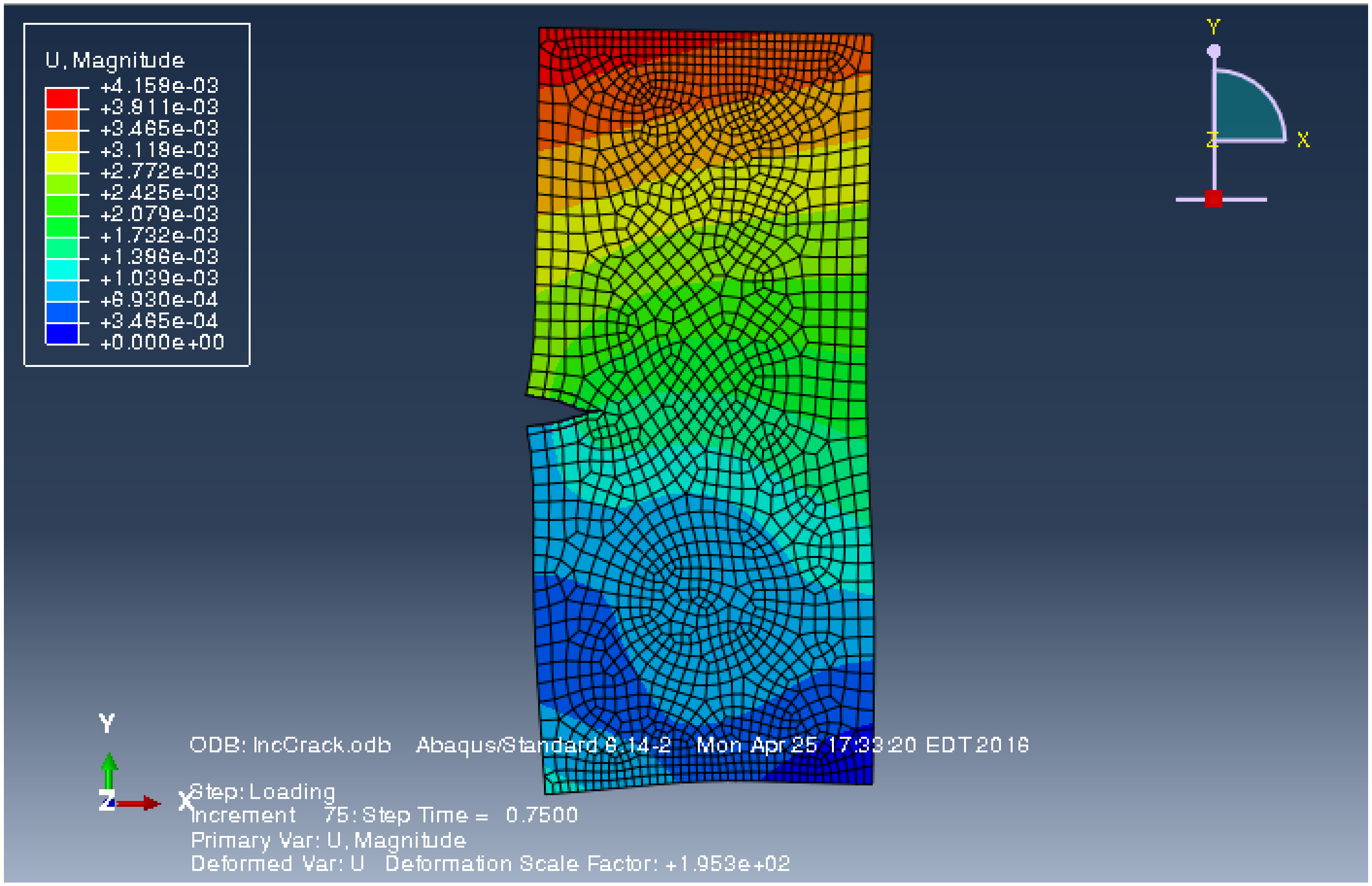
$$
\boldsymbol{u} = \sum_{i\in I} N_i(\boldsymbol{x})\boldsymbol{u_i} + \sum_{j\in J} N_j(\boldsymbol{x})H(\boldsymbol{x})\boldsymbol{a_i} + \sum_{k\in K}\sum_{\alpha=1}^{4} N_k(\boldsymbol{x})F_\alpha(\boldsymbol{x})\boldsymbol{b_k^\alpha}
$$



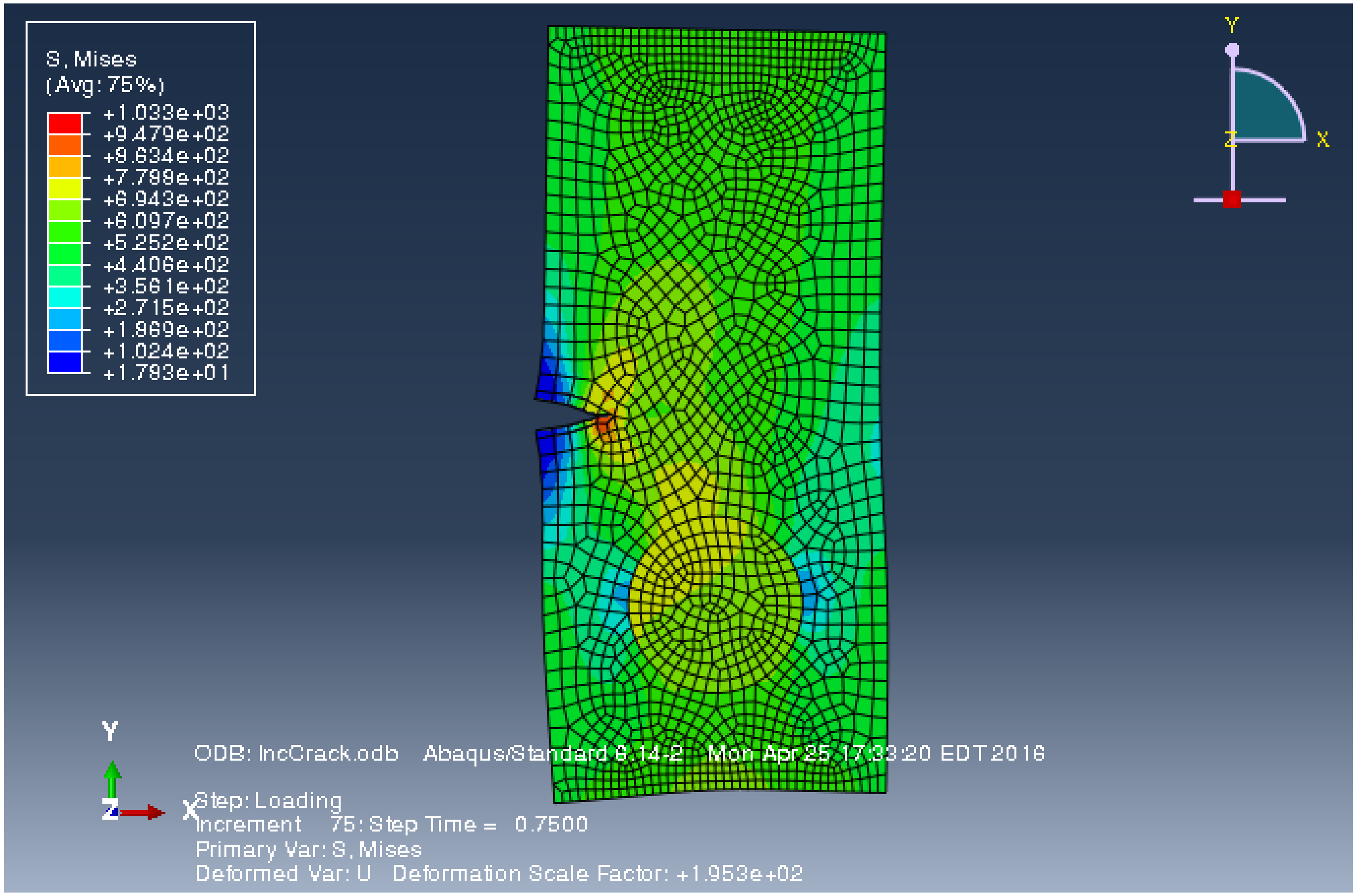
2D Edge Crack with Inclusion



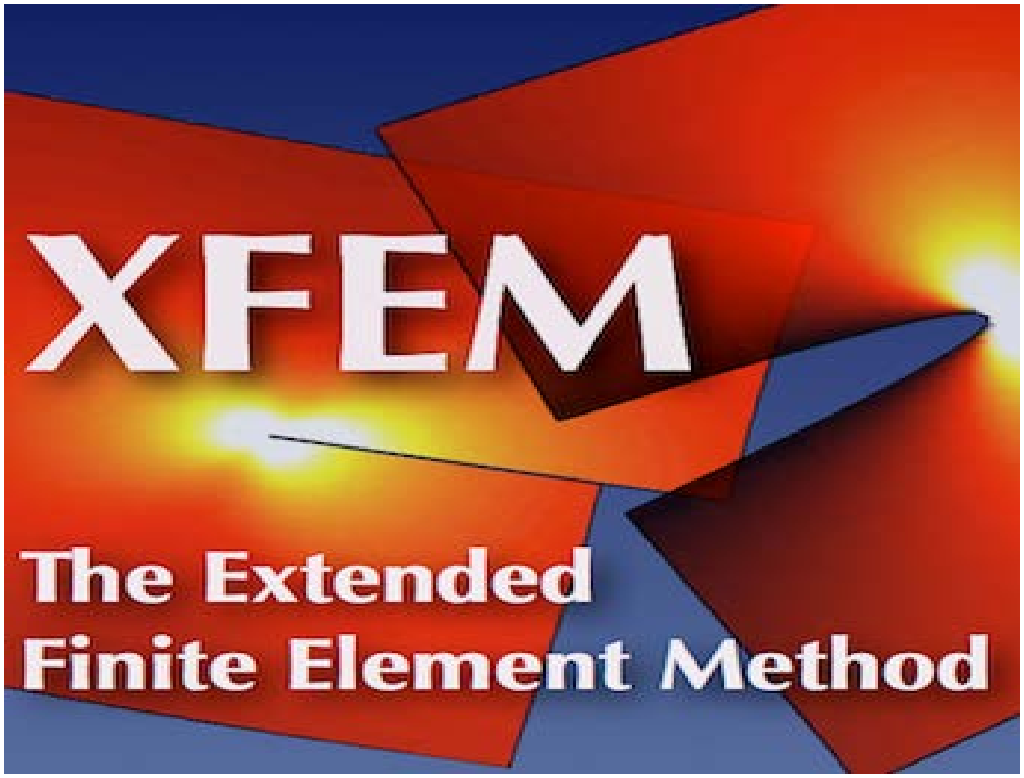
2D Edge Crack with Inclusion



2D Edge Crack with Inclusion



HAND-CALCULATION EXAMPLE



Example Application

- Example simple open hole
https://www.youtube.com/watch?v=4i8TxVix1x0
- Example cooling fin
https://youtu.be/mnJwdD8PdDg?t=4
- Example target temperature
https://www.youtube.com/watch?v=RCLE5pLwMdQ
- Example cracking
https://www.youtube.com/watch?v=thA7Cy1x0vk
https://www.youtube.com/watch?v=nU2bDphLJBc

THANK YOU!